

# Fair Valuation and Hedging of Participating Life-Insurance Policies under Management Discretion

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## Abstract

The fair valuation of participating life insurance policies, also known as With-Profit policies, is considered. Such policies can be seen as European path-dependent contingent claims whose underlying security is the investment portfolio of the insurance company that sold the policy. The fair valuation of these policies is studied under the assumption that the insurance company has the right to modify the investment strategy of the underlying portfolio at any time. Furthermore, it is assumed that the issuer of the policy does not want or can not setup a separate portfolio to hedge the risk associated with the life-insurance policy. Instead, we assume that the management of the insurance company that sold the policy will use its discretion about the investment portfolio to choose the portfolio strategy such that the risk is eliminated or reduced as far as possible. In that sense, the insurer's investment portfolio serves simultaneously as the underlying security and as the hedge portfolio. This means that the hedging problem can not be separated from the valuation problem. We will show how a risk-neutral value of such policies can be calculated and how the management of the insurer can use its discretion about its investment strategy to hedge the contract if the financial market satisfies some assumptions.

Keywords: Participating Life Insurance Policy, With-Profits policy, Risk-Neutral Valuation, Hedging

# 1 Introduction and Literature Review

Participating Life insurance contracts, also called With-Profits contracts, have been issued over the past decades by many insurance companies throughout the world. Although the details of the contracts vary significantly between insurer's there are some common features that all With-Profits contracts share.

In exchange for regular premiums the policyholder receives payoffs at death, surrender of maturity. Since the payoffs of these contracts are based on the performance of a particular investment fund, the risk associated with these contracts is not only mortality risk, but also financial risk. We want to concentrate on the financial risk and, we therefore ignore mortality and surrender, and we assume that all contracts reach maturity. Furthermore, we assume that there is only one premium to be paid by the policyholder at the time the contract is issued.

As we mentioned above the several mechanism to calculate the final payoff to policyholders across the insurance industry. To get an overview about contracts which are common in Europe and the United States see Cummins, Miltersen & Persson (2004).

Since we ignore mortality, buying a With-Profits contract can be seen as an investment by the policyholder into a fund, often called the With-Profits fund, which is managed by the insurance company. In contrast to standard investment funds, With-Profits contracts provide some protection against low or negative returns. Instead of receiving the final value of the fund, the policyholder receives the final balance of an account, called the policyholder's account. The initial balance of this account is a constant depending on the particular contract. At the end of every year the growth rate of the policyholder's account is calculated by the insurance company based on the return of the With-Profit fund during the year. The balance of the account at the end of the year is the balance at the beginning of the year accumulated by the growth rate. At any time before maturity, this balance does not represent a value in an economic sense. The interpretation of that account is, that the balance of the account at the end of any year would be paid to the policyholder if the contract matured immediately.

The calculation of the balance at the end of a particular year is sometimes based on the performance of the fund over the last few years. However, we want to consider contracts in which the growth rate of the policyholder's account only depends on the performance of the With-Profits fund during the current year.

In the literature we find two approaches to calculate fair market-consistent values of With-Profit contracts and derive hedging strategies that the insur-

ance company can apply to protect itself against the risk associated with the minimum guaranteed rate of growth of the policyholder's account.

In the first approach authors treat the With-Profit fund as a fixed investment portfolio. The price process of the fund is therefore a given stochastic process. The payoff at maturity, the final value of the policyholder's account, is then a path-dependent European contingent claim. This approach allows us to directly apply methods known from Financial Mathematics. The first authors to use these methods to price life insurance contracts were Brennan & Schwartz (1976, 1979). These authors considered unit-linked contract for which the payoff to the policyholder is indeed a contingent claim with a payoff depending on a unit, which is an externally given reference portfolio. Since then, Market-consistent valuation of participating insurance contracts has been studied by a number of authors. Among these are Persson & Aase (1997), Miltersen & Persson (1999), Miltersen & Persson (2003), Ballotta (2005). Overviews about the available literature can be found in Kleinow & Willder (2007) and Bauer, Kiesel, Kling & Ruß (2005) and the references therein. A very detailed discussion about the approaches and the results of different authors was carried out by Willder (2004).

A second approach to price and hedge With-Profits funds is based on the assumption that the management of the insurance company has full discretion about the composition of the With-Profits fund. This means that the insurer can change its investment strategy in the With-Profits fund at any time which in contrast to the fixed strategy applied when a reference portfolio is considered. In particular, the insurer can use this discretion to reduce or increase the volatility of the With-Profits fund by changing the proportion of money invested into equity shares and the proportion invested into fixed-income securities. Since the With-Profits fund is the underlying security of the contingent claim that represents the payoff to the policyholder, any change in the With-Profits fund will result in change of the value of the With-Profits insurance contract.

Hibbert & Turnbull (2003) were the first to address this issue. They consider an insurance company in which the management have limited discretion in choosing the assets by applying a fixed rule to increase or decrease the equity exposure of the With-Profits fund depending on the value of the insurer's assets and the guarantees already declared. They calculate the fair value of the With-Profits contract in this situation.

Kleinow & Willder (2007) consider a more realistic setting by assuming that the growth rate of the policyholder's account depends on the actual investment portfolio of the insurer and that the management of the insurer has the right to change their investment strategy whenever and however they want to. Any change in this portfolio strategy will lead to a change in the

underlying price process that is used to calculate the growth rate of the policyholder's account. On the other hand, the insurer is not allowed to set up a separate portfolio to hedge the risks associated with the contract, but whenever the management of an insurance company wishes to hedge the risk associated with a contract it would change the investment portfolio of the company. As this portfolio is the underlying price process for the calculation of the growth rate of the policyholder's account any attempt to hedge the risk associated with the contract leads to a change of the underlying price process of the contract. The hedging and pricing of participating contracts in this setting is a non-standard problem in financial mathematics since the portfolio of the insurer serves simultaneously as the underlying process and the hedge portfolio of the contract.

Kleinow & Willder (2007) seem to be the first to address this problem in a systematic way. They assume a discrete time model for the financial market in which the short interest rate as well as the risky asset follow a binomial tree model. Although this is a relatively simple model for the dynamics of interest rates and risky assets these authors are still able to draw some interesting economic conclusions. First of all they find that the insurance company can perfectly hedge the risk associated with the contract by setting up an appropriate investment strategy. Furthermore, they show that the investment portfolio of the company will only consist of one-year and two-year zero-coupon bonds.

These results have been generalized to a continuous time financial market model by Kleinow (2006). We found that a perfect hedge is not possible unless the market consisting of a particular zero-coupon bond and the bank account is complete. However, even in an incomplete market we were able to derive a fair price for the contract by showing that the contract can be replicated by a sequence of particular contingent claims. These claims were priced using risk-neutral valuation.

In the current paper we want to generalize the results found by Kleinow & Willder (2007) and Kleinow (2006) further. We will particularly emphasize the relationship between hedging and valuation.

The paper is organized as follows. In Section 2 we introduce the financial market model. The contract is described in Section 3. The pricing and hedging problem is introduced in the same Section. In Section 4 we give a precise formulation of the hedging and valuation problem, define the fair value and the risk-neutral value of a participating policy and show some properties of these values. We also investigate the relationship between fair/risk-neutral valuation and hedging in both, complete and incomplete markets. Finally, we provide a summary and some suggestions for further research in Section 5.

## 2 The Financial Market Model

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  denote a probability space, and let  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  for  $T \in \mathbb{N}$  be a right-continuous and complete filtration defined on this space. We assume that  $\mathcal{F}_0$  is the trivial  $\sigma$ -field.

The tradable assets are a bank account and  $d \in \mathbb{N}$  further risky assets. We assume that the value  $S_0(t)$  of the bank account is predictable with respect to  $\mathbb{F}$  and  $S_0(0) = 1$ .

The price processes of the  $d$  risky assets are denoted by  $S_1, \dots, S_d$ . We assume that these are semimartingales with respect to  $\mathbb{F}$  under  $\mathbb{P}$ .

The value of the issuer's assets at time  $t$  is denoted by  $V(t)$ . The issuer can invest into all existing assets and the value of his portfolio at any time  $t \in [0, T]$  is therefore

$$V(t) = \sum_{i=0}^d \xi_i(t) S_i(t) \quad (1)$$

where  $\xi = (\xi_0, \xi_1, \dots, \xi_d)$  is predictable with respect to  $\mathbb{F}$ . The  $(d+1)$ -dimensional process  $\xi$  is called the portfolio strategy.  $\xi_0(t)$  represents the number of units of the bank account and  $\xi_i(t)$  for  $i = 1, \dots, d$  is the number of risky assets that belong to the insurer's portfolio at time  $t$ .

A portfolio strategy is self-financing if

$$\begin{aligned} V(t) &= V(0) + \int_0^t \xi(u) dS(u) \\ &= V(0) + \sum_{i=0}^d \int_0^t \xi_i(u) dS_i(u) \quad \forall t \in [0, T]. \end{aligned}$$

We assume that our model is arbitrage-free, and therefore the set of equivalent martingale measures

$$\mathcal{Q} = \{Q \sim \mathbb{P} : S_i/S_0 \text{ is a } Q\text{-martingale } \forall i = 1, \dots, d\}. \quad (2)$$

is not empty. Using the notation

$$D(u, t) = S_0(u)/S_0(t) \quad 0 \leq u \leq t \leq T$$

for the discount factor, we obtain for the price of any risky asset

$$S(u) = \mathbb{E}_Q[D(u, t)S(t)|\mathcal{F}_u] \text{ for } u < t, \forall Q \in \mathcal{Q}$$

and if the portfolio of the insurance company is self-financing during a time period  $[u, t]$ , we obtain a similar property for  $V$ ,

$$V(u) = \mathbb{E}_Q[D(u, t)V(t)|\mathcal{F}_u] \quad \forall Q \in \mathcal{Q} \quad (3)$$

for all  $u$  and  $t$  with  $0 \leq u \leq t \leq T$ .

### 3 The Contract

We consider an insurance contract with maturity  $T$ . As mentioned in the introduction the payoff to the policyholder at maturity depends on the success of the insurer's investment strategy and some exogenously given contingent claim.

More precisely, for a positive real valued function  $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$  and a constant  $x_0 \in \mathbb{R}_+$  we consider the discrete-time process  $\{X(t), t = 0, \dots, T\}$  which is given by  $X(0) = x_0$

$$\begin{aligned} X(t+1) &= X(t)H\left(\frac{V(t+1)}{V(t)}\right) \\ &= x_0 \prod_{s=0}^t H\left(\frac{V(s+1)}{V(s)}\right) \text{ for } t = 0, \dots, T-1 \end{aligned} \quad (4)$$

where  $V$  is the value process of the insurer's portfolio as defined in (1). Economically,  $X$  represents the balance of an account that the insurance company holds on behalf of the policyholder. The policyholder can access this account at maturity only. Since  $X(T)$  is  $\mathcal{F}_T$ -measurable, it can be seen as a European contingent claim with underlying security  $V$ .  $X(t)$  for all  $t < T$  can be interpreted as the payoff to the policyholder if the contract were to mature at time  $t$  instead of  $T$ . Therefore,  $X(t)$  should not be interpreted as a value in an economic sense for any  $t < T$ . Instead, it should be seen as the "intrinsic value" at time  $t$  of a European contingent claim with maturity  $T$  and payoff  $X(T)$ .

We call the function  $H$  contract function or bonus distribution mechanism since  $H$  describes how  $X$  changes depending on the performance of  $V$ .

To generalize the payoff to the policyholder we assume that the policyholder receives a further derivative at maturity. A typical example for such a derivative is a Guaranteed Annuity Option that allows the policyholder to use the final balance  $X(T)$  of his account to buy an annuity for a price which was fixed at time 0. In general, the inclusion of such a derivative means that the final payoff to the policyholder is

$$X_T \Gamma \text{ where } \Gamma \text{ is a } \mathcal{F}_T \text{ measurable random variable.}$$

To summarize these arguments we define a participating life insurance policy in the following way.

**Definition 1** *Let  $x_0 > 0$  be a constant,  $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$  and  $\Gamma \in \mathcal{F}_T$ . We call the triplet  $(x_0, H, \Gamma)$  a participating life insurance policy. The payoff to the policyholder at maturity is  $X(T)\Gamma$  where  $X(T)$  is the final value of the*

process  $X$  defined in (4). The constant  $x_0$  is called nominal value,  $H$  is called contract function, and  $\Gamma$  is called terminal option of the contract.

We assume that the contract function  $H$  fulfills the following conditions for all  $\tilde{v} \in \mathbb{R}_+$ :

**Assumption 1**

(A1)  $H(y)$  is continuous and non-decreasing in  $y$  with  $H(0) > 0$

(A2)  $y/H(y)$  is strictly increasing in  $y$ .

A typical example for a contract function  $H$  considered by Kleinow & Willder (2007) and Kleinow (2006) is

$$H(y) = \max \{e^\gamma, y^\delta\} \tag{5}$$

where  $\gamma$  and  $\delta \in (0, 1)$  are constant. The growth rate of  $X$  during  $[t, t + 1]$  is then

$$r_X(t + 1) = \log X(t + 1) - \log X(t) = \max \{\gamma, \delta r_V(t + 1)\}$$

where  $r_V(t + 1) = \log V(t + 1) - \log V(t)$  is the return on  $V$  during the same period. The growth rate  $r_X(t + 1)$  of  $X$  is therefore the maximum of a guaranteed rate  $\gamma$  and the return  $r_V(t + 1)$  of the portfolio  $V$  during this period multiplied by a participation rate  $\delta$ .

We assume that the policyholder does not have a surrender option and that there is no mortality risk. Therefore, the policyholder will receive  $X(T)\Gamma$  at maturity  $T$ . Some remarks about a surrender option can be found in Kleinow (2006).

Since  $X(T)\Gamma$  is  $\mathcal{F}_T$ -measurable, the fair price of the contract is given by  $E_Q[S_0(T)^{-1}X(T)\Gamma|\mathcal{F}_0]$ . However, the calculation of this expectation requires knowledge about the investment strategy  $\xi$  of the insurer. We will assume here that the insurance company has the right to decide about this strategy and to change their portfolio at any time  $t \in [0, T]$ .

On the other hand, the insurance company wishes to hedge the risk associated with the contract  $(x_0, H, \Gamma)$ , i.e. the European contingent claim  $X(T)\Gamma$ . We assume that  $V$  represents the entire portfolio of the insurer. This means the insurer can not set up a separate hedge portfolio to hedge the payoff  $X(T)\Gamma$  since this portfolio would become part of the company's assets and therefore, would be part of  $V$  which, in turn, changes the law of the underlying value process  $V$ .

However, the insurance company might be able to choose the portfolio strategy  $\xi$  such that the final value  $V(T)$  of its portfolio at maturity is equal to the value of its liabilities  $X(T)\Gamma$ , in which case the risk associated with

the contract  $(x_0, H, \Gamma)$  is perfectly hedged. In contrast to the hedging of an option with a payoff depending on a given stochastic process, the portfolio  $V$  serves simultaneously as underlying process and hedge-portfolio.

## 4 Hedging and Risk-neutral Valuation

Since there is a close relationship between risk-neutral valuation and hedging of participating contracts we study complete and incomplete markets separately.

### 4.1 Complete Financial Markets

In a complete financial market model our first aim is to find an initial capital  $V(0)$  and a portfolio strategy  $\xi$  such that  $V(T) = X(T)\Gamma$   $\mathbb{P}$ -almost surely, that is the value  $V(T)$  of the insurer's assets is equal to the value  $X(T)\Gamma$  of the insurer's liabilities at maturity  $T$ . Since the market is complete there is a unique martingale measure  $\mathbb{Q} \sim \mathbb{P}$ , and it is sufficient to construct a process  $V$  such that  $V/S_0$  is a  $\mathbb{Q}$ -martingale and  $V(T) = X(T)\Gamma$  almost surely. Once the process  $V$  is constructed, the completeness of the market ensures that there exists a portfolio strategy  $\xi$  with value process  $V$ .

We first define the fair value and the fair relative value of the policy  $(x_0, H, \Gamma)$ .

**Definition 2** *We consider a complete financial market and a policy  $(x_0, H, \Gamma)$ .*

1. *The process  $V = \{V(t), t \in [0, T]\}$  is called fair value process of the policy  $(x_0, H, \Gamma)$  if*

(a)  *$V/S_0$  is a  $\mathbb{Q}$ -martingale and*

(b)  *$P[V(T) = X(T)\Gamma] = 1$  with  $X(T) = x_0 \prod_{t=1}^T H \left( \frac{V(t)}{V(t-1)} \right)$*

2. *If  $V(t)$  is the fair value of  $(x_0, H, \Gamma)$  then  $C(t) = V(t)/X(t)$  is called the fair relative value of  $(x_0, H, \Gamma)$ .*

This definition is inline with the risk-neutral valuation approach in finance. Furthermore, assuming a complete financial market, the existence of a fair value process is equivalent to the existence of a replicating strategy  $\xi$ . We will show how to construct this strategy after we have studied properties of the fair value process and discussed its existence and uniqueness.

To find a fair value process  $V$  we start with the last period and obtain that  $V(T) = X(T)\Gamma$  if and only if  $\Gamma = C(T) = V(T)/X(T)$  and therefore

$$C(T) = \frac{V(T)}{X(T)} = \frac{V(T)}{X(T-1)H(V(T)/V(T-1))} \quad (6)$$

$$= g(X(T-1), V(T-1), V(T)) \quad (7)$$

with

$$g(x, v_0, v) = \frac{v}{xH(v/v_0)}.$$

Note that  $g$  has the property

$$g(x, v_0, v) = g(\alpha x, \alpha v_0, \alpha v) \quad \forall \alpha \neq 0. \quad (8)$$

We obtain from assumption (A2) that for any  $x$  and  $v_0 \neq 0$  there exists the inverse function  $g^{-1}(x, v_0, c)$  of  $g$  with

$$c = g(x, v_0, v) \iff v = g^{-1}(x, v_0, c).$$

It follows that (6) holds almost surely if and only if

$$V(T) = g^{-1}(X(T-1), V(T-1), C(T)) \text{ a.s.} \quad (9)$$

From (8) we obtain that  $g^{-1}(x, v_0, \cdot)$  has the property

$$\alpha g^{-1}(x, v_0, c) = g^{-1}(\alpha x, \alpha v_0, c) \quad \forall \alpha \neq 0$$

and, in particular, for  $x \neq 0$  and  $\alpha = 1/x$ ,  $g$  satisfies

$$\frac{1}{x}g^{-1}(x, v_0, c) = g^{-1}\left(1, \frac{v_0}{x}, c\right).$$

Using Definition 2 we obtain for the fair value process  $V$  and the fair relative value process  $C$  that  $C(T) = V(T)/X(T) = \Gamma$  almost surely and

$$C(T-1) = \frac{V(T-1)}{X(T-1)} \quad (10)$$

$$= \mathbb{E}_Q \left[ D(T-1, T) \frac{V(T)}{X(T-1)} \mid \mathcal{F}_{T-1} \right] \quad (11)$$

$$= \mathbb{E}_Q \left[ D(T-1, T) g^{-1} \left( 1, \frac{V(T-1)}{X(T-1)}, \Gamma \right) \mid \mathcal{F}_{T-1} \right] \quad (12)$$

$$= \mathbb{E}_Q \left[ D(T-1, T) g^{-1} \left( 1, \frac{V(T-1)}{X(T-1)}, C(T) \right) \mid \mathcal{F}_{T-1} \right] \quad (13)$$

Using Definition 2 again we conclude that the fair relative value  $C(T - 1)$  of the policy at time  $T - 1$  must be the solution of the equation

$$C(T - 1) = E_{\mathbb{Q}} \left[ D(T - 1, T) g^{-1} \left( 1, C(T - 1), C(T) \right) \mid \mathcal{F}_{T-1} \right]$$

with  $C(T) = \Gamma$ , and the fair value at time  $T - 1$  of the policy  $(x_0, H, \Gamma)$  is  $V(T - 1) = X(T - 1)C(T - 1)$  which fulfills the equation

$$V(T - 1) = E_{\mathbb{Q}} \left[ D(T - 1, T) g^{-1} \left( X(T - 1), V(T - 1), C(T) \right) \mid \mathcal{F}_{T-1} \right].$$

Using backward induction we obtain the following theorem. The proof can be found in the appendix.

**Theorem 1** *A stochastic process  $V$  is the fair value process of the policy  $(x_0, H, \Gamma)$  if and only if the corresponding fair relative value process  $C(t) = V(t)/X(t)$  has the following two properties*

- (1)  $C(T) = \Gamma$  a.s. and
- (2)  $C(t) = E_{\mathbb{Q}} \left[ D(t, t + 1) g^{-1} \left( 1, C(t), C(t + 1) \right) \mid \mathcal{F}_t \right] \quad \forall t = 0, \dots, T - 1.$

Let us mention that the fair relative value process  $C$  is not a martingale under  $\mathbb{Q}$ . However, the fair value process  $V$  is a  $\mathbb{Q}$ -martingale.

We now come to a critical point in our analysis. For general financial market models  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$  and general participating policies  $(x_0, H, \Gamma)$  we can not ensure that the fair relative value process exists. We will therefore assume in the following that it does exist.

**Assumption 2** *There exists a stochastic process  $C$  that has the properties (1) and (2) in Theorem 1.*

Kleinow (2006) shows that the fair relative value process  $C$  exists for the participating policy  $(x_0, H, \Gamma)$  with  $\Gamma \equiv 1$  and  $H$  given in (5) if the probability space  $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$  fulfills some assumptions.

The existence of the fair relative price process  $C$  ensures that there is a process  $V$ , the fair price process, such that  $V/S_0$  is a  $\mathbb{Q}$ -martingale and  $V(T)$  is equal to the payoff  $X(T)\Gamma$  of the participating policy. Since the market is complete we can apply the Martingale Representation Theorem to conclude that there exists a portfolio strategy  $\xi$  such that

$$V(t) = V(0) + \int_0^t \xi(u) dS(u) \quad \forall t \in [0, T].$$

Let us make some remarks about the structure of  $V$ . Given the fair relative value process  $C$  and the fair value process  $V$  the payoff  $X(T)\Gamma$  of the policy  $(x_0, H, \Gamma)$  can be replicated by a sequence of contingent claims. At any time  $t \in \{0, \dots, T-1\}$  the replicating portfolio consists of a contingent claim that matures at time  $t+1$  and has the payoff  $g^{-1}(X(t), V(t), C(t+1))$ . The fair value of this claim at time  $t$  is  $V(t)$ . Since the financial market is assumed to be complete, there exists a portfolio strategy  $\xi$  such that

$$\begin{aligned} g^{-1}(X(t), V(t), C(t+1)) &= V(t+1) \\ &= V(t) + \int_t^{t+1} \xi(u) dS(u) \\ &= g^{-1}(X(t-1), V(t-1), C(t)) + \int_t^{t+1} \xi(u) dS(u) \end{aligned}$$

for all  $t = 1, \dots, T-1$  with  $X(0) = x_0$ . The hedging problem is therefore reduced to the hedging of the contingent claim  $g^{-1}(X(t), V(t), C(t+1))$  during each of the periods  $[t, t+1]$  for  $t = 0, \dots, T-1$ . The hedging strategies applied in each period are then combined to a hedging strategy  $\xi$  of the policy  $(x_0, H, \Gamma)$ .

In summary, we have shown that in a complete market there exists a self-financing strategy  $\xi$  such that the fair value process of the policy  $(x_0, H, \Gamma)$  as defined in Definition 2 is the value process of this portfolio strategy  $\xi$ . This justifies the term "fair value" since the insurer can perfectly hedge the payoff,  $V(T) = X(T)\Gamma$  almost surely.

## 4.2 Incomplete Financial Markets

In an incomplete financial market the martingale measure  $\mathbb{Q}$  is not unique. Instead, there exists a set of equivalent martingale measures  $\mathcal{Q}$  as defined in (2). There are several ways to choose a measure for the risk-neutral valuation of contingent claims. However, we do not wish to discuss the choice of a particular martingale measure, but we assume that the insurance company has chosen a particular martingale measure  $\mathbb{Q} \in \mathcal{Q}$  already, and this measure is applied for risk-neutral valuation.

Let us now assume that there exists a process  $V$  such that

$$V/S_0 \text{ is a } \mathbb{Q}\text{-martingale and } \mathbb{P}[V(T) = X(T)\Gamma] = 1 \quad (14)$$

where

$$X(T) = x_0 \prod_{t=1}^T H \left( \frac{V(t)}{V(t-1)} \right). \quad (15)$$

This means that  $V$  fulfills condition (b) in Definition 2, and condition (a) is fulfilled for the martingale measures  $Q \in \mathcal{Q}$  that the insurer has chosen. Furthermore, if such a process  $V$  exists we will find with the same arguments as used in the proof of Theorem 1 that the assertions of Theorem 1 hold.

However, since the market is incomplete, a self-financing portfolio strategy  $\xi$  with value process  $V$  might not exist since the contingent claim  $V(T)$  might not be attainable. If such a strategy does not exist, the insurance company can not invest into a portfolio with value process  $V$  and therefore,  $V$  can not be the underlying value process of the payoff  $X(T)\Gamma$ . Instead, the insurer will invest into another portfolio with value process  $V_\xi$  with

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u) dS(u) \quad \forall t \in [0, T].$$

Given this portfolio, the fair value at time 0 of the payoff

$$X_\xi(T)\Gamma$$

with

$$X_\xi(T) = x_0 \prod_{t=1}^T H \left( \frac{V_\xi(t)}{V_\xi(t-1)} \right) \quad (16)$$

should be calculated as the expected discounted value of the payoff under the chosen martingale measure  $Q$ . Since  $V \neq V_\xi$ , the expected discounted value of the payoff is not equal to  $V(0)$ , i.e.

$$E_Q[V_\xi(T)/S_0(T)] \neq E_Q[V(T)/S_0(T)] = V(0).$$

Therefore,  $V(0)$  is not the fair value of the policy  $(x_0, H, \Gamma)$  at time 0. A similar argument holds for all times  $t = 0, \dots, T-1$ .

Comparing these arguments to the situation in a complete market, we see that the existence of a hedging strategy is closely related to risk-neutral valuation.

Even if a process  $V$  that fulfills (14) with  $X(T)$  given by (15) is not the fair value process, it is a lower bound for the value of a super-hedging strategy.

**Lemma 1** *Let  $(x_0, H, \Gamma)$  be a participating policy. Let  $V$  be a stochastic process such that  $V/S_0$  is a  $Q$ -martingale and  $P[V(T) = X(T)\Gamma] = 1$  where  $X(T)$  is defined in (15). Let  $V_\xi$  be the price process of a self-financing portfolio  $\xi$ , i.e.*

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u) dS(u) \quad \forall t \in [0, T]$$

. If

$$P[V_\xi(T) \geq X_\xi(T)\Gamma] = 1$$

where  $X_\xi(T)$  is defined in (16) then  $V_\xi(0) \geq V(0)$ .

If in addition

$$P[V_\xi(T) > X_\xi(T)\Gamma] > 0$$

then  $V_\xi(0) > V(0)$ .

The proof of Lemma 1 can be found in the appendix. The following theorem about the initial value of a super-hedging strategy follows directly from Lemma 1.

**Theorem 2** *Let  $V_\xi$  be the price process of a self-financing portfolio  $\xi$ , i.e.*

$$V_\xi(t) = V_\xi(0) + \int_0^t \xi(u) dS(u) \quad \forall t \in [0, T],$$

and let  $X_\xi(T)$  be as defined in (16). We define the set  $\mathcal{V}$  as

$$\mathcal{V} = \left\{ V : V/S_0 \text{ is a } Q\text{-martingale for some } Q \in \mathcal{Q} \text{ and } \right. \\ \left. V(T) = x_0 \prod_{t=1}^T H \left( \frac{V(t)}{V(t-1)} \right) \Gamma \right\}$$

If  $P[V_\xi(T) \geq X_\xi(T)\Gamma] = 1$  then

$$V_\xi(0) \geq \sup_{V \in \mathcal{V}} V(0).$$

Since a perfect hedge of the participating policy might not exist in an incomplete financial market we suggest to use the following definition for the risk-neutral value of a participating policy.

**Definition 3** *We consider an incomplete financial market and a policy  $(x_0, H, \Gamma)$ .*

1. *The process  $V = \{V(t), t \in [0, T]\}$  is called risk-neutral value process of the policy  $(x_0, H, \Gamma)$  under the measure  $Q \in \mathcal{Q}$  if*

(a) *there exists a self-financing portfolio strategy  $\xi$  with value process  $V_\xi$  such that*

$$V(t) = V_\xi(t) = V(0) + \int_0^t \xi(u) dS(u) \quad \forall t \in [0, T]$$

and

$$(b) E_Q \left[ \frac{1}{S_0} X(T) \Gamma \right] = E_Q \left[ \frac{1}{S_0} V(T) \right] = V(0)$$

$$\text{with } X(T) = x_0 \prod_{t=1}^T H \left( \frac{V(t)}{V(t-1)} \right)$$

2. If  $V(t)$  is the risk-neutral value of  $(x_0, H, \Gamma)$  then  $C(t) = V(t)/X(t)$  is called the risk-neutral relative value of  $(x_0, H, \Gamma)$ .

The study of the existence and uniqueness of a risk-neutral value process is left for further research. However, if for a given measure  $Q \in \mathcal{Q}$  a risk-neutral value process exists we can consider two situations.

Firstly, if we find a risk-neutral value process for which  $Q[V(T) = X(T)\Gamma] = 1$ , then  $V(T)$  is equal to  $X(T)\Gamma$  P-almost surely since  $Q$  is equivalent to  $P$ , This means we are back in the situation of a complete market.

If  $Q[V(T) = X(T)\Gamma] \neq 1$  then  $Q[V(T) < X(T)\Gamma] > 0$  since

$$E_Q [(1/S_0)(V(T) - X(T)\Gamma)] = 0.$$

This means that the insurer faces the risk of making a loss. To avoid the loss in this situation without investing into a super-hedging strategy, the insurer could sell the risk to a third party. This could be done by buying a financial instrument with a value process  $V$  and  $V(T) = X(T)C$  from a third party. This approach was discussed in Kleinow (2006). However, this approach means that the hedging problem is moved away from the insurance company to a third party.

## 5 Summary and Further Research

We have considered the valuation and hedging of a participating insurance policy where we have taken into account that the management of the insurance company that has issued the policy has the right to change the underlying portfolio of the policy at any time. We have furthermore assumed that the insurer does not set up a separate hedge portfolio to hedge the payoff of the policy, but the insurer wishes to hedge this payoff using its discretion about the underlying portfolio.

We have investigated the properties of a hedge-portfolio and the fair value process in a complete financial market. We have found that the definition of the fair value process in a complete financial market can not be used in an incomplete market since the insurer might not be able to set up a portfolio that replicates the fair value process. We therefore suggested an alternative

definition for the risk-neutral value of a participating policy if the financial market is incomplete.

There are several questions for further research. In particular, it might be interesting to find conditions for the existence and uniqueness of the fair and the risk-neutral value process. A particular example of a participating contract was studied in Kleinow (2006) where existence and uniqueness of the fair value process were shown. Another interesting point for further research is to investigate the relationship between prices of participating contracts for which a separate hedge portfolio exists and those contracts for which it doesn't.

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## Proofs

**Proof of Theorem 1:** ” $\implies$ “:  $V(T) = X(T)\Gamma \iff C(T) = \Gamma$  by definition.

Since  $V/S_0$  is a martingale and  $C(T) = \Gamma$  we obtain for  $t = T$  from (9) and the properties of  $G$  that

$$\begin{aligned}
C(t-1) &= \frac{V(t-1)}{X(t-1)} \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) \frac{V(t)}{X(t-1)} \mid \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) \frac{1}{X(t-1)} g^{-1}(X(t-1), V(t-1), C(t)) \mid \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) g^{-1} \left( 1, \frac{V(t-1)}{X(t-1)}, C(t) \right) \mid \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) g^{-1}(1, C(t-1), C(t)) \mid \mathcal{F}_{t-1} \right]
\end{aligned}$$

This shows that for  $t = T - 1$  the fair value is  $V(T - 1) = X(T - 1)C(T - 1)$ . Using the same arguments and backward induction proves the assertion. ” $\impliedby$ “ We have to show that  $V/S_0$  is a martingale for  $V = XC$ . Starting

again with  $t = T$  we obtain

$$\begin{aligned}
V(t-1) &= X(t-1)C(t-1) \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) X(t-1) g^{-1}(1, C(t-1), C(t)) \mid \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) g^{-1}(X(t-1), V(t-1), C(t)) \mid \mathcal{F}_{t-1} \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[ D(t-1, t) V(t) \mid \mathcal{F}_{t-1} \right]
\end{aligned}$$

□

**Proof of Lemma 1:** We only prove the result for a policy with maturity  $T = 1$  and  $x_0 = 1$ . The result for policies with maturity  $T > 1$  follows with a backward induction argument.

Firstly note that  $g(v_0, v) = v/H(v_0, v)$  is a non-decreasing function in  $v_0$ . Assume now that  $V_{\xi}(0) < V(0)$ .

On  $\{\Gamma \leq g(V_{\xi}(0), V_{\xi}(1))\}$

$$g(V(0), V(1)) = \Gamma \leq g(V_{\xi}(0), V_{\xi}(1)) \leq g(V(0), V_{\xi}(1))$$

It now follows from assumption (A1) that

$$\mathbb{Q}[V(1) \leq V_{\xi}(1)] = 1 \tag{17}$$

which is a contradiction to the assumption  $V_{\xi}(0) < V(0)$ .

On  $\{\Gamma < g(V_{\xi}(0), V_{\xi}(1))\}$

$$g(V(0), V(1)) = \Gamma < g(V_{\xi}(0), V_{\xi}(1)) \leq g(V(0), V_{\xi}(1))$$

It now follows from assumption (A1) that

$$\mathbb{Q}[V(1) < V_{\xi}(1)] > 0. \tag{18}$$

(18) and (17), the martingale property of  $V$  and  $V_{\xi}$  and the equivalence of  $\mathbb{Q}$  and  $\mathbb{P}$  prove the assertion. □