DISTRIBUTION OF THE PRESENT VALUE 
OF FUTURE CASH FLOWS

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Abstract

The present value of future cash flows with random interest rates is studied. The first three moments of the present value of future cash flows and an approximation of its cumulative distribution function are presented. Particular results for three stochastic processes that may be used to model the force of interest are presented. The three processes considered are the White Noise, Wiener and Ornstein-Uhlenbeck processes. A n-year certain annuity-immediate is used to illustrate the results.

Key Words: Cash Flows, Force of Interest, Present value, White Noise process, Wiener process, Ornstein-Uhlenbeck process, Annuity-immediate.

1. Introduction

In actuarial science and finance, the present value of future cash flows is obtained by discounting the cash flows by appropriate discount factors. When considering future interest rates that vary in a stochastic fashion, the present value of future cash flows becomes a random variable.

Stochastic interest rates and application has been the subject of many publications. Some papers are mainly concerned with the moments of the present value of certain or contingent cash flows. Examples are Boyle (1976), Panjer and Bellhouse (1980), Waters (1978), Wilkie (1976), Giacotto (1986), Dhaene (1989), Beekman and Fuelling (1990), Parker (1992b)).

Other papers using stochastic models for the interest rates present applications to investment or immunization theory (see, for example, Boyle (1980), Wilkie (1987), Beekman and Shiu (1988)).

Recently, Dufresne (1990), Frees (1990), Parker (1992c) studied the density function or the cumulative distribution function of present val-
The aim of this paper is to present a general approach for finding the moments of the present value of cash flows and to suggest a method for approximating its distribution.

The layout of the paper is the following. Section 2 gives a probabilistic definition of $P$, the present value of future cash flows. Section 3 describes how one can obtain the first three moments about the origin of $P$. These moments are later used to find the standard deviation and the coefficient of skewness of $P$. Three stochastic processes for the force of interest are studied in section 4. They are the White Noise process, the Wiener process and the Ornstein-Uhlenbeck process. Section 5 suggests a method for approximating the cumulative distribution function of $P$. Illustrations for $n$-year certain annuity-immediate are presented in section 6.

2. Present Value of Future Cash Flows

Consider a stream of future cash flows of deterministic amounts $CF_i, i = 1, 2, \ldots, n$ where the $i$th cash flow, $CF_i$, is payable at time $t_i$. Let $P$ be the present value of these cash flows. Then $P$ may be defined as:

$$ P = \sum_{i=1}^{n} CF_i \cdot \exp \{-y(t_i)\} $$

where

$$ y(t) = \int_{0}^{t} \delta_s \cdot ds $$

and $\delta_s$ is the force of interest at time $s$. So, if the force of interest varies stochastically, $P$ is a random variable.

We will assume that the same realization of future forces of interest applies to any common period of time involved in the discounting of the cash flows. In other words, the value that the force of interest will actually take at a future time $s$ will be used (for time $s$) to discount any cash flow payable at or after time $s$. 

3. Moments of $P$

The expected value of $P$ is simply the sum of the expected values of the cash flows. That is:

$$[3] \quad E[P] = E \left[ \sum_{i=1}^{n} CF_i \cdot \exp \left\{ - y(t_i) \right\} \right] = \sum_{i=1}^{n} CF_i \cdot E \left[ \exp \left\{ - y(t_i) \right\} \right].$$

The second moment about the origin of $P$ is given by:

$$[4] \quad E[P^2] = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} CF_i \cdot CF_j \cdot \exp \left\{ y(t_i) - y(t_j) \right\} \right].$$

$$[5] \quad E[P^2] = \sum_{i=1}^{n} \sum_{j=1}^{n} CF_i \cdot CF_j \cdot E \left[ \exp \left\{ - y(t_i) - y(t_j) \right\} \right].$$

The third moment about the origin of $P$ is given by:

$$[6] \quad E[P^3] = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} CF_i \cdot CF_j \cdot CF_k \cdot \exp \left\{ - y(t_i) - y(t_j) - y(t_k) \right\} \right].$$

$$[7] \quad E[P^3] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} CF_i \cdot CF_j \cdot CF_k \cdot E \left[ \exp \left\{ - y(t_i) - y(t_j) - y(t_k) \right\} \right].$$

Parker (1992b, sections 4 and 5) suggests using recursive equations for the numerical evaluation of equations similar to [5] and [7]. These recursive equations are easily adapted for finding the second and third moments of $P$ and such recursive equations were used to obtain the illustrated results presented in section 6 of this paper.

The expected values involved in equations [3], [5] and [7] are still to be determined. If the forces of interest are gaussian, then the function $y(t)$ is normally distributed. Consequently, expressions of the form $\exp \left\{ - y(r) - y(s) - y(t) \right\}$ are lognormally distributed, i.e.

$$[8] \quad \exp \left\{ - y(r) - y(s) - y(t) \right\} \sim \Lambda(\mu, \beta)$$

with parameters

$$[9] \quad \mu = -E[y(r)] - E[y(s)] - E[y(t)]$$
and

\[
\beta = V[y(r)] + V[y(s)] + V[y(t)] + 2\text{cov}[y(r), y(s)] \\
+ 2\text{cov}[y(r), y(t)] + 2\text{cov}[y(s), y(t)].
\]

The expected value of the lognormal variable in [8] is then:

\[E\left\{\exp\{-y(r) - y(s) - y(t)\}\right\} = \exp\{\mu + 0.5 \cdot \beta\}\]

(see, for example, Aitchison and Brown (1963, p.8)).

Finally, [11] can be used, with the fact that by definition \(y(0) = 0\), to obtain the needed expected values in [3], [5] and [7]. Note that the parameters \(\mu\) and \(\beta\) depend on the particular gaussian stochastic process used to model the force of interest.

In the following section we present three gaussian processes that may be used to model the force of interest and derive the specific results needed for finding \(\mu\) and \(\beta\) for each model.

4. FORCE OF INTEREST

4.1. WHITE NOISE PROCESS

The first model to be considered is the White Noise process. Let the force of interest be defined as:

\[\delta_t = \delta + \sigma \cdot dW_t \quad \sigma \geq 0\]

where \(W_t\) is a standard Wiener process.

The forces of interest are therefore gaussian and i.i.d. The function \(y(t)\) is then a Wiener process with drift \(\delta\). Its expected value is:

\[E[y(t)] = \delta \cdot t\]

and its autocovariance function is:

\[\text{cov}[y(s), y(t)] = \sigma^2 \cdot \min(s, t)\]

(see, for example, Arnold (1974, section 3.2)).
4.2. **Wiener Process**

The second model considered for the force of interest is the Wiener process. Let the force of interest be defined as:

\[ d\delta_t = \sigma \cdot dW_t \quad \sigma \geq 0. \quad [15] \]

From section 8.2 of Arnold (1974) we know that conditional on \( \delta_0 \) being the known current value of the process, the mean and autocovariance function of this process are:

\[ E[\delta_t] = \delta_0 \quad [16] \]

and

\[ \text{Cov} [\delta_s, \delta_t] = \sigma^2 \cdot \min(s, t). \quad [17] \]

Then, from the definition of \( y(t) \), it follows that \( y(t) \) is normally distributed with mean:

\[ E[y(t)] = \delta_0 \cdot t \quad [18] \]

and autocovariance function:

\[ \text{Cov} [y(s), y(t)] = \int_0^s \int_0^t \text{Cov}[\delta_u, \delta_v] \, du \, dv \quad [19] \]

\[ \text{Cov} [y(s), y(t)] = \sigma^2 \cdot (s^2t/2 - s^3/6) \quad s \leq t. \quad [20] \]

4.3. **Ornstein-Uhlenbeck Process**

The third model that we will consider for the force of interest is an Ornstein-Uhlenbeck process. Let the force of interest be defined such that:

\[ d\delta_t = -\alpha(\delta_t - \delta) \cdot dt + \sigma \cdot dW_t \quad \alpha \geq 0, \quad \sigma \geq 0 \quad [21] \]

with initial value \( \delta_0 \) (see, for example, Arnold (1974, p.134)).
Then, it can be shown that:

\[ E[\delta_t] = \delta + e^{-\alpha t} \cdot (\delta_0 - \delta) \]
\[ \text{cov}[\delta_s, \delta_t] = e^{-\alpha(s+t)} \cdot \frac{\sigma^2}{2\alpha} \cdot (e^{2\alpha s} - 1) \quad s \leq t \]

and that the function \( y(t) \) is a gaussian process with mean:

\[ E[y(t)] = \delta \cdot t + (\delta_0 - \delta) \cdot \left( \frac{1-e^{-\alpha t}}{\alpha} \right) \]

and autocovariance function:

\[ \text{cov}[y(s), y(t)] = \frac{\sigma^2}{\alpha^2} \min(s, t) + \frac{\sigma^2}{2\alpha^3} \left[ -2 + 2e^{-\alpha s} + 2e^{-\alpha t} - e^{-\alpha|t-s|} - e^{-\alpha(t+s)} \right] \]

(see, for example, Parker (1992b, section 6)).


5. Approximating the Distribution of \( P \)

Parker (1992c) suggests a method for approximating the limiting distribution of the average cost per policy of a portfolio of temporary contracts. This method can be adapted to approximate the distribution of \( P \).

Let \( P(j) \) be the present value of the first \( j \) cash flows, that is:

\[ P(j) = \sum_{i=1}^{j} CF_i \cdot \exp \{-y(t_i)\} \]

and let the function \( g_j(p, y) \) be defined as:

\[ g_j(p, y) = f_{y(j)}(y) \cdot P(P(j) \leq p | y(j) = y) \]
The distribution function of $P$ is then given by:

$$F_P(p) = \int_{-\infty}^{\infty} g_n(p,y) dy.$$  \[28\]

Adapting the results derived and justified in Parker (1992c, sections 4 and 5), the function $g_j(p,y)$ may be approximated using the following recursive integral equation:

$$g_j(p,y) \approx \int_{-\infty}^{\infty} f_y(j) \cdot g_{j-1}(p - CF_j \cdot e^{-y}, x) dx$$

with the starting value:

$$g_1(p,y) = \begin{cases} \phi \left( \frac{y - E[y(1)]}{(V[y(1)])^{1/2}} \right) & \text{if } p \geq CF_1 \cdot e^{-y} \\ 0 & \text{otherwise} \end{cases}$$

where $\phi(\cdot)$ denotes the probability density function of a zero mean and unit variance normal random variable.

And if the forces of interest are gaussian, $y(j)$ given that $y(j-1)$ equal $x$ is normally distributed with mean:

$$E[y(j) | y(j-1) = x] = E[y(j)] + \frac{\text{cov} [y(j), y(j-1)]}{V[y(j)]} \cdot \{x - E[y(j-1)]\}$$

and with variance:

$$V[y(j) | y(j-1) = x] = V[y(j)] - \frac{\text{cov}^2 [y(j), y(j-1)]}{V[y(j-1)]}$$

(see, for example, Morrison (1990, p.92)).

6. Illustrations

Consider a $n$-year certain annuity-immediate with payments equal to one. The present value of this annuity is given by $P$ with $CF_i = 1$ and $t_i = i$ for $i = 1, 2, \ldots, n$. 
Figures 2, 3 and 4 illustrate the expected value, standard deviation and coefficient of skewness (see, for example, Mood, Graybill and Boes (1974, pp.68,76)) respectively of the present value, \( P \), for the annuity under consideration. Each figure presents the results for the three models discussed in section 4 for the force of interest.

Note that although the parameters of each of the three processes used for the force of interest were chosen so that their expected values are the same (.06), the expected responses of the processes to a specific situation are different. This explains their different expected values for the present value \( P \).

To illustrate this, let us assume that the force of interest at time
s is .08. Then the expected future forces of interest beyond time s is unchanged at .06 for the White Noise process. It changes to .08 and remains constant at this level for the Wiener process. For the Ornstein-Uhlenbeck process, the expected force of interest starts at .08 at time s and decreases exponentially to .06.

From figure 2 we can see that the standard deviation of the present value \( P \) varies considerably with the process involved. The White Noise process has independent forces of interest and this leads to the lowest standard deviation. The Wiener process has positively correlated forces of interest with no mechanism to bring the process back to any given value, thus giving the largest standard deviation. Finally, the Ornstein-
Uhlenbeck process has positively correlated forces of interest with a friction force bringing the process back to a long term mean.

It should be pointed out that fitting the parameters of these three processes based on past data is unlikely to give the same values for $\sigma$. The estimated $\sigma$ for the White Noise process might be the largest. This would increase the standard deviation of $P$ for the White Noise process relative to the other two.

![Fig. 3 Coefficient of Skewness](image)

### Present value of a $n$-year certain annuity-immediate
- **White Noise**: $\delta = 0.06$ $\sigma = 0.01$
- **Wiener**: $\delta_0 = 0.06$ $\sigma = 0.01$
- **Ornstein-Uhlenbeck**: $\delta = 0.06$ $\delta_0 = 0.06$ $\alpha = 0.1$ $\sigma = 0.01$

From figure 3, it appears that the distribution of $P$ generated by White Noise forces of interest is fairly symmetric as the coefficient of skewness is close to 0. With a Wiener process, $P$ has a highly asymmetric distribution. For reasons discussed above, the Ornstein-Uhlenbeck
generates a distribution of P with a coefficient of skewness between the other two.

Figures 4 and 5 present the cumulative distribution functions of 5 and 25 years annuity-immediate. The integral equations [28] and [29] were evaluated by an arbitrary discretisation.
7. Conclusion

The present value of future cash flows is treated as a random variable. A general approach for finding its moments when the force of interest is modeled by a gaussian process is presented. A method for approximating its distribution is suggested. The illustrations indicate that the choice of a stochastic process for the force of interest has a significant impact on the distribution of the present value of future cash flows.
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