PORTFOLIO SELECTION MODELS
FOR LIFE INSURANCE AND PENSION FUNDS

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ABSTRACT

This paper outlines the framework and techniques that can be used to determine portfolio selection or asset allocation strategies appropriate for life insurance and pension funds in a multi-period framework. The selection criteria suggested is that of maximising expected utility which contains as a special case the mean-variance criteria used in the existing actuarial literature on matching and portfolio selection. A relatively simple model is used to illustrate the concepts by formulating the asset allocation decision for a single period. The multi-period asset allocation strategy can be solved using stochastic dynamic programming and the techniques used for the single period. Constraints on holdings of assets and the probability of insolvency can be incorporated into the optimal asset allocation.

1. INTRODUCTION

The asset allocation strategy of life insurance and pension funds is a major concern of actuaries involved in the financial management of these funds. Variation in asset values is arguably the most significant component in the determination of the long term surplus of these funds and of the probability of insolvency (ruin). In the actuarial literature the concern with asset allocation strategies has primarily been with immunisation and matching strategies. Immunisation strategies were first discussed by Frank Redington (1952) and are designed to ensure that the change in the value of assets resulting from a change in interest rates is equal to or greater than the corresponding change in the value of the liabilities. The change in interest rates is assumed to occur instantaneously and to implement the strategy in practice requires a

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constant re-balancing of the asset portfolio as time passes and also as
interest rates change. Milgrom (1986), and the discussion of his paper,
provides a more detailed coverage of some of the issues in the theory of
immunisation.

Matching strategies have been considered in an excellent series of
These papers consider the asset allocation problem for a set of liability
cash flows in the presence of stochastic interest and inflation rates. The
best way to characterise the approach to the asset allocation problem
adopted in these papers is that they choose asset proportions, that are
constant throughout the time horizon of the liabilities being considered,
based on the mean and variance of ultimate (end of period) surplus as
well as the initial value of the assets available to meet the liabilities
(which they refer to as the price of the portfolio) and, where applicable,
subject to appropriate constraints (such as positive holdings of all as-
sets). The solution is characterised by efficient portfolios which, at least
for a fixed value of initial assets, maximise the expected value of ulti-
mate surplus for any given variance of ultimate surplus or minimise the
variance for any given expected value of ultimate surplus. The optimal
portfolio for any investor then depends on the preferences of the decision
maker in terms of their trade off between initial assets, expected value of
ultimate surplus and variance of ultimate surplus. One particular case
considered is that of an "unbiased" match in which the asset allocation
is selected to minimise the variance of ultimate surplus for an expected
value of zero.

The Wise-Wilkie papers appear to be multi-period in form. How-
ever this is only partly the case since their approach does not allow the
fund asset allocation to be revised as the actual rate of return on assets
is revealed over each sub-period. The asset allocation decision is treated
as a once and for all decision at the start of the period of the liability
cash flows and the decision is based on parameters of the distribution
of the ultimate surplus. The only factor that is allowed to vary over
sub-periods is the (stochastic) rate of return on the assets. Note that a
strategy involving the revision of the asset allocation, or re-balancing, is
required for the successful implementation of an immunisation strategy.
Such a strategy is also required to provide a match for liabilities con-
taining option features such as maturity guarantees. This feature of the
Wise-Wilkie papers and the extension to a more general multi-period
framework was discussed in Yang (1988).

Wilkie (1985) considers the issue of solvency or the probability of
ruin in terms of ultimate surplus as a constraint on the feasible asset allocations. If the ultimate surplus is negative then the fund is insolvent. A desirable constraint on the solution to the asset allocation problem is that this probability is less than or equal to a specified low probability. This issue is important given the interest that actuaries have in the solvency of the funds that they advise. Such an approach was adopted by McCabe and Witt (1980) to examine a range of issues related to pricing and regulation of a non-life insurer. Neither the Wise-Wilkie models nor the McCabe-Witt model considers the multi-period nature of the insolvency problem.

Against this background, this paper aims to outline and discuss techniques that can be used in the determination of the asset allocation strategy for life insurance and pension funds in a general framework which can readily:

(a) incorporate a general criteria for optimality (in terms of long term surplus);
(b) allow for portfolio revisions (a multi-period framework);
(c) incorporate stochastic liabilities; and,
(d) incorporate the probability of insolvency (ruin).

Such a general framework for addressing the asset allocation problem has been extensively used in the finance and economics literature to examine the optimal consumption strategy of an individual and in the development of general asset pricing models. Papers which consider the discrete time framework include Mossin (1968), Samuelson (1969) and Breeden (1987). The continuous time framework is considered in Merton (1969, 1971) and Richard (1975). Chapter 17 of Jarrow (1988) covers the use of these techniques in deriving multi-period asset pricing models.

The techniques covered in this paper should provide a useful basis for assessing asset allocation decisions in practice. They could be used as a model to assist decision makers in optimising the performance of the funds for which they are responsible. If decision makers in real life do not formally adopt the approach used in this paper then a possibility is that asset allocation decisions are made 'as if' they did. If this was worth investigating then the techniques could also be used to establish results or predictions about asset allocation decisions. These results or predictions could then be tested against decisions that are actually made in practice to see if they have validity. This latter motivation is more in the positive tradition of determining the usefulness of a model by its
predictions. This latter course is not pursued in this paper but is left for future research.

2. BACKGROUND AND ASSUMPTIONS

The fund is assumed to set its investment allocation strategy at discrete points of time during a specified investment decision horizon. The investment decision horizon will depend on many factors and will to a large extent depend on the type of fund and the amount of uncertainty in liability cash flows. At the end of the investment decision horizon an allowance for future periods can be incorporated in a “terminal” function which implicitly allows for periods beyond the time horizon of the model. The multi-period asset allocation problem can be solved using stochastic dynamic programming by commencing in the final period of the investment horizon and solving what is effectively a single period asset allocation. Given the optimal allocation for the final period it is possible to then work backwards in time to the current period to determine the asset allocation for the period of most initial concern. By working backwards from the horizon date the optimal asset allocation strategy will also take into account future periods. Since the techniques used to determine the asset allocation strategy for the final period will be the same as that for a single period model this situation will be considered first.

The simple model outlined in this paper will be a discrete time model for a number of reasons. In practice most asset allocation decisions will be made at discrete points of time as will tests of insolvency. Such models are also more amenable to computer applications which will be necessary to derive practical results for use with actual funds. In theory, the income and outgo of life insurance and pension funds could be modelled as continuous flows in the way that Cummins (1988) models insurer assets and liabilities. For practical applications it will be easier to assume these cash flows occur at discrete points in time.

Consider the final period of a multi-period model. Let the stochastic liability payment to be paid by the insurance or pension fund at the end of the period be denoted by \( L \). Because the nature of the liabilities differ between life insurance and pension funds the stochastic form of \( L \) will differ accordingly. In practice the distribution of \( L \) could be estimated using simulation or approximated by a simple distribution such as the log-normal or normal distribution.

The initial or start of period value of assets available to meet the
liability will be denoted by $A$ and will be assumed to consist of a fixed component of $C$ and a variable component of $K$. In the life insurance fund $C$ could be regarded as the expected or market value of the liability. In such a fund $K$ could be regarded as a provision for adverse deviations which is provided as risk capital or equity. For a pension fund, $C$ could be regarded as the expected value of claim payments under the rules of the fund and $K$ the margin added to this by the actuary in establishing the contribution to the fund either explicitly or, as is more often the case, implicitly by using conservative assumptions. The total of $C$ and $K$ will be taken as the value of liabilities determined by the actuary (or a regulator) for which assets of at least this value must be held by the fund at the start of each period. It will be assumed that $C$ is fixed and independent of the asset allocation choice. The full amount of $K$ is available to meet the payment of the liability of $L$. The initial amount of assets can therefore be written as

$$A = (1 + p) \times C$$

where $p = K/C$ and can be interpreted as the solvency margin. If actual assets are greater than this figure then there is an initial surplus in the fund at the start of the period. For the fund to continue the value of $A$ must be at least this figure otherwise the fund ceases and the assets are distributed since the fund will be considered to be insolvent.

For simplicity it will be assumed that there are two assets available which for a dollar invested at the beginning of the period provide a cash flow at the end of the period of $R_1$ for asset one and $R_2$ for asset two. $R_i$ is therefore one plus the rate of return on asset $i$. For convenience assume that asset 1 provides a certain return and corresponds to a “risk-free” asset and asset 2 provides a stochastic return and corresponds to a “risky” asset. Let $x_i$ be the proportion of the assets which are invested in asset $i$. These $x_i$ will be the main decision variables and represent the asset allocation for the fund. There is no loss of generality in considering only two assets since the extension to multiple assets is straightforward.

The surplus at the end of the period will be a random variable which will depend on the random amount of the liability cash flow and the random rate of return on the risky assets as well as the (endogenous) decision variables $x_1$ and $x_2$. The end of period surplus is given by

$$S_1 = A(x_1 R_1 + x_2 R_2) - L.$$

which, assuming that the surplus at the start of the period $S_0$ is zero,
can be rewritten as

\[ S_1 = C(1 + p) \left[ x_1 R_1 + x_2 R_2 - (1/1 + p)R_L \right] \]

where \( R_L = L/C \) can be considered as one plus the (random) growth rate of the liability from its expected value of \( C \) or the equivalent of one plus the rate of return on the liabilities.

To incorporate the probability of insolvency into the model this surplus value is considered under two “states” of the fund. If the fund is solvent then \( S_1 \geq 0 \). In this event the providers of the risk capital \( K \) will be entitled to receive \( S_1 \) and the claimants will be paid \( L \). In the case where the fund is insolvent then \( S_1 < 0 \). In this event the providers of the risk capital will be assumed to be protected by some form of legal limit to their liability for benefits when the fund is insolvent and will receive (and contribute) nothing at the end of the period so that the claimants will receive \( L + S_1 < L \). Note that the assumed limit to the liability for benefits when the fund is insolvent of the providers of the risk capital results in the claimants payments at the end of the period having an option pay-off structure in the form of minimum \((L, L + S_1)\). In the event of insolvency the fund is assumed to cease to exist and all assets are distributed.

The effect of any legal limit to the liability for benefits when the fund is insolvent is that the value of \( C \) implicitly depends on the asset allocation strategy adopted since this represents the expected value of the claimants payments. As \( C \) will be assumed to be fixed it will be necessary to incorporate rational expectations into the model. This means that \( C \) is assumed to be set based on knowledge of the optimal asset allocation and it will be assumed that there is no moral hazard arising from decision makers acting against the interests of the claimants beyond that already allowed for in the value of \( C \). This could occur by the adoption of a more “risky” asset allocation strategy than that implicit in \( C \) in which event the claimants will suffer the down-side risk of such a strategy but not benefit from the additional surplus on the upside. Such actions could be limited by incorporating a constraint on probability of insolvency on the feasible asset allocation strategies. In practice this might be done through regulatory constraint or through contractual means.

The setting of the asset allocation strategy is assumed to be made by a group of decision makers such as the Board and/or Investment Committee of a life insurance fund or the Trustees of a pension fund.
The decision makers are assumed to have established an appropriate preference or risk function which not only ranks the possible end of period surplus amounts in order from most preferred to least preferred but places a relative cardinal weighting on the possible values of surplus. This function will be denoted by $U(S)$. This function is better known as a utility function and the selection of an optimal asset allocation will be based on the assumption that the decision makers maximise expected utility. More details on the axioms and assumptions required for the utility function can be found in Chapter 7 of Jarrow (1988). In the multi-period case this utility function also provides a value for surplus received at different points in time.

Utility functions characterise the risk aversion or risk tolerance of the decision makers. Pratt (1964) derives measures of risk aversion based on utility functions. These are the absolute risk aversion and the relative or proportional risk aversion defined as

Absolute Risk Aversion $= -\frac{U''(S)}{U'(S)}$

Relative Risk Aversion $= -\frac{SU''(S)}{U'(S)}$

Risk tolerance is defined as the inverse of absolute risk aversion. An important class of utility functions is the Hyperbolic Absolute Risk Aversion (HARA) class which has a linear risk tolerance function so that

$$-\frac{U'(S)}{U''(S)} = a + bS.$$  

The HARA class contains the following:

- Exponential $U(S) = -e^{-S/a}$ for $b = 0$
- Logarithmic $U(S) = \log(S + a)$ for $b = 1$
- Power $U(S) = (b - 1)^{-1}(a + bS)^{(1-1/b)}$ otherwise.

Note that if $b = -1$ then the quadratic utility function is obtained and that for $b = 0$ the utility function exhibits constant absolute risk aversion.

For $a = 0$ the utility function exhibits constant relative risk aversion and the HARA class reduces to:

$$U(S) = \log(S) \text{ for } b = 1 \text{ and }$$

$$U(S) = S^{(1-1/b)} \text{ otherwise.}$$
The use of the utility function is motivated by the fact that such an approach is a fundamental paradigm of economics and of finance theory and also because it provides a rational basis for decision making. Without a utility function (or an equivalent risk function) decisions can only be made using essentially heuristic principles which, as shown by Tversky and Kahneman (1974), can lead to severe and systematic errors. The utility function concept has found its way into the actuarial literature in recent years. Lipman (1990) discusses the use of utility functions in investment decision making. Clarkson (1989, 1990) discusses a risk measure which is for all practical purposes the same as that derived using the notion of expected utility.

3. THE OPTIMAL ASSET ALLOCATION STRATEGY FOR A SINGLE PERIOD

The general multi-period asset allocation problem requires the selection of an asset allocation strategy for each of a number of sub-periods. At the end of each sub-period the asset allocation strategy for the next sub-period, and each future sub-period, is revised by taking into account the actual realisation of the stochastic variables - the rate of return and the liability payments. As already noted, the techniques used to determine the optimal allocation over any sub-period in the multi-period case are similar to those used in a single period model. In the single period case the optimal asset allocation strategy is determined by solving an optimisation problem. Typically the constraints will involve inequalities and Kuhn-Tucker conditions will be used to determine the solution. Lambert (1985) provides details on optimisation for the problems considered in this paper. In general the optimisation problem will be of the form:

maximise $f(x)$ where $x$ is a vector of $n$ decision variables $\{x_i = 1, n\}$

subject to

$x_k \geq 0$ for certain $k$

$g_j(x) \leq b_j$ for $j = 1, 2, \ldots, m$.

The solution to this problem is given by the Kuhn-Tucker conditions which are determined by first forming the Lagrangean

$$L = f(x) - \lambda^T [g(x) - b]$$
where \( \lambda \) is a vector of Lagrange multipliers for each of the \( m \) constraints. The unique maximum is then given by (Lambert, 1985, p128),

\[
\frac{\partial L}{\partial x_i} = 0 \quad \text{for } i \neq k \text{ (i.e. no positive constraint on } x_i) \]

\( x_k \geq 0 \quad \frac{\partial L}{\partial x_k} \leq 0 \quad \text{and} \quad x_k \cdot \left( \frac{\partial L}{\partial x_k} \right) = 0 \quad \text{for all } k \)

\( \lambda_j \geq 0 \quad \frac{\partial L}{\partial \lambda_j} \geq 0 \quad \text{and} \quad \lambda_j \cdot \left( \frac{\partial L}{\partial \lambda_j} \right) = 0 \quad \text{for } j = 1, 2, \ldots, m. \)

The asset allocation is assumed to be determined by selecting that asset allocation (and possibly the solvency margin at the start of the period) which maximises the expected utility of the end of period surplus subject to any constraints which may be imposed. These constraints would include any requirement for all asset holdings to be positive so that short selling (and hence borrowing by disallowing the short selling of any risk free asset) would be excluded and any requirement that the probability that the surplus be negative (the probability of insolvency) be less than or equal to a pre-specified figure.

The problem to be solved for the single period case can be written as:

maximise \( E[U(S_1)] \)

\( \{p, x_1, x_2\} \) subject to

\[ x_1 + x_2 = 1 \quad \text{(budget constraint)} \]

\[ F_*(0) \leq q \quad \text{(probability of insolvency constraint)} \]

\[ p \geq 0 \quad \text{(solvent at commencement of period)} \]

where \( F_*(\cdot) \) is the cumulative distribution function of \( S_1 \). For a “positive” asset allocation the additional constraints

\[ x_i \geq 0 \quad \text{would be required.} \]

This formulation of the problem assumes that negative values of ultimate surplus are given weight in the asset allocation problem. If the providers of risk capital set the asset allocation strategy then they would maximise the expected utility allowing for any legal limit to their liability for benefits in which case the objective function would be

maximise \( E[U(\max\{S_1, 0\})] \)
In order to ensure a unique maximum it is necessary to impose some conditions on the form of $E[U(.)]$. In particular it is necessary to assume it is at least monotonic, twice differentiable and quasi-concave.

In order to obtain analytical closed form solutions to the asset allocation problem it is necessary to place some structure on the probability distribution of $S$ and on the form of the utility function $U(.)$. It is the distribution of the surplus, which is a linear combination of the underlying random components in the model, which is of most interest. If $S$ is assumed to have a normal distribution this would be consistent with rates of return on assets and the rate of growth of the liability being jointly normal. An alternative would be to assume that $S$ had a log-normal distribution. This would preclude negative values for $S$ which would mean that the probability of insolvency was assumed to be zero in the model. This latter assumption is unlikely to acceptable for practical applications.

The assumption adopted for $U(.)$ will be tied to that adopted for the distribution of $S$ if analytical results are to be derived. If $U(.)$ is assumed to exhibit constant absolute risk aversion (an exponential utility function) then assuming $S$ to have a normal distribution makes the problem tractable. On the other hand if $U(.)$ is assumed to exhibit constant relative risk aversion (a log utility function) then assuming $S$ to have a log-normal distribution makes the problem tractable. Quadratic utility is a special case. This assumption leads to the mean-variance criteria adopted in the Wise-Wilkie papers. However it has some undesirable features. The requirements that are necessary for a maximum (monotonicity ($U'(\cdot) > 0$) and quasi-concavity ($U''(\cdot) \leq 0$)) are inconsistent for large enough $S$. The same results can be obtained by assuming that $S$ has a normal distribution in which case only the mean and variance of surplus will count in the expected utility.

One particular case is of interest for the multi-period problem. This is the case where the optimal asset allocation does not vary through time as the realised surplus changes. This case would correspond to the Wise-Wilkie situation where there is no explicit revision of the asset proportions over time. Constant proportional risk aversion (the log utility function) is known to result in such an optimal asset allocation provided the random components in the model have stationary probability distributions. This is not considered further since the aim of this paper is not to give analytical solutions in particular cases.
The problem

\[
\text{maximise } E[U(S_1)] \\
\{p, x_1, x_2\}
\]
subject to

\[
x_1 + x_2 = 1 \quad \text{(budget constraint)}
\]
\[
F_s(0) \leq q \quad \text{(probability of insolvency constraint)}
\]
\[
p \geq 0.
\]

can be solved by substituting the budget constraint into \( S_1 \) and forming the Lagrangean

\[
L = E \left[ U \left( C(1 + p)(R_1 + x_2(R_2 - R_1)) - L \right) \right] - \lambda \left[ F_s(0) - q \right].
\]

Assuming that short-selling is possible and there are no non-negative constraints on \( x_1 \) and \( x_2 \) then the first order conditions which give the maximum expected utility are given by the Kuhn-Tucker conditions

\[
\frac{\partial L}{\partial x_2} = 0
\]
\[
p \geq 0 \quad \frac{\partial L}{\partial p} \leq 0 \quad \text{and} \quad p \cdot \left( \frac{\partial L}{\partial p} \right) = 0
\]
\[
\lambda \geq 0 \quad \frac{\partial L}{\partial \lambda} \leq 0 \quad \text{and} \quad \lambda \cdot \left( \frac{\partial L}{\partial \lambda} \right) = 0
\]

Allowance for non-negative constraints on \( x_1 \) and \( x_2 \) in the case of no short selling or borrowing can readily be incorporated.

Subject to any non-negative constraints, the optimal values of \( x_2 \) and \( p \) are determined so that any marginal gain in expected utility from a marginal variation of either of these decision variables is offset by the resulting effect of this marginal change on the probability of insolvency multiplied by the Lagrange multiplier (\( \lambda \)). The probability of insolvency constraint may or may not be binding. If \( \lambda = 0 \) then the constraint is not binding and the optimal result is identical to the unconstrained problem. If \( \lambda > 0 \) then the constraint is binding. In this situation "risky" asset allocation strategies will be constrained.
4. Optimal Multi-Period Asset Allocation Strategies

The multi-period case is often analysed in continuous time. Continuous time models are of interest since they can provide analytical solutions to the problem for relatively simplistic assumptions. Typically assets and liabilities are modelled using geometric Brownian motion (diffusion processes) and using a constant relative risk aversion utility function. Beyond such simplified cases the solution to the multi-period problem is quite difficult from an analytical point of view.

The analysis of multi-period models in the finance literature suggests that single period asset allocation strategies are not always optimal in the multi-period case. This means that a strategy of investing funds one period at a time without taking into account future periods is not a desirable strategy. Investors need to adjust the single period asset allocation to hedge changes in investment returns (and liability payments) over time. The adjustment required depends on the risk aversion (or risk tolerance) of the fund as compared to the constant proportional risk averse fund.

Discrete time techniques correspond more closely to the practical situation. In the discrete case it is possible to consider a range of numerical techniques to make the selection of the optimal asset allocation easier. Optimisation can use stochastic dynamic programming or the technique originally proposed by Sharpe and used in the multi-period case by Breeden (1987). The discrete time case also allows a more general approach by avoiding the assumptions implied by the use of diffusion processes in the continuous time case.

The problem to be solved is for the decision makers to select an asset allocation for each time period which takes into account the effect of this decision on current end of period surplus as well as on future surplus values. It is assumed that decision makers choose, at the current point of time, optimal $x_t$ and $p_t$ for each future time period $t = 0, 1, \ldots, T-1$ to the decision horizon at time $T$. The optimality criteria is assumed to be the maximisation of multi-period expected utility. To simplify the optimisation it is usual to assume that utility is additive and separable across time periods in which case the asset allocation is chosen to

$$\text{maximise } E \left[ \sum_{t=0}^{T-1} e^{-\delta t} U(S_t, t) + e^{-\delta T} B(S_T, T) \right].$$

The factor $e^{-\delta t}$ allows for a time preference for the emergence of surplus.
$S_t$ is the amount of risk capital that is contributed to or taken from the fund by the decision makers. At the start of the decision period it will equal $K_0$ since this is the amount that could be withdrawn if the fund were terminated. The function $B$ is a function to allow for utility of surplus beyond the decision horizon of $T$ periods. The value for $S_T$ will include $K_T$ since claimants will only be entitled to the expected value of the liability payments of $C_T$ at the end of the decision horizon. Practical application of the model will require a judicious selection of the time period $T$ and the terminal utility function $B$.

This problem is maximised subject to constraints which would include any requirement for positive holdings of assets in each period as well as any constraint on the probability of insolvency. The insolvency constraint will be applied to each period since it will be assumed that the fund could terminate at the end of any period if the fund is insolvent. Any long term solvency requirement will need to be converted into a per period solvency probability. Once the optimal solution has been derived the probability of insolvency over the complete horizon can be checked for reasonableness.

At the start of each time period the fund is either solvent or insolvent. In the case that the fund is solvent the decision makers must select the asset allocation strategy. Let $L_t$ be the claim or liability cash flow at the end of period $t$ and $C_t$ be the expected value of all liability cash flows occurring after time $t$. Let $P_t$ be the premium cash flow at the start of period $t$. The values of $L_t$, $P_t$ and $C_t$ are intimately linked. In practice the model would need to allow these to be determined consistently and as an endogenous part of the model. For simplicity in exposition of the underlying optimisation techniques it will be assumed that $L_t$ is random and that $P_t$ and $C_t$ are determined exogenously but are consistent with the nature of $L_t$.

At the start of the period the assets must equal the liabilities as determined by the actuary of an amount of $C_t + K_t = C_t(l + p_t)$ so that the surplus of the fund sponsors is $S_t = A_t - (C_t, K_t)$. The fund is invested in the safe and risky asset in proportions $x_{1t}$ and $x_{2t}$ respectively so that total assets at the end of the period, after meeting the liability cash flow then due, are

$$A_{t+1} = \left[ C_t (1 + p_t) + P_t \right] \left[ x_{1t} R_{1t} + x_{2t} R_{2t} \right] - L_t.$$

To determine if the fund is solvent at the end of the period the value of the assets $A_{t+1}$ is compared with $C_{t+1}$. If assets are less than this
expected value of the liability payments then the fund can be considered insolvent and claimants would be paid out the value of the assets. If the value of assets $A_{t+1}$ exceeds the expected value of $C_{t+1}$ then the fund can be assumed to continue as above. Because of the potential limit of liability for benefit payments that arises from the possibility of insolvency the actual surplus amount at any time will be

$$S_t = A_t - (C_t + K_t) \quad \text{if} \quad A_t > C_t$$

$$= 0 \quad \text{otherwise}$$

so that the providers of risk capital can be assumed to be always willing to invest risk capital, if necessary, as long as the value of existing assets exceed the expected value of future liability cash flows.

The general form of the multi-period asset allocation problem can be written as

$$\begin{align*}
\text{maximise} \quad & \sum_{t=0}^{T} f_t(S_t, p_t, x_{it}) \\
\{p_t, x_{it}\} \\
\text{subject to} \\
S_{t+1} &= g_t(S_t, p_t, x_{it}) \\
\text{probability} \quad (S_{t+1} \leq K_{t+1}|S_t) &\leq q_t \quad \text{(insolvency)} \\
p_t, x_{it} &\geq 0 \quad \text{(non-negative conditions for control variables)}.
\end{align*}$$

The solution to this problem can be determined using Bellman's principle of optimality. Let $V(S_t, t)$ be the maximum of the objective function from time $t$ to the end of the horizon $T$ given an initial value for $S_t$. This function, $V(\cdot, \cdot)$, is referred to as the derived utility function. The total objective function can then be split into that applying over the next time interval $t$ to $t+1$ and that applying thereafter, or

$$V(S_t, t) = \max_{\{p_t, x_{it}\}} \left[ f_t(S_t, p_t, x_{it}) + V(S_{t+1}, t+1) \right]$$

For $t = T - 1$ there is only one period to maximise over so this is solved as a single period problem based on the values at time $T$ which are chosen to maximise the terminal utility function $B(\cdot)$. Having solved this problem the optimal values in each period can then be solved sequentially using backward induction (dynamic programming) until the entire optimal asset allocation solution path is determined. The model in this paper incorporates a very simple asset model in order to confine
attention to the optimisation techniques. Assets are in effect assumed to be realised at the end of each period and rates of return over each period are holding period returns including interest (dividends) and capital gains/losses. Such an approach is more suitable for equity type assets which do not have fixed cash flows and maturity dates. To handle fixed interest securities the asset model could incorporate the coupon cash flows and the maturity payments of different fixed interest investments. In order to allow for the re-balancing of the portfolio from one time period to another it would then be necessary to model the term structure of interest rates as part of the model. Other factors such as tax could also be incorporated into the model.

5. CONCLUSIONS

This paper has outlined techniques that can be used in practice to determine multi-period asset allocation strategies for life insurance and pension funds. The techniques and models are suitable for computer based applications particularly in the discrete time case. Existing models tend to be based on a single period decision horizon. Setting asset allocation strategies by considering one period at a time is not an optimal strategy. Weight must be given to future periods and the dynamic nature of asset returns and liability values. It is very likely that the optimisation techniques covered in this paper will be increasingly used in practice as asset/liability modelling and computer technology make such techniques more viable.

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