AN ACTUARIAL APPROACH FOR ADJUSTED FORWARD RATES

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ABSTRACT

Interest rates are used to value pension fund and other financial products. The flows are valued at market rates obtained from day to day values. Thus it has been developed procedures to obtain those non risk spot rates from the treasury bonds like bootstrapping procedure. But the rates calculated with this procedure are observed ones and could reflect any anomaly or expectation from the market.

According to Meiselman´s (1962) theory of interest expectancy it is possible to calculate the forward rates derived from the market expectations. In this case we have some common characteristics with actuarial mortality rates used in actuarial mathematics. So, it is developed the Whittaker-Henderson graduation for looking an adjusted future forward rate. With this approach one anomaly of the market could be smoothed by way of adjusted values, as happens with mortality rates. Even more, the adjusted forward rates could be used for valuating liabilities to be funded in the future including the use of the interest rate to be used in future.

These adjusted values could be used to value assets at observed market rates and could be used, for instance, into an immunization procedure. The principal aim of this paper is to establish a simple procedure of estimate those future rates according the market expectances.

The last part of this paper includes a simple application based on the Spanish public bond market.

KEYWORDS:

Interest rates, fair value, adjusted forward rates, market valuation.

AN ACTUARIAL APPROACH FOR ADJUSTED FORWARD RATES

1. INTRODUCTION

Interest rates are used to value pension fund and other financial products. The flows are valued at market rates obtained from day to day values. Even more, it is necessary to valuate the flows at market interest rates into the firm accountancy.

There are a lot of factors that influence the value of interest rates. The crisis and its effect into several firms could give us different values at different periods. The necessity of money (liquidity) in a specific moment could do the value of a bond became higher from one moment to another, but it is due to a punctual anomaly.

Now a day the Central Bank issues bonds to different periods. With those non risk bonds it is common to calculate the term structure of interest rates (non risk spot rates). Thus it has been developed procedures to obtain those non risk spot rates from the treasury bonds like bootstrapping procedure. But the rates calculated with this procedure are observed ones and could reflect any anomaly or expectation from the market.

The reason of use them it is the necessity to take into account the market interest rates. But perhaps the now a day observed interest rates will not be the future real interest rates. Even more, it could happen that the investor could be at a market without the *n*-year spot rate or *n*-year forward rate because in that market there is not a non risk bond redeemable at that *n*-year.

In this sense, the procedure used to calculate the term structure of interest rates has some similar to the procedure used to estimate mortality rates. So, it is developed the Whittaker-Henderson graduation for looking an adjusted future forward rate which could be used, for example, into the immunization procedure for future reinvestment flows.

The graduated forward rates smooth the different expectations at different years that the investors have. As result, the different picks or bottoms diminish translating the expectations to the nearest years. In addition, those graduated forward rates give us an estimated term structure of interest rate where the instability of the market has been smoothed.

The paper is organized as follows. Section 2 and 3 contains a review of Meiselman´s (1962) theory of interest expectancy and a review of certain financial terms and section 4 demonstrates an approach to graduating forward rates according to the market expectations on 2008, April 30.

The last part of this paper includes a simple application based on the Spanish public bond market for valuating an insurance product.

2.   TERM STRUCTURE OF INTEREST RATES

2.1. Definition

The term structure of the interest rates represents the basic alternatives of investment and funding that the investor has in every moment. For a set of financial assets of equal credit quality the term structure of interest rates defines the relation between the value of the interest rate and its term.

The market defines the non risk interest rates. If we bear in mind the difference between the credit-quality of every asset, the premium for risk will define us likewise for every asset quoted in the market.

2.2. Usefulness of the Term Structure of Interest Rates (TSI)

The term structure of interest rates is a fundamental tool for financial decisions.

a)   It serves for the correct valuation of assets and liabilities (lendings, investments, derivatives, insurance products, pension plans, etc.)

b)   It represents the fundamental factor to be used for measurement and management the interest risk (variation of the total value if the interest rate varies). It will make us possible to realize simulations before a wide range of alternative interest rates.

c)   A detailed analysis of the TSI can allow us the comparison between theoretical prices given by the market and the market prices, making arbitration possible.

d)   The TSI is the reference tool when a bond is issued giving the minimal cost with non risk interest rates.

e)   It gives information about the future expectation of future interest rates.

f)   It makes possible the analysis of the influences of the variations of the interest rate (interest risk) in the short term into the long-term and vice versa.

There are a series of factors that affect the conducts of the investors and that define the different theories over the TSI. We will focus on the theory on the pure expectations.

2.3. Theory of the pure expectations

This theory bets [Hicks, 1939], [Lutz, 1940] that the form of the curve of interest rates is determined by the expectations that market investors have on the future interest rates. Definitively it implies that if there is neither uncertainty nor transaction costs, the future forward should be the same that the real spot rates for this term. Using this theory, Meiselman

[Meiselman, 1962] concluded that this theory efficiently describes the market behaviour as if it were determined by investors acting on their expectations about future interest rates. The explanatory factor of the evolution of the term structure of interest rates is, therefore, the consensus of the market.

So, let's suppose the existence of two financial assets (zero bonds) issued to one and two periods respectively. Under an investing horizon of two periods, the investor could expect greater returns into the two year zero bond than into the one year zero bond, reinvesting at the expected forward rate at that moment:

$$\left(1+{}_2i_0\right)^2 \geq \left(1+{}_1i_0\right)\cdot\left(1+E\left({}_1f_1\right)\right)$$

If this inequality was given, all the agents of the market would proceed to invest in the two years bond and as consequence the price of the asset would increase. Equally, the demand of the one year bond would diminish. As consequence, the interest rate for the two year bond would be diminishing progressively and that of one year would be increased until the investment in both financial assets was indifferent. In the situation of balance it would be have:

$$\left(1+{}_2i_0\right)^2 = \left(1+{}_1i_0\right)\cdot\left(1+E\left({}_1f_1\right)\right)$$

So, as result of the theory of the pure expectations, the expected interest rate one year from now coincides with the forward interest rate in that period:

$$E\left({}_1f_1\right)={}_1f_1 = \frac{\left(1+{}_2i_0\right)^2}{\left(1+{}_1i_0\right)} - 1$$

3. REVIEW OF CERTAIN FINANCIAL TERMS

The following terms are commonly used in the finance literature to name different types of interest rates: spot rate, forward rate, and yield to maturity. Their definitions follow Meneu *et al.*, (1992).

3.1. Spot Rate

The *t* period spot rate is the periodical rate of interest that can be earned on an investment paid at time zero and that would be repaid, free of insolvency risk, as a lump sum with interest at some *t* period in the future. This interest rate depends directly on the period in which it is repaid *[0, t]*.

$$ {}_ti_0 = \left(\frac{P_t}{P_0}\right)^{1/t} - 1 $$

where:

${}_ti_0$ = Spot rate at *t* periods;

P_0 = Price of zero-coupon bond (at zero moment);

P_t = Price of zero-coupon bond at maturity (*t* periods in the future).

If we have a set of zero-coupon bonds with different periods to maturity and different prices but with no insolvency risk, we can establish a relationship, called the *interest rate term structure,* which determines the spot rate as a function of *t* as *t* increases from zero.

- If the spot rate is constant for all the periods, we have a flat term structure;
- If the spot rates for the short periods are greater than the spot rates for the long term, then it is an inverted term structure;
- If the spot rates for the short period take values less than for the long periods, then it is a positive or increasing term structure.

Figure 1 shows the observed spot rates.

Figure 1
Observed Spot Rates at 2008 April 30 in Spain.
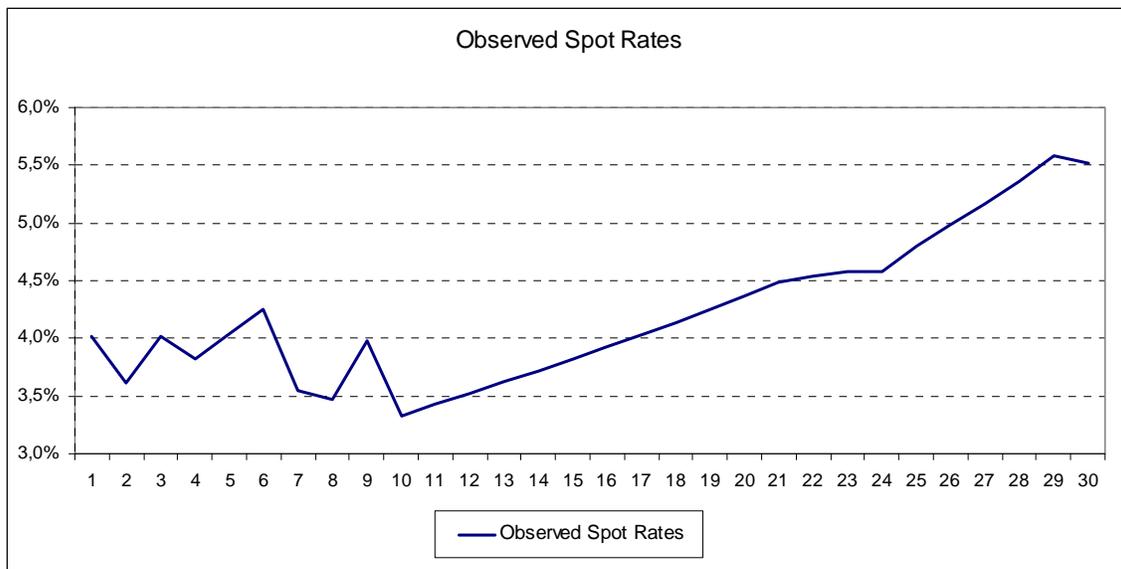
3.1.1.   *Forward Rate*

The forward rate is the rate of interest, implicit in currently spot rate that could be applicable from one point of time in the future to another point of time in the future, which equates to:

$$(1+_{t+1}i_0)^{t+1} = (1+_t i_0)^t \cdot (1+f_t)$$

where:

$_{t+1}i_0$ = The spot rate at the *[0, t+1]* period;

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$_t i_0$ = The spot rate at the *[0, t]* period;

f_t = The forward rate at the *t* period point.

Therefore, it is an interest rate that, applied to the amount of a monetary unit valued at the *t* period spot rate, produces a value equal to the one that would be obtained with the amount of a monetary unit valued at the *t + 1* period spot rate.

If the forward rate for each period is known, we can determine the *t* period spot rate as a product of the forward rates:

$$(1+_t i_0)^t = (1+f_1)\cdot(1+f_2)\cdot \ ... \ \cdot(1+f_{t\text{-}1})$$

where:

$$_t i_0 = \left[(1+f_1)\cdot(1+f_2)\cdot \ ... \ \cdot(1+f_{t\text{-}1})\right]^{1/t} - 1$$
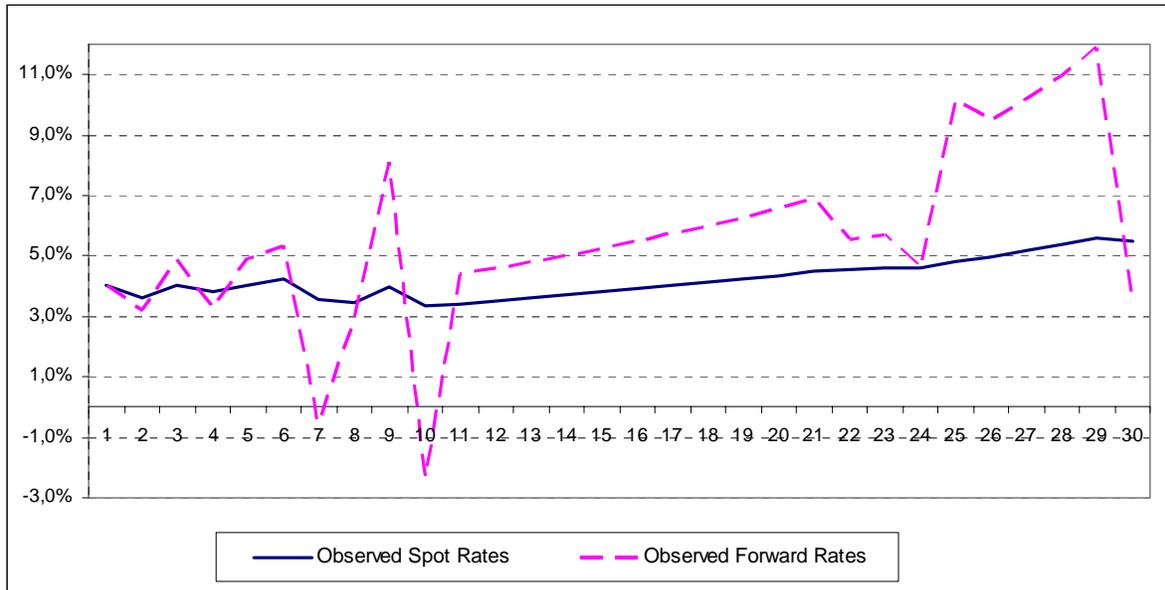
as result we have the *[0, t]* spot rate.

Figure 2 shows the implied forward rates derived from the spot rates of Figure 1.

Figure 2
Observed Forward Rates at 2008 April 30 in Spain.

And Figure 3 shows both, the implied forward rates derived from the spot rates and these spot rates.

Figure 3
Term Structure of Interest Rates at 2008 April 30 in Spain.

3.1.2.   *Yield to Maturity*

For a bond, with current price *P* with coupon payments of *C* per period and redeemable at the end of *t* periods at its maturity value (*M*), the yield to maturity (*y*) is obtained as the solution to the equation:

$$P = \sum_{j=1}^{t} \frac{C}{(1+y)^j} + \frac{M}{(1+y)^t}$$

Thus the yield to maturity of a bond is the interest rate that equates the future payments to the current price. The yield to maturity coincides with the *[0, t]* spot rate when the bonds are zero coupon bonds. The curve of the *y* as a function of *t* is called the yield curve.

4.   GRADUATED FORWARD RATE

The current *t* period forward rate is calculated using current information. When period *t* is reached (*t* periods from now), however, the actual forward rate will likely differ from what was calculated. Not surprisingly, as we get closer to *t*, that one period rate becomes more accurate. There is less time in order that changes affect it.

Usually the main factor that influences the term structure of interest rates effect is the inflation, but [Vanderhoof, 1983] "in spite of considerations of academicals over the unlikelihood of

parallel changes in the TSI, this is what happens in reality. The dominant force in the variations of the interest rates in the last years has been the inflation. Its effects have provoked parallel movements in the curves and these movements generally have reflected the inflation in it entirely".

The observed interest rates obtained at a specific moment are according at the expectations that the market has at that moment and influence the value of interest rates.

i)   The crisis and its effect into several firms could give us different values at different periods.

ii) The necessity of money (liquidity) in a specific moment could do the value of a bond became higher from one moment to another, but it is due to a punctual anomaly.

iii) Political factors could affect interest rates at different periods

iv) Even more, it could happen that the investor could be at a market without the n-year spot rate or n-year forward rate.

So, the observed interest rates will not be the real interest rates. It is necessary to develop a method for looking an adjusted future forward rate smoothing the effect of several anomalies detected in the market.

When determining these future forward rates we should be aware of two constraints:

i) The future forward rates should be close to the currently observed values.
ii) The forward curve is not to have discontinuities.

These constraints are similar to those that exist in the construction of mortality tables (for example see Slaney, 1994) where an observed series of rates is used to estimate other rates.

- The observed rates fit to a certain periodicity (annual).
- The observed rates are based on past information.
- The observed rates depend on the reality.
- For different periods (ages) the rates are different and depend on past values.

Thus, we will use a common Whittaker-Henderson graduation formula (a standard actuarial technique) to graduate the observed forward rates.

Whittaker-Henderson graduation (Whittaker 1923 and Henderson, 1924) requires minimizing a function F, which is the sum of a fitness measure and a smoothness measure:

An Actuarial Approach For Adjusted Forward Rates

$$F = \sum_{j=1}^{n} \left(f_j - \overline{f_j} \right)^2 + h \cdot \sum_{j=1}^{n \cdot z} \left(\Delta^z f_j \right)^2$$

where:

f_j = The future (unknown) forward rate at the j period;

$\overline{f_j}$ = The observed (or implied) forward rate at the j period;

Δf_j = $f_{j+1} - f_j$ = The forward rate variation for two consecutive periods;

h = Relative emphasis of smoothness over fit;

z = Degree of difference between consecutive observed forward rates.

n = The last period of the observed forward rates.

If h tends to infinite the graduated forward rate would be constant for all periods and if h tends to zero the graduated forward rate would be the same than the observed forward rate. The parameter z takes 3 and implies the grade of the polynomial function. Usually it is necessary several proves to choose the best parameters for the model, but it is suitable to choose $z = 2, 3$ or 4 and h bigger than 2.

Other payment patterns (monthly or quarterly) can be used. For example, if monthly payments are used, we can use monthly spot rates, forward rates and graduated forward rates. Although some months may be without bonds to calculate their spot rates, we use a procedure developed by Slaney (1994) to estimate spot rates where there are discontinuities at the interest rate term structure.

In addition, if no bonds exist beyond a certain maturity, we can view the price, spot, and forward rates as constant beyond that maturity. For example, if there are no bonds with a 20 year maturity or longer, we view the price, spot, and forward rates as constant for $t > 20$.

Table 1
Graduation Results at 2008 April 30 in Spain.

| t	Observed Spot Rates	Observed Forward Rates	Graduated Forward Rates	Estimated Spot Rates
1	4,014978%	4,015%	3,87445%	3,87445%
2	3,607912%	3,202%	3,82785%	3,85115%
3	4,020745%	4,851%	3,98968%	3,89730%
4	3,823831%	3,235%	3,86356%	3,88887%
5	4,043985%	4,929%	3,94725%	3,90054%
6	4,256938%	5,328%	3,70322%	3,86763%
7	3,553088%	-0,571%	2,91760%	3,73137%
8	3,474299%	2,924%	3,29480%	3,67670%
9	3,979412%	8,110%	3,79538%	3,68988%
10	3,332282%	-2,314%	2,85786%	3,60638%
11	3,428501%	4,396%	3,64364%	3,60976%
12	3,525201%	4,595%	4,17904%	3,65708%

An Actuarial Approach For Adjusted Forward Rates

t	Observed Spot Rates	Observed Forward Rates	Graduated Forward Rates	Estimated Spot Rates
13	3,622869%	4,802%	4,57661%	3,72753%
14	3,721920%	5,018%	4,89859%	3,81074%
15	3,822736%	5,244%	5,18125%	3,90155%
16	3,925685%	5,482%	5,44387%	3,99728%
17	4,031140%	5,733%	5,69386%	4,09632%
18	4,139489%	5,999%	5,93112%	4,19742%
19	4,251151%	6,282%	6,14635%	4,29909%
20	4,366588%	6,584%	6,31592%	4,39902%
21	4,486320%	6,910%	6,39578%	4,49325%
22	4,533803%	5,536%	6,30543%	4,57495%
23	4,583190%	5,676%	6,47199%	4,65672%
24	4,583190%	4,583%	6,90371%	4,74939%
25	4,800738%	10,160%	8,10830%	4,88172%
26	4,976849%	9,477%	8,62896%	5,02343%
27	5,164726%	10,169%	8,86655%	5,16332%
28	5,366659%	10,968%	8,66992%	5,28658%
29	5,585578%	11,903%	7,70729%	5,36914%
30	5,512185%	3,406%	5,34620%	5,36838%

Table 1 shows the graduated forward rates for a Whittaker-Henderson model with $n = 30$; $z = 2$ and $h = 3$. Figure 4 shows the graduated and observed forward rates.

These graduated forward rates represent annual interest rates used, for instance, for the reinvestment pension annual plan contributions and other assets.

Figure 4
Observed and Graduated Forward Rates at 2008 April 30 in Spain.

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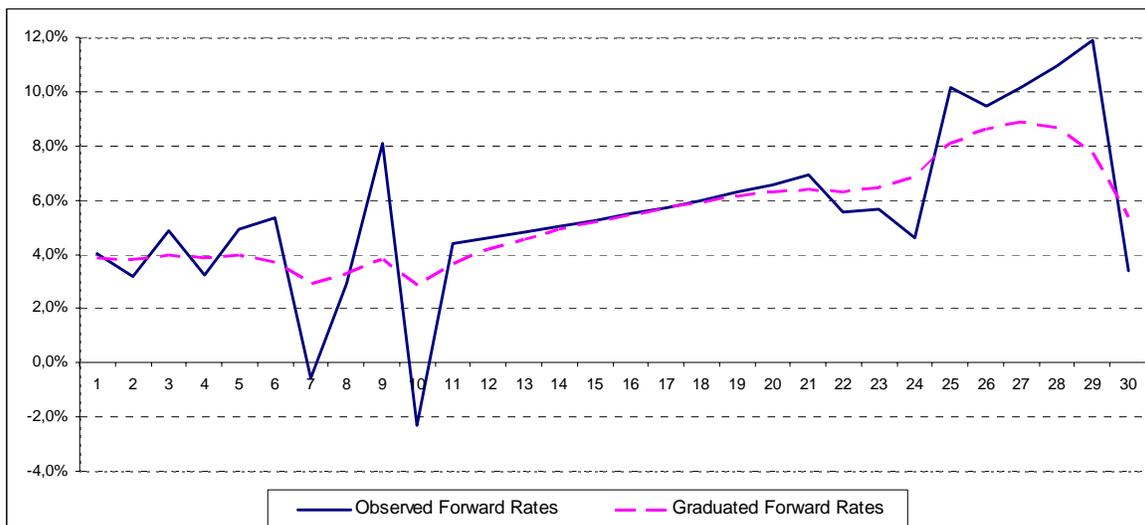
In addition, figure 5 shows the estimated term structure of interest rates using the graduated forward rates at 2008, April 30. This curve could be used to valuate liability flows in insurance industry, or into a pension scheme where the actuarial liability must be valued at market value. The same estimated curve could be used to value the assets whom fund the liability into an immunization program.

Figure 5
Observed and Estimated Term Structure of Interest Rates at 2008 April 30 in Spain.
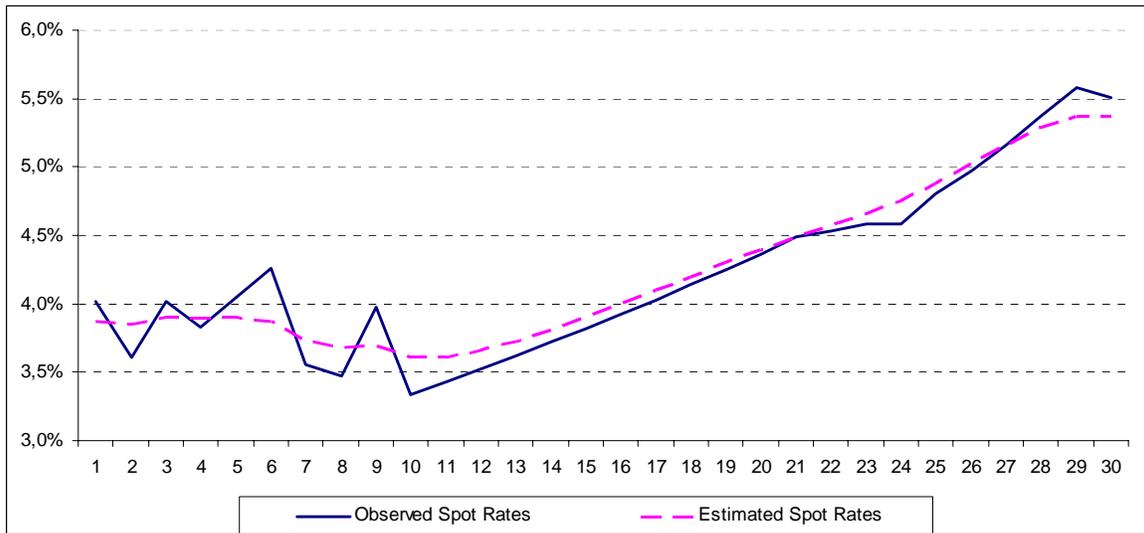
5. CONCLUSIONS

- The adjusted interest rates include the expectations of the market but smoothed throughout different years.

- The Whittaker – Henderson graduation method is fast and easy to apply

- This graduation method allows realizing estimations and future valuations accord to the expectations that, in a certain moment, exist on the interest rate.

- We can use of the inefficiencies observed in the market to anticipate to them before they arrive.

- There is necessary a periodic follow-up that determines the inefficiencies of the market on the expectations and that they could affect the valuations to be realize with the adjusted interest rates.

- For very long valuations or estimations, the Spanish financial market has problems offering bonds with durations longer than 15 years. This strategy is difficult to put into practice, at least globally if the period for liabilities is very long.

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7. APPLICATION

In this part of the paper we include a simple application of a portfolio of fixed income bond using an immunization procedure: duration matching. For this intention we are going to use the observed interest rates and later the graduated interest rates. For every case we proceed to assign Treasury Bonds with different maturities. Between one year to twenty maturity year.

We will apply to an endowment insurance that will pay 200.000 €at 65 years old if the insured is alive and the same amount if the insured dies from 55 to 65 years old, using the GR-95 mortality tables for males.

The problem is to find a combination of bonds that satisfies certain constraints and provides the lowest priced bond portfolio, [Kocherlakota et al, 1988],[Christensen and Fabozzi, 1995], where the duration of the liabilities should be equal to the duration of the asset flows.

The principal characteristics of the Treasury Bonds are the following ones:

Table 2
Characteristics of the Treasury Bonds.

Maturity	Price	Cupon	Maturity	Price	Cupon
1	99,616	3,62%	11	100,684	3,56%
2	100,748	4,01%	12	100,735	3,64%
3	99,491	3,83%	13	100,783	3,71%
4	100,738	4,03%	14	100,829	3,79%
5	100,861	4,22%	15	100,873	3,87%
6	96,734	3,60%	16	100,915	3,94%
7	99,544	3,52%	17	100,954	4,02%
8	102,925	3,95%	18	100,991	4,10%
9	95,880	3,40%	19	101,026	4,17%
10	100,631	3,48%	20	101,058	4,25%

With the observed interest rates on April 30, 2008, we obtain the following results,

Table 3
Results of a Duration Matching to 10 years.

| t | | Observed Rates | | Adjusted Rates | | % |
	L_t	Balance	Surplus	Balance	Surplus	Difference
1	1.349	4.593	4.593	4.743	4.743	**3,266%**
2	1.450	4.491	9.232	4.641	9.566	**3,618%**
3	1.560	4.382	14.061	4.531	14.478	**2,966%**
4	1.679	4.263	18.779	4.413	19.451	**3,578%**
5	1.807	4.135	23.840	4.285	24.503	**2,781%**
6	1.944	3.998	29.108	4.147	29.558	**1,546%**
7	2.091	3.851	32.792	4.000	34.421	**4,968%**
8	2.235	3.707	37.458	3.857	39.412	**5,217%**
9	2.367	3.575	44.072	3.725	44.632	**1,271%**

10	184.762	(43.052)	(0)	(45.908)	(0)	**0,000%**

Where

L_t :   It is the expected liability (probable payment) at time t for an insured.

The principal values and indicators obtained under both cases (Observed rates and adjusted or graduated rates) are the following ones:

Table 4
Principal indicators and values.

	Observed Rates	Adjusted Rates	DIF (%)
DEM=	9,251	9,217	**-0,3675%**
DM=	9,251	9,217	**-0,3675%**
CXEM=	96,892574	96,245349	**-0,6680%**
CXM=	133,864	105,86530	**-20,9158%**
L(i)o =	146.600	143.158	**-2,3479%**
A(i)o=	141.352	144.929	**2,5146%**

Where,
DEM   =   Expected modified duration of liabilities.
DM   =   Modified duration of the bond portfolio.
CXEM   =   Expected modified convexity of liabilities
CXM   =   Modified Convexity of the bond portfolio.
$L(i)_0$ =   Present value of future liabilities (probable payments) to be done at 0-moment..
$A(i)_0$ =   Present value of future incomes from the bond portfolio at 0-moment.

The solution for minimized bond portfolio, subject to the classical constraints where the immunization strategy has been applied is the following bond distribution:

Table 5
Bond Portfolio structure

	Observed Rates	Adjusted Rates	(%)
Maturity	X_t	X_t	DIF
10	0,26	-	**-**
12	0	0,01	**-**
17	0,08	-	**-**
18	285,53	941,30	69,67%
19	1.427,79	2.873,97	50,32%
20	139.638,59	141.113,32	1,05%

Where,

X_t =   Invested amount in the bond with maturity in t year.

Through this process, the financial income generated by the fund does not depend on the future values of the interest rate in the valuation of those flows. It is possible to pay liabilities each year, without proceeding to anticipated sales which could produce losses affecting the financial equilibrium of the fund. This procedure follows De La Peña (1999).