LONGEVITY BOND PRICING MODELS: AN APPLICATION TO THE ITALIAN ANNUITY MARKET AND PENSION SCHEMES

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ABSTRACT
The paper focuses on the securitization of longevity risk through mortality-linked securities. Alternative mortality-linked securities have been proposed in literature (see Cairns-Blake-Dowd (2006)) and among these we considered the longevity bond as the most appropriate to hedge longevity risk. The paper aim at comparing two different approaches developed in the actuarial literature for the pricing of mortality-linked securities: the one based on a distortion operator as the Wang transform and the other based on the arbitrage-free pricing framework used for financial derivatives.
We underline drawbacks and advantages of each method, with reference to the Italian population. Each approach is applied to the case study adopting a Lee-Carter log-bilinear model to represent the evolution of mortality. Later on, we calculate the risk adjusted market price of a longevity bond with a constant fixed coupon. Numerical results confirm models' different features.

KEYWORDS
Stochastic mortality, longevity risk, longevity bonds, annuity market, interest rate term structure

1 INTRODUCTION

Pension plans and life insurance companies are usually exposed to longevity risk, the risk that mortality rates of their reference population might differ from that expected, affecting their pricing and reserving calculations. The unanticipated mortality improvements have most significance at higher ages, leading annuity providers to experience losses in their life annuity business. The annuity providers are also heavily exposed to interest-rate risk because their investment portfolios are predominantly fixed income.

Several authors (see, for example, Milevsky and Promislow (2001), Dahl (2004), and Biffis (2005)) have observed that there are important similarities between the force of mortality and interest rates. In particular, both are positive processes, have term structures, and are fundamentally stochastic in nature. It is now widely accepted that the similarities between mortality and interest rate risks allows to model mortality risks and to price mortality-related instruments using adaptations of the arbitrage-free (or risk-neutral) pricing frameworks developed for interest-rate derivatives.

This paper deals with the securitization of longevity risk through the mortality-linked securities. Securitization offers great opportunities for hedging the longevity risk for annuity providers. It consists in isolating the cash flows linked to the longevity risk, repackaging them into cash flows traded on capital markets. Several mortality-linked securities have been proposed in literature (see Cairns-Blake-Dowd (2006)), among these longevity bonds can be considered more appropriate to hedge longevity risk. When considering a cohort of individuals born in the same year, the longevity risk can be hedged with longevity bonds having coupons proportional to the number of cohort' survivors at each year.

We focus on the longevity bonds pricing. Prices are calculated as the expected present value of future cash flows under a risk-adjusted probability measure. Since the market of mortality-linked securities is incomplete, the risk-adjusted measure cannot be estimated consistently with observed market prices of longevity bonds.

To price the longevity bond we consider two different pricing methods: the risk-neutral approach and the distortion approach suggested by Wang (2002).

When concerning the mortality models, a lot of models have been developed to forecast mortality rates over time. One of the most famous has been proposed by Lee and Carter (1992). We refer here to the extention of such a model proposed by Brouhns-Denuit-Vermont (2002).

One of the main disadvantage of mortality-linked securities is the basis risk that might be too high relative to the price being charged. Basis risk can arise because annuitants are likely to experience more rapid mortality improvements than is reflected in the population wide index on which the payments are determined, or when the security allows for level annuity payments, while most pension schemes provide inflation-linked payments (see Blake-Cairns-Dowd (2006)). The choice of the reference population underlying the longevity bond is therefore a key issue. If the reference population is inappropriate, the investors that use such contracts to hedge their mortality risk could be exposed to a significant basis risk. In this case the mortality derivative might not provide an adequate hedge.

The paper is organised as follows. In Section 2 we describe the longevity bond framework: the main features, cash flows and prices. In Section 3 we define the stochastic modelling for mortality, while in Section 4 two different valuation models used to price the longevity bond. In Section 5 we present a numerical application to Italy and, finally, in Section 6 some remarks and conclusions.

2 THE MODEL OF LONGEVITY BOND

Longevity bonds are mortality-linked securities traded on organized exchanges. Compared with others mortality-linked securities, they have two main advantages: allow a tailor-made hedging and do not involve credit risk. The credit risk is handled by the exchange standing between traders and imposing margin requirements on traders.
Several alternatives about longevity bonds have been proposed in the literature. We consider a longevity bond with the following features:

- The bond is issued to hedge the longevity risk of a portfolio of immediate annuities;
- The bond has coupons proportional to the survivors of a given cohort. The coupon payments are therefore mortality-dependent: if the members of the cohort live longer than expected, the extra payment would be offset with the payment of the coupon from the bond;
- The bond guarantees the principal repayment.

An example of such a bond was the EIB/BNP Paribas longevity bond issued by the European Investment Bank (EIB) in November 2004 with BNP Paribas both structurer and manager of the bond. The bond had a maturity of 25 years and coupon payments depending on a survivor index based on the realized mortality experience (published by the UK Office for National Statistics) of a cohort of males from England and Wales aged 65 in 2003. Such a bond was not well received by investors (as UK pension funds and life offices) and did not generate the necessary demand.
Following Lin and Cox (2005) we consider a longevity bond with coupons based on the difference between the realized and the expected survivors of a given cohort.
Let us define the cash flows from the coupon bond contract. Let us consider an annuity provider that has to pay immediate annuities to a cohort of l_{x_0} annuitants all aged x_0 at initial time. Let R be the annual payment of the individual annuity and l_{x_0+t} the number of survivors to age x_0+t, for $t = 1,2,...,\omega - x_0$, where ω is the maximum attainable age in the cohort. It follows that in t the annuity provider will pay the amount $R\,l_{x_0+t}$, where l_{x_0+t} is a random value at evaluation time $t = 0$. Given $\widehat{l}_{x_0+t}$ the expected number of survivors to age x_0+t, the annuity provider is exposed to risk of systematic deviations between l_{x_0+t} and $\widehat{l}_{x_0+t}$ at each time t. Those deviations represent the losses experienced by the annuity provider at each time t.
Such losses can be hedged through securitization. Let us consider a Government coupon bond (the straight bond) with maturity T, paying the aggregate cash flow $R\,C_t$ at each time t and the principal $R\,F$ at maturity T. By means of a Special Purpose Company (SPC) the coupons are split between the annuity provider and the investors into two financial instruments depending on the realized mortality at each future time t, for $t = 1,2,...T$. In formula we have:

$$R\,C_t = R(B_t + D_t) \tag{1}$$

where $R\,B_t$ represents the benefits received by the annuity provider used to cover the experienced loss up to the maximum level $R\,C_t$ and $R\,D_t$ the payments received by the investors. Note that $R\,D_t$ is specular to the payments $R\,B_t$.
According to Lin and Cox (2005), we consider a longevity bond with constant fixed coupons

RC. The benefits received by the annuity provider are RB_t, where B_t is equal to:

$$B_t = \begin{cases} C & l_{x_0+t} - \widehat{l}_{x_0+t} > C \\ l_{x_0+t} - \widehat{l}_{x_0+t} & 0 < l_{x_0+t} - \widehat{l}_{x_0+t} \leq C \\ 0 & l_{x_0+t} - \widehat{l}_{x_0+t} \leq 0 \end{cases} \tag{2}$$

while payments to the investors are RD_t, where D_t is equal to:

$$D_t = \begin{cases} 0 & l_{x_0+t} - \widehat{l}_{x_0+t} > C \\ C - (l_{x_0+t} - \widehat{l}_{x_0+t}) & 0 < l_{x_0+t} - \widehat{l}_{x_0+t} \leq C \\ C & l_{x_0+t} - \widehat{l}_{x_0+t} \leq 0 \end{cases} \tag{3}$$

The annuity provider's payout at time t seen from time 0 depends from l_{x_0+t}, while premiums paid by each annuitants of the cohort depends from the expected cash flows $\widehat{l}_{x_0+t}$. Therefore, before buying the longevity bond the random loss experienced by the annuity provider at time t is given by:

$$Loss(t) = R\left(l_{x_0+t} - \widehat{l}_{x_0+t}\right) \tag{4}$$

After buying the longevity bond, equation (4) becomes:

$$Loss(t)^{LB} = Loss(t) - RB_t \tag{5}$$

Let us suppose that the SPC buys, at the price W, a straight coupon bond with annual coupon RC and principal RF refunded at maturity $t = T$. Let P be the premium that the annuity provider pays to the SPC to hedge its longevity risk and V the price paid by the investors to purchase the longevity bond issued from the SPC with face value RF and coupons RD_t. The structure of such transactions is shown in Figure 1. The SPC meets its obbligations and has a profit if $P + V \geq W$. In this paper we assume $P + V = W$.

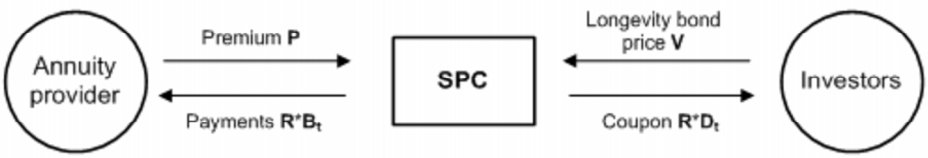

Figure 1: Longevity bond cash flows scheme

Let $d(0,t)$ be the risk-free discount factor at time 0. The price of longevity bonds is obtained with the expected present value technique, where the expected present value of future cash flows is evaluated under a risk-adjusted probability measure. Therefore, assuming the independence between interest rate and mortality, the bond price is given by:

$$W = RFd(0,T) + RC\sum_{t=1}^{T} d(0,t) \tag{6}$$

While, the premium P and the longevity bond price V are given by:

$$P = R \sum_{t=1}^{T} \tilde{E}[B_t] d(0,t) \tag{7}$$

$$V = R F d(0,T) + R \sum_{t=1}^{T} \tilde{E}[D_t] d(0,t) \tag{8}$$

Where $\tilde{E}[B_t]$ and $\tilde{E}[D_t]$ are the risk-adjusted expected value of B_t and D_t, respectively.

3 STOCHASTIC MODELLING FOR MORTALITY

Let ${}_tq_{x_0}$ be the probability that an individual of the reference cohort, aged x_0 at time 0, will die before reaching the age x_{0+t}. Given the corresponding survival probability ${}_tp_{x_0}$, the stochastic number of survivors l_{x_0+t} follows a Binomial distribution with parameters:

$$E(l_{x_0+t}) = l_{x_0} {}_tp_{x_0} \quad \text{and} \quad Var(l_{x_0+t}) = l_{x_0} {}_tp_{x_0}(1 - {}_tp_{x_0}) \tag{9}$$

where l_{x_0} is the initial number of individuals in the reference cohort. If l_{x_0} is large, for example more than 30, according to the Central Limit Theorem, l_{x_0+t} is approximately distributed as a Normal with the same parameters. Furthermore, the coefficient of variation $(\sqrt{Var(l_{x_0+t})}/E(l_{x_0+t}))$ tends to a value close to zero. The expected number of survivors, $\widehat{l}_{x_0+t}$, is obtained with the pointwise projection of the death probability ${}_t\widehat{q}_{x_0}$.
To obtain the death probabilities ${}_tq_{x_0}$ we model and forecast the period death rates at age x and time t, $m_{x_0}(t)$, with an extension of the well known Lee-Carter model (Lee and Carter (1992)), specifically the one suggested by Brouhns-Denuit-Vermont (2002). Considering the higher variability of the observed death rates, at ages with a smaller number of deaths, they assumed a Poisson distribution for the random component, against the assumed Normal distribution in the original Lee-Carter model.
Therefore, they assumed that the number of deaths, $D_x(t)$, is a random variable following a Poisson distribution:

$$D_x(t) \sim Poisson(N_x(t)m_x(t)) = \frac{[N_x(t)m_x(t)]^{D_x(t)} e^{-[N_x(t)m_x(t)]}}{D_x(t)!} \tag{10}$$

where $N_x(t)$ is the mid-year population observed at age x and time t and $m_x(t)$ is the central death rate at age x and time t. The central death rates $m_x(t)$ follows the model suggested by Lee and Carter:

$$\ln m_x(t) = \alpha_x + \beta_x k_t \tag{11}$$

where the parameters are subject to the following constraints:

$$\sum_t k_t = 0 \text{ and } \sum_x \beta_x = 1 \tag{12}$$

and α_x refers to the average shape across ages of the log of mortality schedule; β_x describes the pattern of deviations from the previous age profile, as the parameter k_t changes, and k_t can be seen as an index of the general level of mortality over time. The parameters can be estimated by maximizing the log-likelihood based on model (10) and, because of the presence of the bilinear term $\beta_x k_t$, an iterative method is used to solve it (see Goodman (1979)).

The log-likelihood function is equal to:

$$logL(\alpha,\beta,k) = \sum_{x,t}\{D_x(t)(\alpha_x+\beta_x k_t) - N_x(t)e^{\alpha_x+\beta_x k_t}\} + constant \tag{13}$$

To evaluate the fitting of the model, we calculate the deviance residuals:

$$r_D = sign\left[D_x(t)-\widehat{D}_x(t)\right]\cdot\left[D_x(t)\ \ln\left(\frac{D_x(t)}{\widehat{D}_x(t)}\right)-\left(D_x(t)-\widehat{D}_x(t)\right)\right]^{1/2} \tag{14}$$

To forecast future death rates, Lee and Carter (1992) assume that α_x and β_x remain constant over time and forecast future values of the time factor k_t, intrinsically viewed as a stochastic process, using a standard univariate time series model. Box-Jenkins identification procedures are used here to estimate and forecast the Autoregressive Integrated Moving Average model (ARIMA). With the mere extrapolation of the time factor k_t, it is possible to forecast the entire matrix of future death rates.
Uncertainty in the forecasts is the result of combined sources: the Poisson variability enclosed in the data, the sample variability of the parameters of the Lee-Carter model and the ARIMA model and the uncertainty of the extrapolated values of the model's time index k_t. An analytical equation of the prediction intervals, that would account for all the three sources of uncertainty simultaneously, is impossible to derive due to very different sources of uncertainty to combine. An empirical solution to the problem is found through the application of the bootstrap method, a computationally intensive approach used for the construction of prediction intervals, first proposed by Efron (1979). More precisely, we apply here the non-parametric or residual bootstrap, following a recent work by Koissi et al. (2006). It is assumed that the theoretical distribution of innovations is approximated by the empirical distribution of the observed residuals and random innovations are generated sampling from the empirical distribution of past fitted errors. Residuals are assumed to be independent and identically distributed.
The simulation procedure we followed consists of two steps. In the first one, we evaluated the sampling variability of the estimated coefficients of the model, sampling N times from the deviance residuals of the Lee-Carter model. In the second step, for each of the N k_t simulated, we evaluate the variability of the projected model's time index, sampling M times from the residuals of the ARIMA model. Overall $N \cdot M$ simulations are performed.

4 LONGEVITY BOND PRICING MODELS

The development of a market for mortality-linked securities also depends on the prices that should make the securities attractive to potential buyers and sellers. The longevity bonds price is a critical issue because it cannot be estimated using the standard spot yield curve and zero arbitrage methods due to the market incompleteness. To face this problem we consider two different approaches (see Blake-Cairns-Dowd-MacMinn (2006)). One consists in adapting the arbitrage-free pricing framework of interest-rate derivatives to the valuation and securitization of mortality risk. The price of mortality-linked securities is therefore given by the expected present value of future cash flows under a risk-neutral probability measure $\mathbb{Q}$ (or equivalent-martingale measure). The mortality dynamic is specified under a risk-adjusted pricing measure $\mathbb{Q}$ that is equivalent to, in the probabilistic sense, the current real-world measure $\mathbb{P}$ (often called physical measure). Example of this approach are the papers of Milevsky and Promislow (2001), Dahl (2004), Cairns-Blake-Dowd (2006), Biffis and Denuit (2006) and Biffis-Denuit-Devolder

(2005). Here the main problem is how to find a risk-neutral pricing measure, given the lack of mortality-linked securities.
The other approach is based on a distortion operator - the Wang transform (see Wang (2002)) - that distorts the distribution of projected death probability to generate risk-adjusted expected values that can be discounted at the risk-free rate. Example of this approach are reported in the papers of Lin and Cox (2005), Cox, Lin and Wang (2006) and Denuit, Devolder and Goderniaux (2007).

4.1 Risk-neutral approach

Biffis-Denuit-Devolder (2005) and Biffis-Denuit (2006) propose a continuous-time version of the Lee-Carter model and describe a class of measure changes (equivalent martingale measures) under which stochastic intensities of mortality remain of the generalized Lee-Carter type. By a change in the intensity process and assuming the independence between mortality and interest rate risks, they provide a risk-neutral version of the standard Lee-Carter model.
Let us define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where all filtrations are assumed to satisfy the conditions of right-continuity and completeness. We focus on a portfolio of insureds aged x_i for $i = 1, ..., n$ at the time 0. For a single insured at time t the death time is modelled as a $\mathbb{F}$-stopping time τ^i, where τ^i is a nonnegative random variable and where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ carries information about whether τ^i has occurred or not by each time t in the interval $[0, T^*]$. We assume that $\mathbb{F} = \mathbb{G} \vee \mathbb{H}$, where $\mathbb{H} = \vee_{i=1}^{n} \mathbb{H}^i$ and $\mathbb{H}^i = (\mathcal{H}_t^i)_{t\in[0,T^*]}$ is the minimal filtration making τ^i a stopping time and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T^*]}$ is a strict subfiltration of $\mathbb{F}$ carrying information about mortality dynamics and other relevant factors. In other words we see $\mathbb{G}$ as a filtration carrying the available information about relevant factors driving the evolution of mortality. Since it is a strict subfiltration of $\mathbb{F}$, we have that τ^i is not a stopping time with respect to the filtration $\mathbb{G}$, which can thus be seen as carrying information about the likelihood of deaths happening, but not about their actual occurrence.
According to Biffis-Denuit-Devolder (2005) the following assumptions hold:

- $\mathbb{P}(\tau^i \leq t \,|\, \mathcal{G}_t)$ is continuous;
- $\mathcal{H}_t^i, \mathcal{G}_{T^*}$ are conditionally independent given $\mathcal{G}_t$ for all t and i;
- $\mathcal{H}_t^1, ..., \mathcal{H}_t^n$ are conditionally independent given $\mathcal{G}_{T^*}$ for all t.

Further, under $\mathbb{P}$ the n stopping time τ^i are assumed to have stochastic intensities $\mu_{x_i}(t)$ following the generalized continuous-time Lee-Carter model:

$$\mu_{x_i}(t) = \exp(\alpha_{x_i+t} + \beta_{x_i+t}\, k_t) \tag{15}$$

for some continuous functions $\alpha(.)$ and $\beta(.)$ and R^d-valued $\mathbb{G}$-predictable process k. We assume that k have the dynamics described by the following stochastic differential equation:

$$dk_t = \delta(t, k_t)dt + \sigma(t, k_t)dW_t \tag{16}$$

where W is a d-dimensional standard Brownian motion generating the filtration $\mathbb{G}$.
In this framework α_x and β_x of equation (12) are the pointwise estimates of the functions $\alpha(.)$ and $\beta(.)$. Following Biffis-Denuit-Devolder (2005) it is possible to define a probability measure $\mathbb{Q}$ equivalent to the physical probability $\mathbb{P}$ on the space $(\Omega, \mathcal{F}_t)$. Under $\mathbb{Q}$ the n stopping time τ^i

have stochastic intensity of the generalized Lee-Carter type as follows:

$$\tilde{\mu}_{x_i}(t) = \mu_{x_i}(t)(1+\phi_i) \tag{17}$$

where $\phi_i(t)$ is a strictly positive process given by:

$$\phi_i(t) = \exp(a_{x_i+t} + b_{x_i+t}\, k_t) - 1 \tag{18}$$

Assuming that $\tilde{\alpha} = \alpha + a$ and $\tilde{\beta} = \beta + b$, we can write:

$$\tilde{\mu}_{x_i}(t) = \exp(\tilde{\alpha}_{x_i+t} + \tilde{\beta}_{x_i+t}\, k_t) \tag{19}$$

Under $\mathbb{Q}$ the dynamics of the time-trend k are described, instead of (16) that holds under $\mathbb{F}$, by the following differential equation:

$$dk_t = (\delta(t,k_t) - \eta_t \sigma(t,k_t))dt + \sigma(t,k_t)d\tilde{W}_t \tag{20}$$

The change of measure affects the drift of the time-index k through the process η as well as the intensity process itself through the process ϕ^i. Note that in this framework, the process η represents the market price of systematic risk.
Now, let us define $r(t)$ the interest rate process adapted to $\mathbb{G}$ and consider an insurance market involving mortality-contingent securities with prices adapted to $\mathbb{F}$.
Let τ be a stopping-time admitting intensity μ and representing the random time of death of a single individual in the cohort. We have that the mortality dynamics of the reference population is described by $\mathbb{P}(\tau > t|\mathcal{G}_t) = \exp(-\int_0^t \mu(s)ds)$.
Under this framework the market prices P and V are given by (see equations (7) and (8)):

$$P = R\sum_{t=1}^{T} E_{\mathbb{Q}}(B_t|\mathcal{G}_t)d(0,t) \tag{21}$$

$$V = RFd(0,T) + R\sum_{t=1}^{T} E_{\mathbb{Q}}(D_t|\mathcal{G}_t)d(0,t) \tag{22}$$

where $E_{\mathbb{Q}}(B_t\,|\,\mathcal{G}_t)$ and $E_{\mathbb{Q}}(D_t\,|\,\mathcal{G}_t)$ are the expected value of payments B_t and D_t received by the annuity provider and investors respectively, under the risk-neutral measure $\mathbb{Q}$ conditional on sub-filtration $\mathcal{G}_t$ and the risk-free discount factor, $d(0,t)$, is given by $E_{\mathbb{Q}}(e^{-\int_0^t r(s)ds})$.

4.2 Distortion approach: the Wang transform

The Wang transform is based on the idea that the annuity market price takes into account the uncertainty in the mortality table once the table is given. The underlying assumptions are that the investors accept the same transformed distribution and that mortality and interest rate risks are independent.
Let $\Phi(x)$ be the standard normal cumulative distribution function. Wang defines the distortion operator as:

$$g_\lambda(u) = \Phi\left[\Phi^{-1}(u) - \lambda\right] \tag{23}$$

for $0 < u < 1$ and a parameter λ. Now, given a distribution with cumulative density function $F(t)$, a distorted distribution $F^*(t)$ is determined by λ according to the equation:

$$F^*(t) = g_\lambda(F(t)) = \Phi\left[\Phi^{-1}(F(t)) - \lambda\right] \tag{24}$$

The parameter λ is called the market price of risk, reflecting the level of systematic risk. Given an insurer's liability X over a time horizon $[0,T]$, the fair price of the liability is the discounted expected value under the distribution obtained from the distortion operator ($E^*[X]$). Thus, the Wang transform will produce a risk-adjusted "fair-value" of X at time T, which can be further discounted using the risk-free rate.
Following Lin and Cox (2005), we use observed annuity prices to estimate the market price of risk for annuity mortality, that is employed to price the mortality bond. Therefore, the value of λ is the solution of the following equation:

$$a_{x_0}^{market}(t_0) = \sum_{t>1} \left\{1 - \Phi\left[\Phi^{-1}({}_t\widehat{q}_{x_0}) - \lambda_{x_0}(t_0)\right]\right\} d(0,t) \tag{25}$$

It is worthy noting that ${}_t\widehat{q}_{x_0}$ plays the role of the distribution function F in equation (24) and represents the probability that an annuitant aged x_0 does not reach age $x_0 + t$, according to a reference projected lifetable. The corresponding transformed probability of death is denoted by:

$${}_t\widehat{q}^*_{x_0} = \Phi\left[\Phi^{-1}({}_t\widehat{q}_{x_0}) - \lambda_{x_0}(t_0)\right] \tag{26}$$

Under this framework, P and V are computed as follows (see equations (7) and (8)):

$$P = R\sum_{t=1}^{T} E^*[B_t]d(0,t) \tag{27}$$

$$V = RFd(0,T) + R\sum_{t=1}^{T} E^*[D_t]d(0,t) \tag{28}$$

where $E^*(.)$ is the expectation operator associated with the Wang transformed distribution function ${}_t\widehat{q}^*_{x_0}$. In practice, annuity providers adjust the distribution of death probability to compensate for the risk that the annuitants live longer than expected. As a result the market price of risk should be positive, i.e. the expected number of survivors under the transformed distribution should be higher than what a given table suggests.

5 NUMERICAL SIMULATION

The models above described are here applied to the Italian population. We consider two cases:

1. the longevity bond is built on the forecast of the Italian insured mortality, obtained from the general population data and then applying self-selection factors taken from ANIA (the National Association of Insurance Companies)

2. the longevity bond is built on the forecasts of the Italian general population mortality

Note that when using population data and not specific of the insurer annuitants (case 2) a basis risk is introduced. Thus, the hedge does not perfectly match the longevity risk of the annuity provider. Conversely, in case 1 the basis risk is strongly mitigated by the self-selection factors.

It must be noticed that the ANIA self-selection factors are taken from the English experience and could not exactly represent the self-selection of the Italian insured.
In this section we first present a detailed description of the data used in the numerical simulation as well as the mortality projection estimates. Later on we give the longevity bond price evaluated on a cohort of Italian males aged 65 in the year 2005 under both the martingale approach and the distortion approach.

5.1 Mortality projection

The data used to perform the Italian population mortality forecasts come from the Human Mortality Database (www.mortality.org), an open source database, whose data come directly from the official vital statistics and the census counts published by ISTAT (the Italian National Institute of Statistics). Population size at the 1st of January and death counts are considered from 1950 till 2004, by single years of age, in the age range 50-110 and for males.
Therefore, the quantities l_{x_0+t} and $\widehat{l}_{x_0+t}$ are not specific of a single insurer but come from national statistics, i.e. reliable and easily accessible public sources. This can help investors to have full access to the data and reassure them that annuity providers can not manipulate their reported death rates, avoiding moral hazard.
Figure 2 shows α_{x+t}, β_{x+t} and k_t, the estimated parameters of the Lee-Carter model (11) fitted on the Italian males insured (case 1) and general population (case 2). Applying Box-Jenkins estimation procedures, an ARIMA(0,1,0) is used to estimate and forecast the series k_t and consequently the future age-specific death rates. The number of simulations performed is 37,500 (N=150 and M=250 (for details on mortality simulation procedure see Section 3)).
Results of mortality projection are reported in Figure 3: on the left side we have the future value of the time index k_t up to the year 2050 and on the right side the projected death probabilities ${}_tq_{x_0}$ for the cohort born in 1940, aged 65 in 2005. Confidence intervals are set to 95%. It can be noted that k_t are similar for the two cases, while we observe a difference in death probabilities due to the self-selection factors.

5.2 Longevity bond pricing: simulation results

Let us consider an initial cohort of $l_{x_0} = 10,000$ males, aged 65 in the year 2005, following a Binomial distribution at age $x_0 + t$. The coefficient of variation of the distribution of survivors l_{x_0+t} is close to zero due to the large size of l_{x_0}. Therefore, a very small number S of simulations of l_{x_0+t}, for each of the $N \cdot M$ simulated ${}_tq_{x_0}$, is enough to combine systematic deviations and random fluctuations. We simulate $S = 30$ trajectories. Then, the total number of simulations $N \cdot M \cdot S$ is equal to 1,125,000.

5.2.1 Risk-neutral approach

We calibrate the risk-neutral intensities to the survival probabilities of the IPS55 life table that refers to the mortality of Italian annuity policyholders born in 1955 (see ANIA (2005)). It is the reference life table currently used by the Italian insurers for pricing and reserving.
The IPS55 life tables are based on mortality projections performed by the Italian National Statistical Institute (ISTAT). These projections are obtained from the Lee-Carter model, fitted to

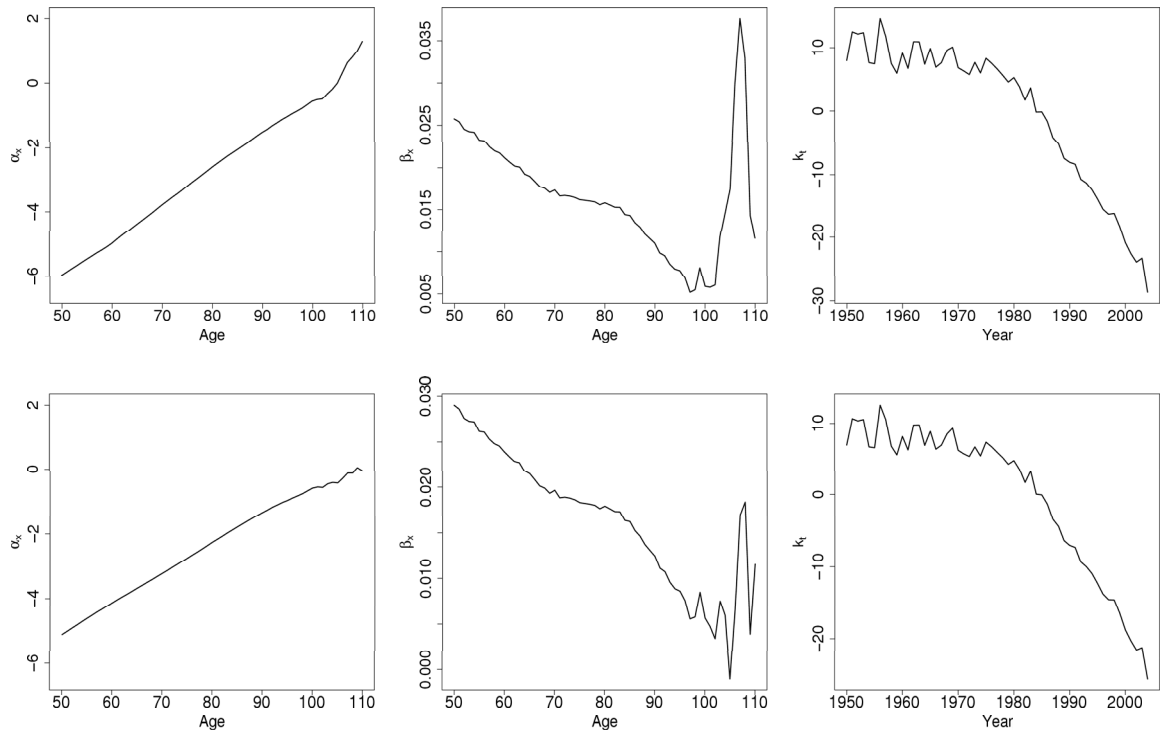

Figure 2: Parameters α_{x+t} and β_{x+t} and k_t of the Lee-Carter model for insured (up) and general population (down) - Italian males

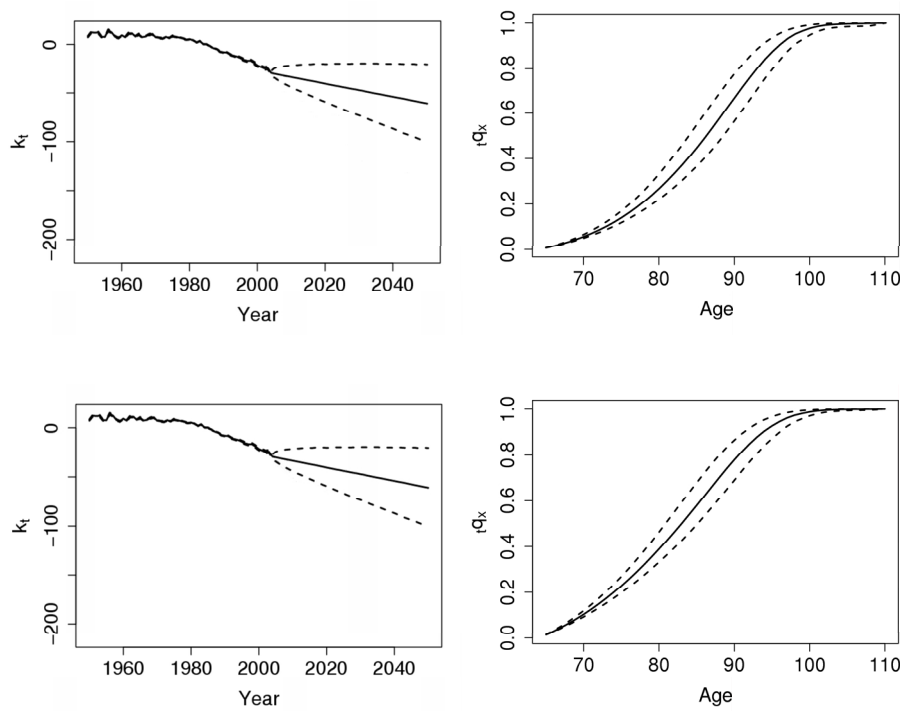

Figure 3: Estimate of future k_t and ${}_tq_x$ for insured (up) and general population (down) - Italian males

general population data. ANIA then applied English self-selection factors to the death probabilities of the projected 1955 cohort life table. Note that the ANIA self-selection factors are the same we used in projecting mortality in case 1.
The dynamics of μ under $\mathbb{Q}$ can be represented, instead of equation (19), by the following (see Biffis-Denuit-Devolder (2005)):

$$\tilde{\mu}_{x_i}(t) = \exp(\tilde{\alpha}_{x+t} + \beta_{x+t}\tilde{k}_t) \tag{29}$$

where $\tilde{k}_t$ under $\mathbb{Q}$ follows the dynamics:

$$d\tilde{k}_t = \delta(t,\tilde{k}_t)dt + \sigma(t,\tilde{k}_t)d\tilde{W}_t \tag{30}$$

The equality between equations (19) and (29) is verified for $[\beta_{x+t} + b_{x+t}]\,k_t = \beta_{x+t}\tilde{k}_t$ and $b_{x+t} = \beta_{x+t}[\sigma(t,\tilde{k}_t)/\sigma(t,k_t) - 1]$.
Therefore, to provide estimates of the functions a_{x+t} and b_{x+t}, and the parameter η_t changing the drift of k under $\mathbb{Q}$, we fix the population β_{x+t}'s and estimate the $\tilde{\alpha}_{x+t}$'s and the time-index $\tilde{k}_t$ implied by the IPS55 table. Given $\tilde{k}_t$'s parameters we are able to calculate η_t:

$$\eta_t = \frac{\delta(t,k_t)}{\sigma(t,k_t)} - \frac{\delta(t,\tilde{k}_t)}{\sigma(t,\tilde{k}_t)} \tag{31}$$

The adjustment function a_{x+t} and b_{x+t} resulting from the fit of the Lee-Carter model to the IPS55 data are presented in Figure 4. Parameters σ and δ under both the physical (see equation (16)) and risk-neutral (see equation (30)) measure are reported in Table 1.

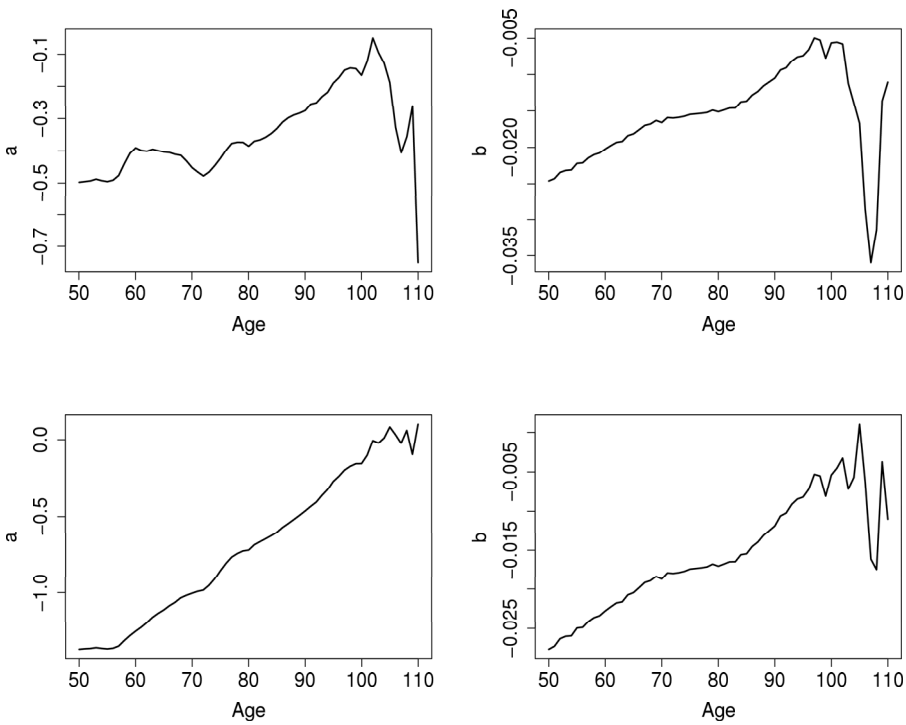

Figure 4: Adjustment functions a_{x+t} and b_{x+t} of Lee-Carter model - insured (up) and general population (down) - Italian males

Table 1: Estimated parameters of the model under both physical and risk-neutral measure

	case 1		case 2	
Parameter	k_t	$\tilde{k}_t$	k_t	$\tilde{k}_t$
σ	2.3242	0.1034	1.9256	0.0913
δ	-0.6788	-0.1570	-0.6033	-0.1384

Therefore, applying equation (31) we obtain $\eta_t = 1.2269$ for the insured population and $\eta_t = 1.2026$ for the general population. Results highlight that the market price of risk η_t does not strongly change in the two cases considered. The differences between insured and general population come out from the adjustment function a_{x+t} as shown in Figure 4. In Figure 5 we show the mortality parameters under both $\mathbb{P}$ and $\mathbb{Q}$ measures.

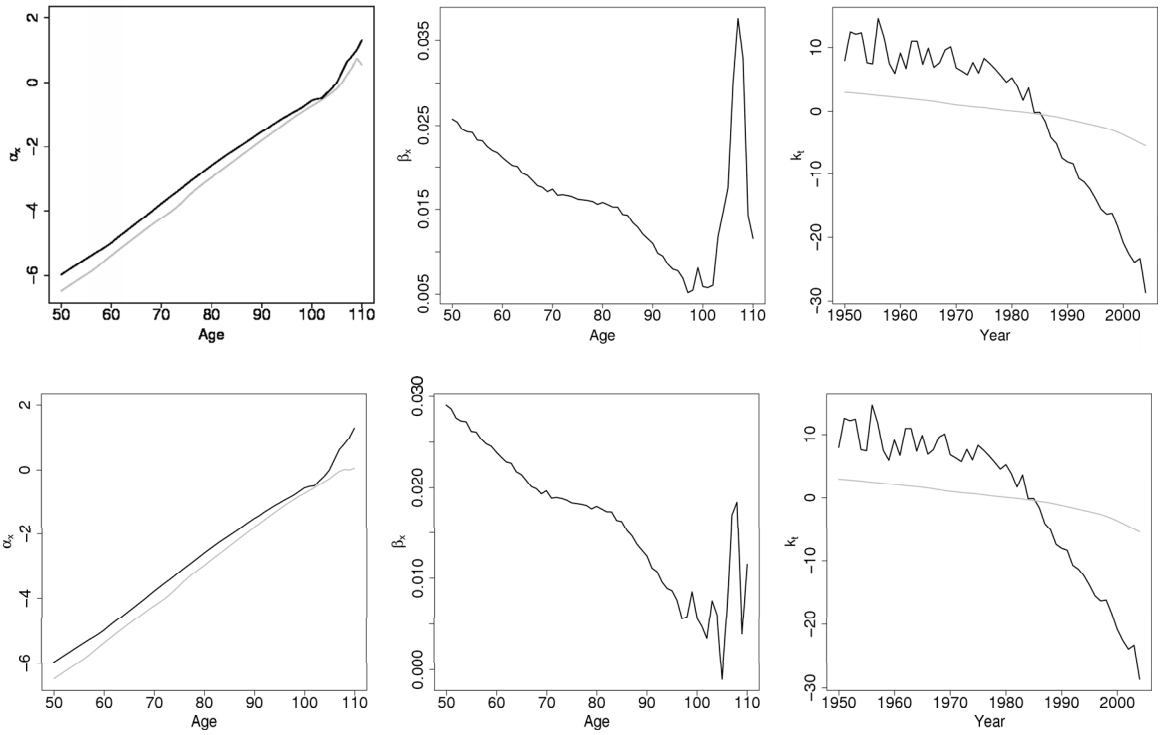

Figure 5: Parameters α_{x+t}, β_{x+t} and k_t (dark line) and $\tilde{\alpha}_{x+t}$ and $\tilde{k}_t$ (light line) - insured (up) and general population (down) - Italian males

5.2.2 Distortion approach

When considering the distortion approach, it is important to bear in mind that the estimate of market price of risk λ is a critical issue in longevity bond pricing. Lin and Cox (2005) estimate λ solving equation (25) using the annuity price observed on the U.S. annuity market. The lack of a secondary annuity market in Italy makes necessary to use the IPS55 life table to obtain the corresponding measure for the Italian market. On the right side of equation (25), the probabilities ${}_t\widehat{q}_{x_0}$ are the result of our estimation procedure that refers to both insured (case 1) and general population (case 2). Therefore, in case 1 λ accounts only for longevity risk, while accounts also for selection in case 2. The solution is obtained iteratively.
Given $a_{65}^{market}(2005) = 12.3606$ we find $\lambda = 0.0467$ (case 1) and $\lambda = 0.3675$ (case 2). As the

IPS55 is the result of an estimation procedure similar to the one we used for probabilities ${}_t\widehat{q}_{x_0}$ we obtain a small value of λ when selection factors are considered (case 1). Actually, according to ANIA the IPS55 is the most realistic estimate of Italian annuitants mortality trend without including an explicit or implicit safety loading. Consequently, it is questionable wether the IPS55 is adequate to catch the market price of risk.
Once λ is found, the Wang transform (see equation (26)) allows to compute the transformed death probabilities ${}_t\widehat{q}^*_{x_0}$ and then the premium P and the bond price V.
Figure 6 shows the pointwise estimate of death probabilities under physical measure ${}_t\widehat{q}_{x_0}$, under risk-neutral measure ${}_t\tilde{q}_{x_0}$ and the Wang transformed probabilities ${}_t\widehat{q}^*_{x_0}$.

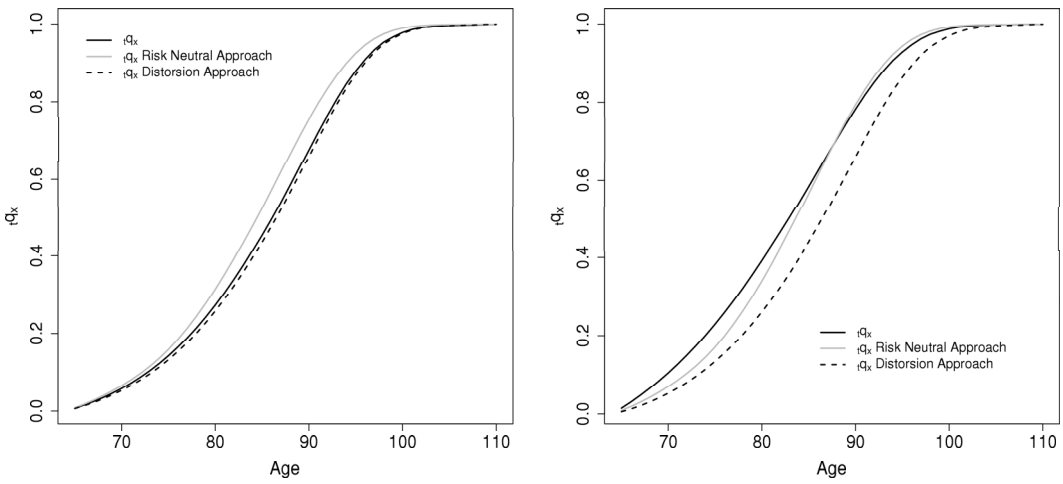

Figure 6: Death probabilities ${}_t\widehat{q}_{x_0}$, ${}_t\tilde{q}_{x_0}$ and ${}_t\widehat{q}^*_{x_0}$ - insured (left) and general population (right) - Italian males

As can be seen from Figure 6 (left side) with regards to case 1, the Wang transformed probabilities are slightly lower than the physical ones, while the risk-neutral probabilities are greater. Such a result may be due to the age-shifting used in the IPS55 life table, providing mortality rates that are constant at intervals. This procedure causes a bias in the risk-neutral approach, but not in the distortion approach that uses data of only one cohort. When considering different cohorts the behaviuor of λ is not regular, sometimes becoming negative (see Levantesi-Torri (2008)).
Namely, differences in the results can be explained by a methodological distinction: distortion approach determines a cohort-dependent distortion operator applied to all ages, while the risk-neutral approach works on age-dependent adjustment functions a_{x+t} and b_{x+t}.

6 Pricing results and concluding remarks

Let us define now the longevity bond features. The constant coupon RC is calculated as RFc, where c is the coupon rate, equal to the par yield and equal to 4.8% (the bond is issued at par) and RF is the face value of the straight bond. We fix the maturity T to 25. According to the number of annuitants we set F =10,000 and the annuity annual payout per person R =1,000; the face value of the longevity bond refunded at maturity is therefore 10,000,000. The longevity bond features and prices are reported in Table 2.

Table 2: Longevity bond price

Number of annuitants l_{x_0}	10,000	
Annuity payment R	1,000	
Face value RF	10,000,000	
Case 1: insured population		
Pricing method	Risk-neutral	Wang transform
Straight bond price W	10,000,000	10,000,000
Longevity bond price V	9,999,233	8,339,279
Premium P	767	1,660,721
Case 2: general population		
Pricing method	Risk-neutral	Wang transform
Straight bond price W	10,000,000	10,000,000
Longevity bond price V	5,511,534	4,100,409
Premium P	4,488,466	5,899,591

The different prices obtained with the risk-neutral appoach and the distortion approach are the direct consequence of the difference in death probabilities observed in Figure 6. The small value of premium P in the case 1 is due to the pointwise estimate of risk-neutral death probabilities that are greater than the physical ones.
Note that when longevity bond is built on general population (case 2), premium P incorporates both longevity risk and annuitants self-selection.
The choice of the reference population is a critical issue when constructing longevity bonds. To provide a real hedge the bond's cash flows have to match the payments to the annuitants from the annuity provider. Besides, the reference population must be chosen taking into account both hedging needs and speculative interest.
To this purpose we analyse two different cases: in the first one the longevity bond is built on the insured population realizing the hedging needs of the annuity provider, in the second case the bond is built on the general population and then more adequate for speculative purposes. On the other hand, in case 2 the bond requires a greater premium P and introduces basis risk.
Our analysis shows that the lack of a secondary annuity market in Italy causes problems in the estimate of the market price of longevity risk giving contradictory pricing results. Therefore, it is diffult to establish the best pricing method. A unique risk-neutral measure might not exist due to the incompleteness of the mortality-linked securities market. Distortion approach sounds more appropriate for incomplete markets, but it is not coherent with pricing model mark to market applied to other securities.

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