

A DYNAMIC CONTROL STRATEGY FOR PENSION PLANS IN A STOCHASTIC FRAMEWORK

Colivicchi Ilaria

Dip. di Matematica per le Decisioni, Università di Firenze
(Presenting and corresponding author)

Via C. Lombroso, 6/17 - 50124 Firenze

ilaria.colivicchi@gmail.com

Mulinacci Sabrina

Dip. di Matematica per le Scienze Economiche e Sociali, Università di Bologna

Viale Filopanti, 5 - 40126 Bologna

sabrina.mulinacci@unibo.it

Vannucci Emanuele

Dip. di Statistica e Matematica Applicata all'Economia, Università di Pisa

Via C. Ridolfi, 10 - 56124 Pisa

e.vannucci@ec.unipi.it

ABSTRACT

In recent years the study of the optimum management of pension plans is a main issue.

The two principal alternatives in the design of pension plans are the defined contribution plan and the defined benefit plan.

The aim of this paper is to study a stochastic model both for investment returns and income dynamics in order to propose a contribution strategy that allows fixed substitution rates.

The control strategy is based on the definition of a temporal grid defining the times at which it is possible to change the level of the contribution (expressed through a ratio of the annual salary), according to the temporal information flow and inside pre-definite bounds.

Numerical simulations in order to show some sensitivity analysis on featuring variables are proposed.

KEYWORDS

Supplementary pension plans, income substitution rate, stochastic dynamic control strategy, numerical simulation

1. INTRODUCTION

Problems on pension fund management can be generally divided in two different categories considering defined benefit pension plans or defined contribution pension plans.

In this paper we would like to mix the characteristics of these two types of pension plans. Indeed, using a capitalization method for the worker contributions, as usual in the defined contribution pension plans is done, we want to define a dynamic control strategy in order to have aimed levels of the instalment annuity, that is generally the aim of a defined benefit pension plans.

We are going to use the income substitution rate to measure the efficiency of the supplementary pension investment as usual measured in the actuarial practice in terms of the income substitution rate that is the ratio between the instalment annuity and the last salary before the retirement.

The income substitution rate has different uncertainty elements that create sources of risk on its final level: the performance of the fund in which the contributions are invested in and the evolution over time of the salary during the working period that is linked to the inflation rate and the personal working carrier.

The last salary before retirement, used to evaluate the efficiency of the investment in the pension fund, is a random variable too, that needs to be estimate.

About the uncertainty rate of return of the fund to which the evolution of the pension plan is linked, we have to consider that the worker can change, during the accumulation period, the risk profile of the investment transferring the capitalization of the contributions from a fund to another with different mix of the assets of the portfolio. The optimal fund composition to reach a fixed target is considered in a lot of papers as Battocchio, Menoncin (2004), Josa-Fombellida, Rincon-Zapatero (2006), Haberman, Vigna (2001).

Haberman, Vigna (2001) considered the problem of the capitalized level management during the annuity payment period; the rate of return uncertainty of the investment in the fund is obviously present during the instalment annuity payments.

In our paper we will consider a fixed constant technical rate and a fixed demographic table with which we can evaluate a substitution rate to convert the final capital into an instalment annuity.

We will focus on the possibility of finding the constant contribution rate that the worker should pay yearly till the retirement time, to have the final capital expected value enough for having a fixed substitution rate when he/she will finish working.

Time by time we will evaluate this contribution rate, we will need to consider the level of the accumulated capital and the salary level at that moment; the real contribution will depend on the maximum and the minimum contribution aliquota constrains the worker would like to fix.

Our work will stress also on the trade off between the income substitution rate the worker can obtain and his/her behaviour on the investment. Obviously the higher is the level of the investment in the pension fund, the higher is the

possibility to reach interesting level of the supplementary pension but on the contrary lower is the worker solidity during his/her working life.

The choice between optimal consumption and portfolio selection is considered in a lot of frameworks, see for example Merton (1971). Josa-Fombellida, Rincon-Zapatero (2006) studied the cases in which the main objective of the worker is to minimize both the contribution rate risk and the solvency risk.

Our paper contemplates a worker investment capability changing over the accumulation period so that this capability represents a constraint for the investment in the pension fund.

The pension model under consideration refers to a temporal interval strictly linked to the working life period in discrete time and year by year.

Numerical results are obtained by a simulation method with which we can generate the rate of salary variation, the rate of return of the fund and the investment capability year by year, while the other variables of the model can be obtained by an analytical way.

In this paper other pension fund benefits, as annuity for accident and/or invalidity and/or death insurance are not considered.

This evaluation model doesn't implement the effects of death before the retirement, but if we think about the Italian mortality dynamics where the probability of death are going to be lower and lower over time, this case is not so important to consider.

The paper is organized as follows: in the first paragraph the model for the accumulation period of the contributions is described. In the second paragraph a periodic control strategy is implemented. In the third the contribution aliquota constraints are introduced. In the fourth interesting results are obtained from the model. In the fifth numerical results are provided. In the last paragraph new ideas for implementing the paper are suggested.

2. THE CONTRIBUTION CAPITALIZATION MODEL

Assuming e_1 the time at which the investment in the pension fund starts and e_2 the time at which the worker retires from work, having the final capital realized by the investment at his disposal.

Let $[0, n] := [0, e_2 - e_1]$ be the time interval considered in the model, being this interval divided in n periods (years) and let 0 be the evaluation time. Let s_0 be the deterministic salary known at time 0 and S_i with $i = 1, 2, \dots, n$ be the sequence of the random salary the worker should have during his/her working carrier: such sequence, with starting value $S_0 := s_0$, is recursively defined by

$$S_i = S_{i-1}K_i \text{ with } i = 1, 2, \dots, n$$

where K_i is the random salary variation rate in the i -th year and we assume the independence among the K_i 's.

If α_i is the contribution rate for the i -th year with $i = 1, 2, \dots, n$ then the contribution to the pension fund in that year with monetary value at time i , is

$$\alpha_i S_i$$

Let M_i with $i = 0, 1, \dots, n$ the sequence of of the random capitalized contributions for each worker. The first one, M_0 , is deterministic and we define

$M_0 := m_0$, on the way that the worker could transfer in the pension fund other possible contributions. The sequence is defined as follows

$$M_i = M_{i-1}G_i + \alpha_i S_i \text{ with } i = 1, 2, \dots, n$$

where G_i is the random capitalization factor for the i -th year and we assume the independence among the G_i 's and between G_i and K_i for each i .

The aim of this paper is to estimate the random variable distributions S_n and M_n from which we will analyze the supplementary pension investment in term of the income substitution rate.

The income substitution rate is in fact defined as follows

$$\frac{\varphi(e_2, g)M_n}{S_n}$$

where $\varphi(e_2, g)$ is the coefficient for transforming the final capital M_n in the instalment annuity with a technical rate g and a demographic table expressed by the function φ .

Notice that if $m_0 = 0$ as we assume in what follows, the income substitution rate doesn't depend on the initial salary s_0 .

3. THE DYNAMIC CONTROL STRATEGY

As announced in the Introduction the investment efficiency in the supplementary pension is measured in terms of income substitution rate, that is the ratio between the annuity instalment, and the last salary.

Note that, given the level of the salary at each date $i = 1, 2, \dots, n-1$, S_i , the expected value of the last salary $E[S_n | S_i]$ is given by

$$E[S_n | S_i] = S_i(1 + E[K_i])^{n-i}$$

This expected value represents the reference comparison for the substitution rate β that the worker aims to reach with the information available at time i : the amount of a constant annuity the worker aims to is $l_i = \beta E[S_n | S_i]$.

Notice that, starting from this value, it is possible to compute the capital, $M_n(l_i)$, needed at the retirement time in order to guarantee a substitution rate β , given by

$$M_n(l_i) = \frac{l_i}{\varphi(e_2, g)}$$

In particular, at each date $i = 1, 2, \dots, n-1$, the constant contribution rate to be invested up to the retirement time in order to guarantee a final capital given by $M_n(l_i)$ is evaluated.

The expected value of the final capital M_n , given the capital at time i , M_i , is computable through

$$M_n = M_i \prod_{w=i+1}^n G_w + \sum_{w=i+1}^n S_w \alpha_w \prod_{j=w+1}^n G_j$$

By this, assuming $\alpha_w^* = \alpha_{i+1}^*$ for $w = i+1, i+2, \dots, n$, we have

$$E[M_n | M_i, S_i] = E \left[M_i \prod_{w=i+1}^n G_w + \sum_{w=i+1}^n S_w \alpha_{i+1}^* \prod_{j=w+1}^n G_j \mid M_i, S_i \right] =$$

$$= M_i E \left[\prod_{w=i+1}^n G_w \mid M_i, S_i \right] + \alpha_{i+1}^* \sum_{w=i+1}^n E \left[S_w \prod_{j=w+1}^n G_j \mid M_i, S_i \right]$$

By the independence among the G_w 's and among the K_w 's and between G_w and K_w for each w , we have

$$E \left[\prod_{w=i+1}^n G_w \mid M_i, S_i \right] = \prod_{w=i+1}^n E[G_w]$$

and

$$E \left[S_w \prod_{j=w+1}^n G_j \mid M_i, S_i \right] = E[S_w \mid M_i, S_i] \prod_{w=i+1}^n E[G_w]$$

from which

$$E[M_n \mid M_i, S_i] = M_i E \left[\prod_{w=i+1}^n G_w \right] + \alpha_{i+1}^* \sum_{w=i+1}^n E[S_w \mid M_i, S_i] \prod_{w=i+1}^n E[G_w]$$

Now, it remains to compute α_{i+1}^* in such a way that

$$E[M_n \mid M_i, S_i] = M_n(l_i)$$

holds, that is

$$\alpha_{i+1}^* = \frac{M_n(l_i) - M_i E \left[\prod_{w=i+1}^n G_w \right]}{\sum_{w=i+1}^n E[S_w \mid S_i] \prod_{w=i+1}^n E[G_w]}$$

We stress that the contribution rate really invested could not be the one we just computed, since the constraints introduced in the following Section would be considered.

4. THE CONTRIBUTION CONSTRAINTS

In evaluating the contribution rate, we make two main assumptions.

The first consists in defining at time 0 some features of the contract that cannot be modified during the lifetime of the accumulation period. More precisely, we assume that the contract fixes the dates at which to check the investment in order to evaluate the new contribution rate; for example every d years: if i is such a date we have $\alpha_i^* = \alpha_{i+2}^* = \dots = \alpha_{i+d-1}^*$.

Moreover we assume the worker to define two bounds to the contribution rate: a global minimum α_{\min} and a maximum $\alpha_{\max}(i)$, depending on time, in such a way that $\alpha_i^* \in [\alpha_{\min}, \alpha_{\max}(i)]$, $\forall i = 1, 2, \dots, n$ and such that $\alpha_{\max}(i) = f(i, S_i, P_i)$ meaning that it depends on time, on the salary S_i and on the wealth P_i .

The second consists in allowing the contract to take into account some specific "liquidity" requirements of the worker during the lifetime of the accumulation period. Since future liquidity requirements are not predictable we model the future upper bounds of the saving capability through a stochastic process

$(\delta(i))_{i=0,1,\dots,n}$ with $\delta(i) \in [\alpha_{min}, \alpha_{max}(i)]$ and such that the distribution of each $\delta(i)$ is different and depends on the date i , on the $\delta(i-1)$ value observed at time $i-1$ and on $\alpha_{max}(i)$.

This way, in our model, the salary contribution rate really invested in the year i , α_{i+1} is

$$\alpha_{i+1} = (\alpha_{i+1}^* \vee \alpha_{min}) \wedge \delta(i)$$

5. THE LAYOUT OF THE RESULTS

Let Q be the number of simulations for each different scenario. The substitution rate obtained in the q -th simulation is

$$\beta_q^* = \frac{M_{q,n}\varphi(e_2, g)}{S_{q,n}} \text{ with } q = 1, 2, \dots, Q,$$

where $M_{q,n}$ and $S_{q,n}$ are respectively the capital at the end of the accumulation period and the final salary obtained in such q -th simulation. For each scenario we report the sample average and the mean square error of the substitution rate given by

$$E[\beta^*] = \frac{\sum_{q=1}^Q \beta_q^*}{Q}, \quad \sigma[\beta^*] = \frac{\sum_{q=1}^Q (\beta_q^*)^2}{Q} - \left(\frac{\sum_{q=1}^Q \beta_q^*}{Q} \right)^2$$

We also report the estimates of the probabilities that the actual substitution rate will be below fixed proportions of the substitution rate that the worker aims to reach, that is

$$N_\gamma, \quad \frac{\#\{q : \beta_q^* < \gamma\beta\}}{Q} \quad \text{with } \gamma = 1, 0.9, 0.75$$

Such results describe the efficiency of the control strategy in order to reach the aimed substitution rates. It is also necessary to take count of the cost to obtain such results in terms of average contribution rate during the accumulation period.

Given $\alpha_{i,q}$ the contribution rate in year $(i-1, i]$, with $i = 1, 2, \dots, n$, in the q -th simulation, the average contribution rate in this q -th simulation is given by

$$\alpha_q^* = \frac{\sum_{i=1}^n \alpha_{i,q}}{n}$$

For each scenario we report the sample average and the mean square error of the average contribution rate given by

$$E[\alpha^*] = \frac{\sum_{q=1}^Q \alpha_q^*}{Q}, \quad \sigma[\alpha^*] = \frac{\sum_{q=1}^Q (\alpha_q^*)^2}{Q} - \left(\frac{\sum_{q=1}^Q \alpha_q^*}{Q} \right)^2$$

6. NUMERICAL EXAMPLES

The demographic table used for the coefficient for transforming the final capital M_n in the instalment annuity is the the most recent table projected by the Ragioneria Generale dello Stato, known as IPS55, which is the table that has to be used for the fixing of annuities rates.

Let c be the annual inflation rate, the standard scenario is

$$e_1 = 30, e_2 = 65, g = 0.01$$

$$\alpha_{\min} = 0, \alpha_{\max}(i) = \alpha_{\max} = 0.15 \text{ for each } i = 1, \dots, n, d = 1, \beta = 0.3$$

$$c = 0.02, K_i = \begin{cases} 0 & \text{pr. } 0.4 \\ 1 + c & \text{pr. } 0.3 \\ 1 + 2c & \text{pr. } 0.2 \\ 1 + 3c & \text{pr. } 0.1 \end{cases} \text{ from which } E[K_i] = 1 + c$$

$$G_i \sim \text{Logn}(\mu, \sigma) \text{ with } \mu = c, \sigma = 0.15$$

$$\delta(i) \sim N(\lambda\delta(i-1) + (1-\lambda), \xi), i = 1, \dots, n, \text{ with } \delta(0) = \alpha_{\max}, \lambda = 0.5, \xi = 0.02$$

In each following table we present the results concerning the standard scenario (in bold type) except for the value of one parameter in order to describe some sensitivity analysis of the results. For each scenario $Q = 100000$ simulations are made.

Inflation rate

c	0	0.01	0.02	0.03	0.04
$E[\beta^*]$	0.264	0.269	0.274	0.279	0.285
$\sigma[\beta^*]$	0.141	0.147	0.154	0.162	0.171
$E[\alpha^*]$	0.127	0.129	0.130	0.131	0.132
$\sigma[\alpha^*]$	0.017	0.015	0.014	0.013	0.012
N_1	0.706	0.692	0.678	0.664	0.650
$N_{0.9}$	0.619	0.605	0.592	0.580	0.566
$N_{0.75}$	0.460	0.450	0.444	0.435	0.426

With the setting used in these numerical examples, an increase of the inflation rate implies an increase both of the investment expected return (since we have assumed $\mu = c$) and of the expected final salary (since $E[K_i] = 1 + c$). In our numerical examples it is stronger the first effect and hence, an increase of the inflation rate implies a slight increase of the expected substitution rate. Anyhow, the simultaneous increase of its volatility forbids to have big improvements in terms of probability of reaching low substitution rates.

Volatility of saving capability

ξ	0	0.01	0.02	0.03	0.04
$E[\beta^*]$	0.293	0.284	0.274	0.265	0.255
$\sigma[\beta^*]$	0.155	0.152	0.154	0.143	0.139
$E[\alpha^*]$	0.139	0.134	0.130	0.126	0.121
$\sigma[\alpha^*]$	0.016	0.015	0.014	0.013	0.012
N_1	0.628	0.653	0.678	0.703	0.730
$N_{0.9}$	0.535	0.563	0.592	0.621	0.651
$N_{0.75}$	0.376	0.404	0.444	0.469	0.502

Since for the saving capability is assumed a maximum value, increasing the volatility of such random variable implies a decrease of the expected saving capability (as described by the results concerning the average contribution) and, hence, a decrease of the expected substitution rate.

Volatility of the returns

σ	0.05	0.1	0.15	0.2	0.25
$E[\beta^*]$	0.241	0.251	0.274	0.311	0.363
$\sigma[\beta^*]$	0.043	0.084	0.154	0.318	0.453
$E[\alpha^*]$	0.138	0.135	0.130	0.125	0.117
$\sigma[\alpha^*]$	0.004	0.009	0.014	0.017	0.020
N_1	0.905	0.760	0.678	0.635	0.609
$N_{0.9}$	0.759	0.641	0.592	0.569	0.557
$N_{0.75}$	0.390	0.430	0.444	0.455	0.462

In our model increasing returns volatility implies an increase of the expected return and, hence, an increase of the expected substitution rate. Anyhow, the simultaneous increase of the volatility of the substitution rate even increases the probability of low substitution rates.

Aimed substitution rate

β	0.1	0.2	0.3	0.4	0.5
$E[\beta^*]$	0.125	0.227	0.274	0.284	0.287
$\sigma[\beta^*]$	0.052	0.110	0.154	0.158	0.164
$E[\alpha^*]$	0.060	0.108	0.130	0.135	0.137
$\sigma[\alpha^*]$	0.014	0.019	0.014	0.009	0.006
N_1	0.346	0.489	0.678	0.829	0.908
$N_{0.9}$	0.210	0.377	0.592	0.773	0.874
$N_{0.75}$	0.063	0.215	0.444	0.651	0.788

Let observe that high substitution rates (from 0.3 up) are not reachable with the standard scenario. The low volatility of the average contribution points out that the contribution is as high as possible.

Persistency of the previous saving capability

λ	0	0.25	0.5	0.75	1
$E[\beta^*]$	0.280	0.278	0.274	0.266	0.209
$\sigma[\beta^*]$	0.150	0.149	0.154	0.144	0.138
$E[\alpha^*]$	0.133	0.132	0.130	0.126	0.098
$\sigma[\alpha^*]$	0.014	0.014	0.014	0.013	0.026
N_1	0.662	0.667	0.678	0.701	0.813
$N_{0.9}$	0.574	0.580	0.592	0.618	0.759
$N_{0.75}$	0.418	0.425	0.444	0.466	0.650

If the persistency of the previous saving capability increases (namely if λ increases) the average saving capabilities decreases (because the "default" saving capability corresponds to the maximum capability at each age). Hence, the consequences are easily predictables.

Periodicity of the control

d	1	2	3	5	10
$E[\beta^*]$	0.274	0.271	0.267	0.263	0.254
$\sigma[\beta^*]$	0.154	0.147	0.147	0.146	0.144
$E[\alpha^*]$	0.130	0.128	0.127	0.125	0.121
$\sigma[\alpha^*]$	0.014	0.013	0.012	0.011	0.010
N_1	0.678	0.688	0.697	0.710	0.732
$N_{0.9}$	0.592	0.604	0.616	0.633	0.659
$N_{0.75}$	0.444	0.453	0.467	0.487	0.520

Decreasing the periodicity of the control does not significantly influences the results. From a commercial point of view to not invite the worker to frequent changes of the contribution rate could be a positive factor.

Maximum contribution rate

α_{\max}	0.09	0.12	0.15	0.18	0.21
$E[\beta^*]$	0.163	0.222	0.274	0.313	0.336
$\sigma[\beta^*]$	0.094	0.123	0.154	0.162	0.166
$E[\alpha^*]$	0.078	0.106	0.130	0.148	0.159
$\sigma[\alpha^*]$	0.004	0.008	0.014	0.020	0.026
N_1	0.924	0.808	0.678	0.570	0.501
$N_{0.9}$	0.892	0.749	0.592	0.470	0.392
$N_{0.75}$	0.815	0.620	0.444	0.305	0.229

As widely expected an increase of the maximum contribution rate implies an increase in the expected substitution rate and even a more significant decrease of the probability of low substitution rates. It is anyhow important to take count of the costs of such results in terms of average contribution.

Further comments of the previous results that we leave to the readers concern the comparison of different scenarios which give the same results or in terms of the average contribution (and in such a case it could be interesting to analyze what happens in terms of expected substitution rates) or in terms of the expected substitution rate (and in such a case it could be interesting to analyze what happens in terms of the average contribution).

An example of such kind of comparison could be the one between the two scenarios that differ from the standard one or for the saving capability volatility $\xi = 0.04$, or for the periodicity of the control $d = 10$. In both scenarios the average contribution is 0.121 but the results in terms of reaching the aimed substitution rate are slight better in the case of $\xi = 0.04$.

7. CONCLUSION

About the concluding remarks of this paper we underline that even a simple model as the one here considered puts in evidence some crucial aspects about reaching satisfying income substitution rates.

The results proposed in this paper alert the potential subscribers of pension fund that the randomness of their income substitution rates cannot be neglected and hence the knowledge of the expected substitution rate is not sufficient to

take proper decisions. Even, an increase of the expected substitution rate could imply a higher probability of reaching low substitution rates.

Another fundamental aspect of the paper is to stress the role of the control strategy of the investment in order to reach aimed income substitution rates.

We point out some possible developments of the subject considered in this paper.

The first provides to consider the expectations on the future investment returns depending on the preceding evidences using well-known models based on credibility theory.

The second concerns the modelling of the possibility of the periodic switching between the available lines (which are different in terms of financial risk level), of the pension fund.

The third concerns the fact that the demographic scenario can (or perhaps will) change during the accumulation period, so making necessary to review the annuity conversion rates. This fact could make necessary to change the target value for the final capital in order to reach the aimed substitution rate.

It could be an interesting subject to measure the risk that the pension fund manager would assume if he promises fixed annuity conversion rates before the retirement time without a complete information on the evolution of the demographic aspect.

REFERENCES

Battocchio P., Menoncin F. (2004); Optimal pension management in a stochastic framework; *Insurance: Mathematics and Economics*, 34, pp. 79-95.

Haberman S., Sung J.H. (1994); Dynamics approaches to pension funding; *Insurance: Mathematics and Economics*, 15, pp. 151-162.

Haberman S., Vigna E. (2001); Optimal investment strategy for defined contribution pension schemes; *Insurance: Mathematics and Economics*, 28, pp. 233-262.

Josa-Fombellida R., Rincon-Zapatero J.P. (2006); Optimal risk management in defined benefit stochastic pension funds; *Insurance: Mathematics and Economics*, 34, pp. 489-503.

Merton R.C. (1971); Optimal consumption and portfolio rules in a continuous-time model; *Journal of Economic Theory*, 3, pp. 373-413.