EFFICIENT ALGORITHMS FOR MEAN-VARIANCE PORTFOLIO OPTIMIZATION WITH HARD REAL-WORLD CONSTRAINTS

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ABSTRACT

The Markowitz mean-variance optimization model is a widely used tool for portfolio selection. However, in order to capture real world restrictions on actual investments, a Limited Asset Markowitz (LAM) model with the introduction of quantity and cardinality constraints has been considered. These two constraints have been modelled by adding binary variables to the Markowitz model, thus resulting in a Mixed Integer Quadratic Programming problem that is considerably more difficult to solve.

We propose a new method for solving the LAM model based on a reformulation as a Standard Quadratic Program and on some recent theoretical results by the last two authors. We report optimal solutions of some previously unsolved benchmark problems used by several other authors and available from Beasley's OR-Library. We also test our method on five new data sets involving real-world capital market indices from major stock markets. On these data sets we have been able to evaluate, on out-of-sample data, the performance of the portfolios obtained from the LAM model and to compare them to the classical Markowitz portfolio, and to the market index. This comparison seems to point in favour of the solutions obtained with the LAM model.

We made our data sets and the solutions that we found publicly available for use by other researchers in this field.

KEYWORDS

Portfolio Optimization, Efficient Frontier, Mixed Integer Quadratic Programming, Standard Quadratic Programming, Cardinality Constrained Optimization.

1 INTRODUCTION

The classical Mean-Variance portfolio selection model of Markowitz [21, 22, 23] has been widely recognized as one of the cornerstones of modern portfolio theory. However, its success has inevitably drown many criticisms and proposals of alternative or more refined models, see, e.g., [11, 15, 16, 24, 25, 27, 28] and references therein.

Among the many refinements that have been proposed to make the Markowitz model more realistic, we analyze in this paper the one that limits the number of assets to be held in an efficient portfolio, and that also prescribes lower and upper bounds on the fraction of the capital invested in each asset. These requirements come from real-world practice, where the administration of a portfolio made up of a large number of assets, possibly with very small holdings for some of them, is clearly not desirable because of transactions costs, minimum lot sizes, complexity of management, or policy of the asset management companies. We call *Limited Assets Markowitz* (LAM) model the Markowitz model with the above restrictions. Because of its practical relevance, this model (often called *cardinality constrained portfolio optimization*), and some variations thereof, have been fairly intensively studied in the last decade especially from the computational viewpoint [1, 3, 4, 8, 9, 10, 12, 14, 19, 20, 26, 29, 31, 32].

From these studies it appears that the computational complexity for the solution of the LAM model is much greater than the one required by the classical Markowitz model or by several other of its refinements. Indeed, real-world problems of this type involving markets with less than one hundred assets have not yet been solved to optimality, while the standard Markowitz model is routinely solved for markets with thousands of assets. The practical difference in computational complexity is also theoretically justified by the fact that the classical Markowitz model is a convex quadratic programming problem that has a polynomial worst-case complexity bound, while the LAM model falls into the class of considerably more difficult NP-hard problems (see, e.g., [4, 31]).

We present here a new solution method for the LAM model that is based on a reformulation as a Standard Quadratic Programming problem and exploits recent theoretical results for Quadratic Programming by Tardella [34] and by Scozzari and Tardella [30]. Our method is able to solve to optimality the five benchmark problems described in [8] and publicly available in Beasley's OR Library. These problems have been used by several authors but no optimal solutions seems to have been reported until now. Indeed, Jobst, Horniman, Lucas and Mitra in 2001 state that solving these problems: "remains a computationally intractable task" [14, p. 498], and Di Gaspero, Di Tollo, Roli, Schaerf in 2007 still believe that it is: "intractable to solve real-world instances of the problem with proof of optimality" [10, p. 47]. In addition to these five problems, we also report solutions of much larger real-world problems with more than 2000 assets also taken from the OR-Library.

Furthermore, we have tested our method on other real-world data sets drawn from five important world markets. On these data sets we have been able to evaluate, on out-of-sample data, the performance of the portfolios obtained from the LAM model and to compare them to the classical Markowitz portfolio, and to the market index. This comparison seems to point in favour of the solutions obtained with the LAM model.

We made our data sets and the solutions that we found publicly available for use by other researchers in this field.

2 THE LIMITED ASSETS MARKOWITZ MODEL

The classical Mean-Variance (MV) portfolio optimization model introduced by Markowitz aims at determining the fractions x_i of a given capital to be invested in each asset i belonging to a predetermined set or market so as to minimize the risk of the return of the whole portfolio, identified with its variance, while restricting the expected return of the portfolio to attain a specified value.

More precisely, we assume that n assets are available, and we denote by μ_i the expected return of asset i, and by σ_{ij} the covariance of returns of asset i and asset j for $i, j = 1, \dots, n$. We also denote by ρ the required level of return for the portfolio. The classical MV model is:

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ st \quad & \\ & \sum_{i=1}^{n} \mu_i x_i = \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad i = 1, \dots, n \end{aligned} \tag{1}$$

This is a convex quadratic programming problem which can be solved by a number of efficient algorithms with a moderate computational effort even for large instances. We denote by $\phi(\rho)$ the optimal value of (1) as a function of ρ. Let ρ_{min} denote the value of $\sum_{i=1}^{n} \mu_i x_i$ at an optimal solution of the problem obtained by deleting the first constraint in (1), and let $\rho_{max} = \max\{\mu_1, \dots, \mu_n\}$. Then the graph of $\phi(\rho)$ on the interval $[\rho_{min}, \rho_{max}]$ coincides with the set of all non-dominated (or efficient) portfolios (*efficient frontier*), and is usually approximated by solving (1) for several (equally spaced) values of ρ in $[\rho_{min}, \rho_{max}]$.

The convexity of (1) implies that, for $\rho \geq \rho_{min}$, the function $\phi(\rho)$ is increasing and convex, so that the solution of (1) does not change if we replace the first constraint with $\sum_{i=1}^{n} \mu_i x_i \geq \rho$.

We now add to the MV model the realistic constraint that no more than K assets should be held in the portfolio (a *cardinality constraint*), and furthermore that the quantity x_i of each asset that is included in the portfolio should be limited within a given interval $[\ell_i, u_i]$ (a *quantity constraint* or *buy-in threshold*). Thus we obtain the following Limited Assets Markowitz model:

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ st \quad & \\ & \sum_{i=1}^{n} \mu_i x_i = \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i = 0 \text{ or } \ell_i \leq x_i \leq u_i, \quad i = 1, \dots, n \\ & |supp(x)| \leq K, \end{aligned} \tag{2}$$

where $supp(x) = \{i : x_i > 0\}$.

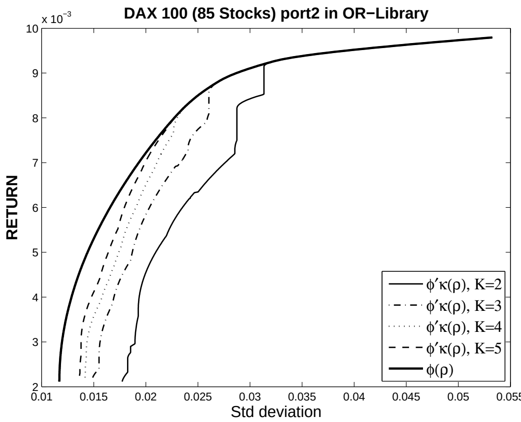

Figure 1: Graphs of $\phi(\rho)$ and $\phi'_K(\rho)$

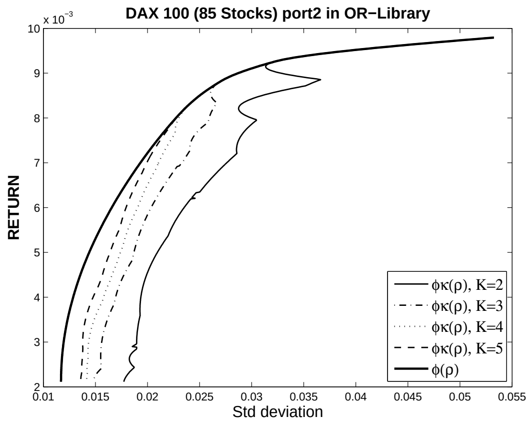

Figure 2: Graphs of $\phi(\rho)$ and $\phi_K(\rho)$

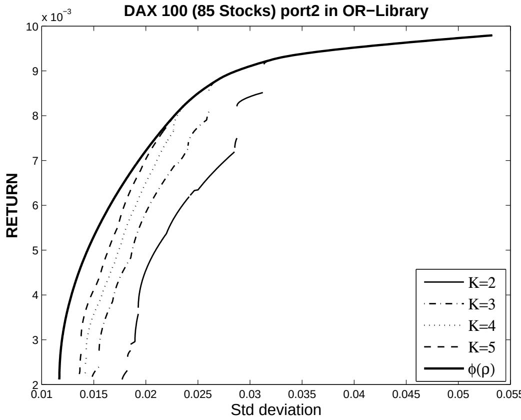

Figure 3: Efficient frontiers

Problem (2) is no longer a convex optimization problem because of the non-convexity of its feasible region. As a consequence the optimal value function $\phi_K(\rho)$ of (2) need not be increasing nor convex. Furthermore, $\phi_K(\rho)$ does not any longer coincide with the optimal value function $\phi'_K(\rho)$ of problem (2) where the first constraint is replaced by $\sum_{i=1}^{n} \mu_i x_i \geq \rho$. Indeed, $\phi'_K(\rho) = \phi_K(\rho)$ if and only if the point $(\rho, \phi_K(\rho))$ is on the efficient frontier.

Figures 1, 2 and 3 illustrate the graphs of $\phi(\rho)$ and $\phi'_K(\rho)$, of $\phi(\rho)$ and $\phi_K(\rho)$, and the efficient frontiers for some values of K, in an instance based on real-world data. Note that these figures are based on the *exact* optimal solutions to problem (3) obtained with the algorithm described in Section 4 below. Figure 3 can be compared with Figure 9 in [8] that is based on *approximate* solutions to (3) found with heuristic algorithms.

As observed by several authors [4, 8, 14], problem (2) can be reformulated as a Mixed Integer Quadratic Program (MIQP) with the addition of n binary variables:

$$
\begin{array}{lll}
\text{Min} & \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j & \\
st & & \\
 & \sum_{i=1}^{n} \mu_i x_i = \rho & \\
 & \sum_{i=1}^{n} x_i = 1 & \\
 & \sum_{i=1}^{n} y_i \le K & \\
 & \ell_i y_i \le x_i \le u_i y_i & i = 1, \ldots, n \\
 & x_i \ge 0 & i = 1, \ldots, n \\
 & y_i \in \{0, 1\} & i = 1, \ldots, n
\end{array} \tag{3}
$$

A number of exact approaches have been proposed to solve problem (3). Bienstock [4] proposes a branch-and-cut algorithm and reports good computational results for some real-life problems (not available for comparison). However, his method seems to become extremely slow for small values of K. Bertsimas and Shioda [3] extend the algorithm of [4] presenting a *tailored procedure*, based on Lemke's pivoting algorithm [18], that takes advantage of the special structure of the problem. They present computational results only on randomly generated data for fairly large values of K. A branch-and-bound algorithm for mixed integer nonlinear programs, including portfolio selection problems, is presented in [6]. Li, Sun and Wang [19] propose a convergent Lagrangian method as an exact solution scheme for a problem slightly more general than (3) and they describe some computational results for problems with at most 30 assets. Another Lagrangian relaxation method is proposed in [31] with application to some undisclosed real-life problems with up to 500 assets. Lee and Mitchell [17] develop an interior-point algorithm within a parallel branch-and-bound framework for solving nonlinear mixed-integer programming problems. Preliminarily computational results on three randomly generated quadratic portfolio models are reported. Frangioni and Gentile [13] use a method based on perspective cuts to solve randomly generated LAM problems (3) without cardinality constraints involving up to 400 assets. Furthermore, some commercial or free optimization softwares provide tools to solve general MIQPs, and thus (3), although only for problems with few hundreds variables at most.

Since exact methods are able to solve only a fraction of practically useful LAM models, a variety of heuristic procedures have also been proposed for solving (3). Local search techniques are discussed in [29], while [8] presents three heuristics based upon genetic algorithms, tabu search, and simulated annealing. In [14] two heuristic solution approaches are proposed for problems subject to buy-in threshold, cardinality and roundlot constraints. A hybrid local search algorithm combining principles of Simulated Annealing and of evolutionary strategies is used in [20] to solve problem (3) in the absence of quantity constraints. Other evolutionary algorithms combined with local search techniques in order to improve the quality of the solutions are described in [32, 33]. Finally, [12] introduce a parallel solution method by extending techniques developed in the multi-objective evolutionary optimization domain.

We should mention that, while several authors experiment their algorithms on undisclosed or randomly generated data, a selection of the cited papers [1, 10, 14, 26, 29, 32, 33] report results obtained on the five real-world data sets introduced in [8] that have been made available by Beasley in his OR-Libray [2].

3 REDUCTION TO A STANDARD QUADRATIC PROGRAMMING PROBLEM

We propose a new method for solving (2) that avoids the explicit use of additional binary variables. Our approach is based on the reduction of the LAM model (2) to a Standard Quadratic Programming (StQP) problem, as defined by Bomze [5], and is able to solve to optimality Beasley's problems and problems of greater dimension. A StQP is the problem of minimizing a (possibly indefinite) quadratic form over the standard simplex Δ, that is

$$\begin{array}{ll} \text{Min} & x'Qx \\ st & \\ & x \in \Delta = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \dots, n\} \end{array} \tag{4}$$

Despite its formal simplicity, this problem is theoretically difficult to solve (NP-hard) when Q is indefinite [5]. Indeed, its actual optimal solution for instances with more than 40 variables have not been reported in the literature until the recent paper by Scozzari and Tardella [30], where instances with more than 1000 variables have been solved.

We also point out that there is no loss of generality in restricting to quadratic *forms* instead of considering a general quadratic objective function. Indeed, over Δ, a quadratic function $f(x) = x'Px + 2q'x$ coincides with the quadratic form $x'Qx$, where $Q = P + eq' + qe'$, and e denotes the all-ones vector.

Problem (1) can easily be transformed into a (convex) StQP problem by using a quadratic penalty for the return constraint:

$$\begin{array}{ll} \text{Min} & f_M(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j + M[\sum_{i=1}^{n} \mu_i x_i - \rho]^2 \\ st & \\ & x \in \Delta \end{array} \tag{5}$$

Adding the cardinality constraint $|supp(x)| \leq K$ to (5) amounts to minimizing f_M on the faces of dimension not greater than K of the standard simplex Δ. If we further add the condition that $\ell_i \leq x_i \leq u_i$ for $i = 1, \dots, n$, then we obtain a StQP with cardinality and upper and lower bound constraints which is equivalent to (3).

In the next Section we describe how to solve a StQP with cardinality and upper and lower bound constraints by adapting the algorithms developed by Scozzari and Tardella [30] for the unconstrained case.

4 THEORETICAL RESULTS AND SOLUTION METHOD

We consider the cardinality constrained StQP problem:

$$\begin{array}{ll} \text{min} & f(x) = x'Qx \\ st & \\ & x \in \Delta = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \dots, n\} \\ & |supp(x)| \leq K \end{array} \tag{6}$$

In order to restrict the search for its global minimizers, we use the following QP extension of the fundamental theorem of Linear Programming

Theorem 1 *[30, 34]. A quadratic function f that is bounded below on a (pointed) polyhedron P attains its minimum on P in the relative interior of a face of P where f is strictly convex.*

Let $N = \{1, \ldots, n\}$. Every face of Δ has the form $\Delta_I = \{x \in \Delta : \sum_{i \in I} x_i = 1\}$, where $I \subseteq N$ is a subset of indices. Furthermore, the dimension $\dim(\Delta_I)$ of Δ_I coincides with the cardinality $|I|$ of I. Let $\mathcal{I}_K$ denote the family of all subsets of N with cardinality at most K. Then the cardinality constrained StQP (6) can be reformulated as:

$$\min_{x \in \bigcup_{I \in \mathcal{I}_K} \Delta_I} f(x) = x'Qx \tag{7}$$

Hence we obtain the following straightforward consequence of Theorem 1:

Corollary 2 *At least one global minimizer of (6) must be in the relative interior of a face Δ_I of Δ where f is strictly convex and $|I| \leq K$.*

For $I \subseteq N$, let Q_I denote the submatrix of Q formed by those elements with row and column indices in I. When Q_I is positive definite, the unique global minimizer of the quadratic form $x'Q_I x$ on the hyperplane $\sum_{i \in I} x_i = 1$ is attained at the point $x_I^* = \frac{1}{2} Q_I^{-1} e$. Thus the quadratic form $f(x)$ has a global minimizer on Δ in the relative interior $rint(\Delta_I)$ of a face Δ_I where f is strictly convex only if $x_I^* \in rint(\Delta_I)$.

To every subset $I \subseteq N$ we associate the (nonlinear) weight

$$w(I) = \min\{f(x) : x \in \Delta_I\}.$$

Corollary 2 and simple matrix algebra imply that

$$\min_{x \in \bigcup_{I \in \mathcal{I}_K} \Delta_I} x'Qx = \min_{I \in \mathcal{C}_K} w(I) = \min_{I \in \mathcal{C}_K} f(x_I^*) = \min_{I \in \mathcal{C}_K} \frac{1}{4} e' Q_I^{-1} e, \tag{8}$$

where $\mathcal{C}_K$ is the subset of $\mathcal{I}_K$ defined by

$$\mathcal{C}_K = \{I \in \mathcal{I}_K : Q_I \text{ is positive definite and } x_I^* \in rint(\Delta_I)\}.$$

In view of (8), the cardinality constrained StQP could be solved by evaluating $\frac{1}{4} e' Q_I^{-1} e$ for all elements $I \in \mathcal{C}_K$, but this is clearly not practical for large values of n and K. However, another recent theoretical result can be used to restrict the search for a global minimizer:

Theorem 3 *[30]. If x^* is a global minimizer of a quadratic function f on a polyhedron P, then there exists a nested sequence of faces $F^1 \subset F^2 \subset \ldots \subset F^k$ of P, with dimension $dim(F^i) = i$, where f is strictly convex, has an interior global minimizer $\hat{x}_{F^i}$, and $x^* = \hat{x}_{F^k}$.*

For $j \leq K$, let $\overline{\mathcal{C}}_j = \{I \in \mathcal{C}_K : |I| = j\}$. Then the above theorem guarantees that for any I^* minimizing $w(I)$ on $\mathcal{C}_K$ there exists a sequence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_h = I^*$, such that $I_j \in \overline{\mathcal{C}}_j$ for all $j = 1, \ldots, h$. Thus we can apply the following algorithm to solve the cardinality constrained StQP (6) or, equivalently, to minimize $w(I)$ on $\mathcal{C}_K$:

Increasing Set Algorithm()

1 Set $\mathcal{C}_0 \leftarrow \emptyset,\ \mathcal{C}_1 \leftarrow \{\{i\}, i \in N\}$
2 $\mathrm{MIN}(1) \leftarrow \min_{I \in \mathcal{C}_1} w(I) = \min_{1 \le i \le n} q_{ii}$
3 **for** $j = 1$ to K
4 **do** construct $\overline{\mathcal{C}}_{j+1}$ by increasing, if possible, all elements in $\overline{\mathcal{C}}_j$
5 **if** $\overline{\mathcal{C}}_{j+1} = \emptyset$
6 **then** $\mathrm{MIN}(h) \leftarrow \mathrm{MIN}(j)$ for $h = j+1, \ldots, K$, **return** $(\mathrm{MIN}(K))$
7 **else** $\mathrm{MIN}(j+1) \leftarrow \min\{\mathrm{MIN}(j)\ ,\ \min_{I \in \overline{\mathcal{C}}_{j+1}} w(I)\}$
8 **return** $(\mathrm{MIN}(K))$

Note that, by Theorem 3, at any iteration j, $\mathrm{MIN}(j)$ contains the minimum value of $w(I)$ among all sets in $\mathcal{C}_j$. Furthermore, if $\overline{\mathcal{C}}_{j+1} = \emptyset$ in step 5, then, again by Theorem 3, $\overline{\mathcal{C}}_h$ must be empty for all $h \ge j+1$. Hence the algorithm correctly stops with the global minimizer in $\mathcal{C}_K$. In fact, at each iteration j the Increasing Set Algorithm provides in $\mathrm{MIN}(j)$ the solution to the StQP problem with cardinality constraint $|supp(x)| \le j$.

We have proved that the increasing set algorithm is exact. Unfortunately, it has exponential complexity in the worst case, and may be too slow in practice for large size problems. However, we obtain a very good heuristic if we bound at each iteration the size of $\overline{\mathcal{C}}_j$ by keeping only a limited number of sets I with the best values of $w(I)$. Form the theoretical viewpoint, we can achieve polynomial time complexity in this way, but of course we lose the guarantee of optimality. In practice, however, we have observed considerable reduction in the execution time without losing optimality in all real-world instances, described in Section 5, that have been solved with both our algorithm and with CPLEX.

We should point out that in order to apply our algorithm to the (reformulated) LAM model that also includes lower and upper bounds ℓ_i and u_i on the variables $x_i > 0$, we need to further modify the basic Increasing Set Algorithm described above. Indeed, to solve a StQP with cardinality and lower and upper bound constraints, we find the sets $\overline{\mathcal{C}}'_j$ and $\overline{\mathcal{C}}''_j$, where $\overline{\mathcal{C}}'_j = \{I \in \overline{\mathcal{C}}_j : \ell \le x^*_I \le u\}$ and $\overline{\mathcal{C}}''_j = \overline{\mathcal{C}}_j \setminus \overline{\mathcal{C}}'_j$. We replace $\min_{I \in \overline{\mathcal{C}}_{j+1}} w(I)$ in step 7 of the algorithm with $\min_{I \in \overline{\mathcal{C}}'_{j+1}} w(I)$, and we memorize the list of all sets I in $\mathcal{C}''_K = \bigcup_{j=1}^{K} \overline{\mathcal{C}}''_j$. At the end of the algorithm, we then replace $\mathrm{MIN}(K)$ with $\min\left\{\mathrm{MIN}(K), \min_{I \in \mathcal{C}''_K} w(I)\right\}$. This can be done efficiently by observing that, for all $I \in \mathcal{C}''_K$, $w(I)$ can be computed by solving a convex quadratic programming problem of dimension $|I|$, and that we only need to solve such problems for those $I \in \mathcal{C}''_K$ for which $f(x^*_I) < \mathrm{MIN}(K)$.

5 DATA SETS AND COMPUTATIONAL RESULTS

5.1 DATA SETS

An important issue for evaluating computational results for a class of problems is the availability of benchmark data sets, possibly with solutions, that can be used by researchers to compare the efficiency of their algorithms, and the quality of the solutions obtained in the case of heuristics. Unfortunately, in the case of the LAM model, such benchmark data sets are currently only partially available.

The most popular publicly available data sets based on real-world data for the LAM model seem to be the ones described by Chang, Meade, Beasley and Sharaiha in [8]. They, include covariance matrices and expected return vectors of sizes ranging from 31 to 225 built from weakly price data from March 1992 to September 1997 for the Hang Seng, DAX, FTSE 100, S&P 100, and Nikkei 225 capital market indices. The weakly price data are contained in the files indtrack1, indtrack2,..., indtrack5 available from Beasley's OR-Library [2] at http://people.brunel.ac.uk/ ~mastjjb/jeb/orlib/indtrackinfo.html, where one can also find weakly price data for the S&P 500 (457 assets), Russell 2000 (1318 assets), and Russell 3000 (2151 assets) capital market indices in the files indtrack6, indtrack7, indtrack8, as described in [7]. The covariance matrices and return vectors for the first five data sets are contained in the files port1, port2,..., port5 also available from Beasley's OR-Library at http://people.brunel.ac.uk/~mastjjb/jeb/orlib/ portinfo.html.

We believe that making these data sets available to the scientific community is a laudable service. It should be mentioned however that, for commercial reasons, the data sets have been anonymized, in the sense that the names of the stocks associated to the data are not disclosed. Thus we decided to construct, and to make available in the web page http://w3.uniroma1.it/ Tardella/homepage.html, five additional data sets that refer to the EuroStoxx50 (Europe), FTSE 100 (UK), MIBTEL (Italy), S&P 500 and NASDAQ (USA) capital market indices. These data sets contain the names of all the stocks included. For each stock we obtained 265 weakly price data, adjusted for dividends, from Yahoo Finance for the period from March 2003 to March 2008. Stocks with more than two consecutive missing values were disregarded. The missing values of the remaining stocks were interpolated. We thus obtained data sets of 48 stocks for EuroStoxx50, 79 for FTSE 100, 226 for MIBTEL, of 476 for S&P 500, and of n=2196 for NASDAQ. We then computed (logarithmic) weekly returns $\ln(P_t/P_{t-1})$, expected returns, and covariance matrices based on the (in-sample) data for the period March 2003-March 2007. The remaining data, for the period April 2007-March 2008, have been used as out-of-sample data to evaluate the performance of the portfolios obtained with our methods.

Another drawback of Beasley's data sets is the lack of optimal (or best known) solutions to the LAM model based on them, although some statistics are presented in [8, 10, 14, 26, 29]. We fill this gap by providing the optimal (or best known) solutions to the LAM model both for our data sets and for the ones contained in Beasley's OR-Library.

5.2 COMPUTATIONAL RESULTS

We now provide some computational results comparing our heuristic algorithm with the exact QMIP solver in CPLEX 11.0. We point out that although optimality is not guaranteed for our algorithm, we have observed that in all instances where CPLEX could solve the problem, the solutions found by the two algorithms coincided up to numerical precision. Hence we need not report the accuracy of the solutions found by our algorithm.

Our algorithm is coded in MATLAB 7.4 and executed on a workstation with Intel Core2 Duo CPU (T7500, 2.2 GHz, 4Gb RAM) under Windows Vista. CPLEX 11.0 is also called from MATLAB with the TOMLAB/CPLEX toolbox.

For each data set, we computed ρ_{min} and ρ_{max} as described in Section 2 by solving the classical (unconstrained) Markowitz model. We then repeatedly solved the LAM model (3) for 500 equally spaced returns between ρ_{min} and ρ_{max} thus obtaining 500 values of the function $\phi_K(\rho)$. A simple post-processing of these values allowed us to compute also $\phi'_K(\rho)$, and to determine the points on the Efficient Frontier of the LAM model (3), also called LAMEF. The graphs obtained for some data sets are shown in Figures 4.

As in [8, 10, 14, 20, 26, 29], we report results for problems with cardinality constraints of $K = 5$ and $K = 10$, lower bound $\ell_i = 0.01$, and upper bound $u_i = 1$ for all $i = 1, \ldots, n$.

The choice of $K = 10$ as the largest cardinality constraint is also justified by the observation that for several data sets the optimal portfolio in the classical Markowitz

		Number of assets (N)	$K = 5$		$K = 10$	
			CPLEX	INCR. SET	CPLEX	INCR. SET
OR-Library	Hang Seng	31	12	39	10	58
	DAX 100	85	906	363	135	811
	FTSE 100	89	3190	638	1414	2155
	S&P 100	98	12055	1363	39600	4844
	Nikkei	225	186	389	83	724
	S&P 500	457	-	2625	-	4127
	Russell 2000	1318	-	14819	-	15763
	Russell 3000	2151	-	47964	-	56499
	EuroStoxx50	48	20	93	14	191
	FTSE 100	79	231	210	65	483
	MIBTEL	226	-	4163	-	21450
	S&P 500	476	-	5292	-	32007
	NASDAQ	2196	-	738094	-	-

Table 1: Execution times in seconds to solve the LAM model for 500 return values with K assets

model does not include more than 10 stocks for more than half of the ρ values (see also Figure 5).

Furthermore, we observed that the number of stocks with positive weight in the optimal portfolio for the classical Markowitz model might be an important indicator of the practical computational complexity for most exact algorithms for the LAM model. This is certainly the case both for CPLEX and for our Increasing Set algorithm, as clearly results by comparing in Table 1 the computation time for S&P 100, (98 assets) with the one for Nikkei (225 assets). The computation for S&P 100 takes much longer because it has many more assets in the optimal portfolio for the classical Markowitz model, as shown in Figure 5.

We should make some important remarks concerning the execution times presented in Table 1. First, the Increasing Set algorithm is currently a prototype algorithm coded in

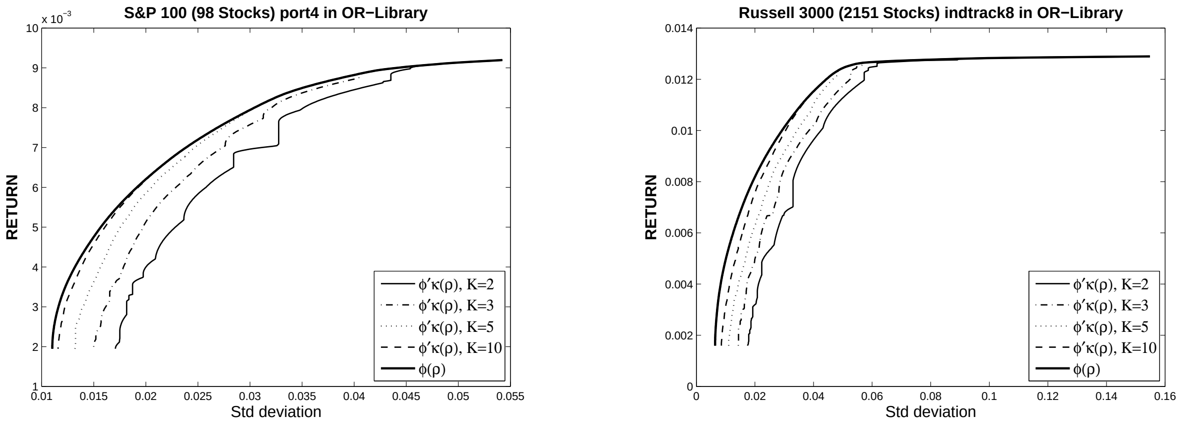

Figure 4: Examples of Efficient Frontiers of the LAM model

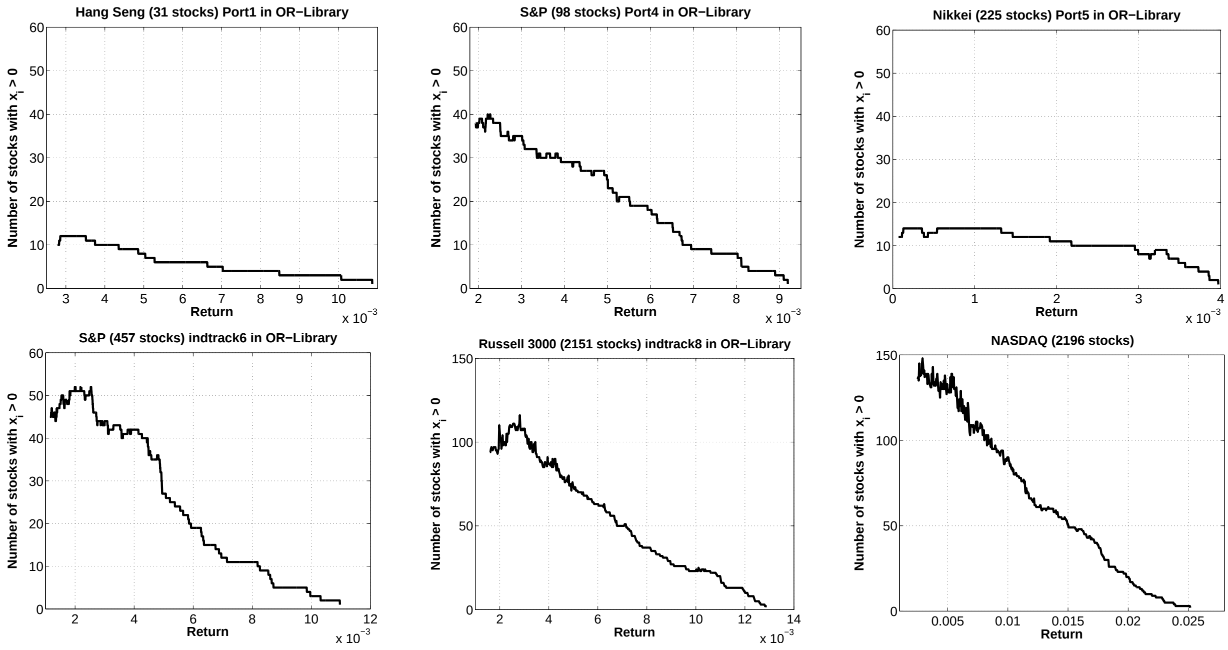

Figure 5: Number of assets in the unconstrained Mean-Variance optimal portfolio

MATLAB tailored for the LAM model, while the solver in CPLEX is a highly optimized general purpose QMIP solver. Furthermore, the times reported for the Increasing Set algorithm to solve the LAM model for a given K should be actually read as the times required to solve the model for all $K' \leq K$, as observed in Section 4. Thus such times are clearly increasing with K, but they refer to solving a family of problems. On the other hand, the execution times of CPLEX seem to almost always decrease with K. Hence, CPLEX might be used as a complementary tool with respect to the Increasing Set algorithm. However, it should be noted that CPLEX is currently unable to solve the largest problems in our data sets.

Since the optimal solutions to the LAM model for the five benchmark data sets from the OR-Libray described in [8] were previously unknown, some authors [10, 26, 29] have measured the quality of the results obtained by their heuristic algorithms by computing an *Average Percentage Loss* comparing the risk obtained by the algorithms for the LAM model with a given required return ρ to the optimal risk for the same return in the classical (unconstrained) Markowitz model. More precisely, with our notation, the Average Percentage Loss is obtained as

$$\mathrm{APL} = \frac{100}{|\mathrm{J}|} \sum_{\mathrm{j} \in \mathrm{J}} \frac{\phi_{\mathrm{K}}(\rho_{\mathrm{i}}) - \phi(\rho_{\mathrm{i}})}{\phi(\rho_{\mathrm{i}})}, \tag{9}$$

where the returns ρ_j, $j \in M = \{1, \ldots, 100\}$, are equally distributed in the interval $[\rho_{min}, \rho_{max}]$, $J = \{j \in M : \phi_K(\rho_j) = \phi'_K(\rho_j)\}$ is the set of indices of the points on the efficient frontier, $K = 10$, $\ell_i = 0.01$ and $u_i = 1$ for all i.

Since we could compute the exact values of $\phi_K(\rho_j)$ and $\phi'_K(\rho_j)$, the Average Percentage Loss that we obtain is the best possible (we call it Exact APL), and thus it is not greater than the one computed in [10, 26, 29] on the basis of heuristic algorithms. Hence, the difference between these two values of the APL give a measure of the quality of the heuristic algorithms proposed in [10, 26, 29], showing that in some case such algorithms have actually obtained the optimal solutions.

Data set	# of Assets	Exact APL	[10]	[26]	[29]
Hang Seng	31	0.00321	0.00321	0.00321	0.00344
DAX 100	85	2.47386	2.53139	2.53180	2.53845
FTSE 100	89	1.90233	1.92133	1.92150	1.92711
S&P 100	98	4.69339	4.69371	4.69507	4.69426
Nikkei	225	0.20197	0.20198	0.20197	0.20478

Table 2: Comparison of Average Percentage Loss

6 EVALUATION ON OUT-OF-SAMPLE DATA

It is interesting to evaluate and compare the performances of the optimal portfolios obtained from the classical Markowitz model (1), from the LAM model for different values of K, and of the official capital market index in the same period. We performed such evaluation using out-of-sample data for the EuroStoxx 50, FTSE 100, MIBTEL, S&P 500 and NASDAQ capital market indices. The results are illustrated in Figure 6.

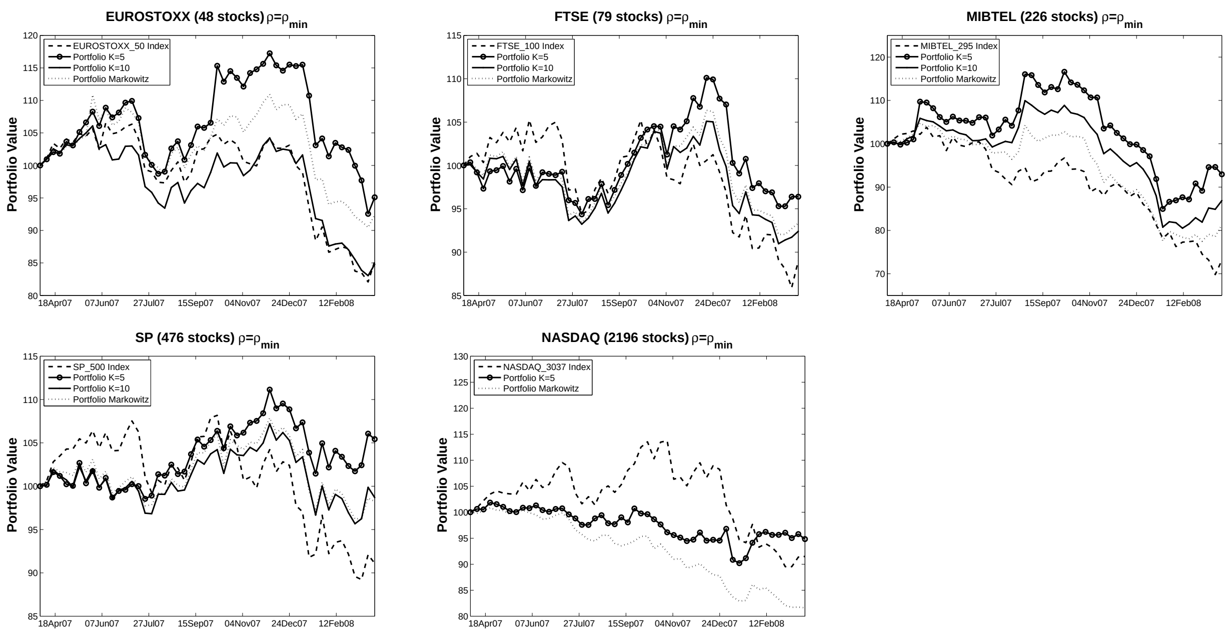

Figure 6: Evaluation on Out-of-Sample data

Although more thorough computational experience should be needed, the preliminary results seem to indicate that the portfolios obtained from the LAM model are almost always better than the ones provided by the classical Markowitz model. Furthermore, portfolios with less assets ($K = 5$) seem to be preferable to those with more assets ($K = 10$). This might be explained by observing that a strong limitation on the number of assets to hold in the optimal portfolio could provide more robust portfolios with respect to the estimation errors for returns and covariances.

7 CONCLUSIONS AND FURTHER RESEARCH

We have described an efficient algorithm for a Mean-Variance portfolio selection model that incorporates some mathematically hard constraints coming from real-world practice. Our algorithm is able to solve to optimality some previously unsolved benchmark problems, and can also solve problems with more than 2000 variables.

Our algorithm is based on a completely new approach that starts from a pair of assets and tries to add one asset at the time in an optimal manner by exploiting some recent theoretical results on Quadratic Programming.

In a forthcoming paper we will extend this approach to the problem of tracking a benchmark index by minimizing a quadratic (variance) error, possibly considering additional complex constraints.

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