MODELLING THE ASSETS AND LIABILITIES 
OF A PENSION FUND 

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ABSTRACT 

In the context of the historical development of portfolio theory the authors describe a stochastic asset/liability modelling exercise for a closed pension fund portfolio, illustrating the reduction in variance of estimated future surplus which can be achieved by investment in index–linked securities. Equity investment would increase the expected level of surplus, but the outcome would be more uncertain and there could be a not insignificant probability that the assets may ultimately prove insufficient to meet the liabilities. 

1. INTRODUCTION 

Pension funds in the United Kingdom are usually of a defined benefit nature. The main pension benefit is related to the earnings of the member at or near retirement age. Pensions in payment are increased in line with the consumer price index in the public sector. In private sector schemes which are contracted out of the State earnings-related additional pension, guaranteed minimum pensions in payment must be increased by at least 3% a year if they accrued after April 1988. Further increases, up to full protection against movements in the consumer price index, are usually provided on a discretionary basis, if the finances of the scheme permit. 

Pension schemes are usually financed by contributions from both employees and the employer. The employees will normally pay a fixed percentage of earnings, sometimes restricted to earnings above a certain level. The employer will typically be committed to meeting the balance of the cost of the scheme. The level of contribution by the employer will be recommended by the scheme actuary, and reviewed every 3 years in the light of the experience of the scheme. 

Investment policy is relatively unconstrained. The assets of the
scheme are held in a trust fund, which is quite separate from the assets of the employer. Responsibility for administering the trust fund is in the hands of a group of trustees, which may include nominees of the employer, the members and the pensioners, but whose responsibilities are personal and fiduciary to manage the assets of the trust in accordance with the rules and in the best interests of the beneficiaries. Trust deeds, which define the scope of investment policy, usually permit the trustees a wide range of discretion to manage the assets.

The majority of trustees are laymen in respect of actuarial and investment matters and will normally delegate day-to-day investment decisions to an investment manager. However, they cannot abrogate their personal responsibility for investment policy and must therefore provide guidelines to the investment manager concerning the overall investment strategy which they would wish to see implemented. Most trustees also consider it necessary to monitor the investment performance, often by subscribing to an industry-wide portfolio performance measurement service, in order to satisfy themselves as to the adequacy of the investment management services which are being provided.

This paper considers an asset-liability modelling exercise for such a pension fund in which risk is measured in terms of failure to meet the future liabilities.

2. RISK AND RETURN

Portfolio selection models have been extensively discussed in the finance literature, in particular following the ideas of Markowitz (1952, 1959) and Sharpe (1964, 1970). Traditional actuarial concepts in the UK also emphasized the trade-off between rate of return and risk, usually expressed in terms of the volatility of the outcome. An essentially risk-free rate of return to redemption could be obtained on government bonds, where the risk of default was negligible. Equity and property investment were expected to offer a higher yield, since they were intrinsically more risky, both as regards income and capital.

As inflationary pressures developed in the late 1950s and 1960s it was recognized that equity dividends and property rentals could be expected to go up in line with inflation, providing some sort of hedge against earnings- or price-related liabilities. Current yields on these types of investment shifted to below the redemption yield on government fixed interest securities (bonds), leading to the so-called reverse yield gap. The expectation of total return \( r \) from equities and property
could be expressed in the form

\[ r = (1 + i)(1 + g) - 1 \]

where \( i \) is the running yield, and \( g \) is the expected rate of growth of dividends (or rental yield).

This total return would almost always be expected to exceed the redemption yield on government bonds, in recognition of (i) the risk of falls in the value of shares (or property values), (ii) the risk of falls, or fluctuations, in dividend (or rental) income or (iii) the lack of marketability, especially in the case of property.

### 3. IMMUNIZATION

In the life insurance context, Redington (1952) discussed the immunization of a portfolio of known (or reasonably predictable) liability outflows. His idea was to compare the cash flows of liability outgo and total investment income. The problem was that precise matching of these flows was unduly constraining, but, if they were not matched precisely, changes in market values could create severe strains on the balance sheet valuation of assets and liabilities. Redington showed that it was sufficient for two conditions to be met:

i) the mean terms of the value of the asset proceeds must equal the mean term of the value of the liability outgo; and

ii) the spread of the value of the asset proceeds about the mean term should be greater than the spread of the value of the liability outgo.

Mathematically, the conditions can be stated as

i) \[ \Sigma t \cdot v^t \cdot A_t = \Sigma t \cdot v^t \cdot L_t \]

ii) \[ \Sigma t^2 \cdot v^t \cdot A_t > \Sigma t^2 \cdot v^t \cdot L_t \]

Redington's simple formulation assumed a flat yield curve (term structure of interest rates), that asset proceeds are received and liability payments made at a single fixed time each year, and that only infinitesimal changes occur in the market rate of interest.

Shedden (1976) generalized Redington's ideas to allow for a yield curve which was not flat, but assets and liabilities were still assumed to be deterministic. Boyle (1978) established conditions for immunization
with a stochastic interest rate model of the term structure of interest rates. This followed the development by Richard (1976), Vasicek (1977), Cox, Ingersoll and Ross (1977) and Brennan and Schwartz (1977) of stochastic models of the interest rate structure.

4. CAPITAL ASSET PRICING MODEL

Financial economists pursued the concept of the trade-off of risk and return between different classes of investment. An efficient portfolio was defined as a feasible portfolio which was not dominated by any other. A portfolio is dominated by another if for the same total expected return \( r \) the other has a lower variance of return \( \sigma^2 \), or if for the same variance \( \sigma^2 \) the other has a higher expected return \( r \). The efficient frontier is defined as the locus of efficient portfolios. If the expected return \( r \) on a mix of assets is plotted on the \( y \)-axis and the variance \( \sigma^2 \) on the \( x \)-axis, a curve is obtained to represent the efficient frontier, such that portfolios represented by points to the right of the curve are not efficient, whilst points to the left cannot be obtained with that combination of assets.

The capital asset pricing model is developed by introducing the concept of a risk-free rate of interest and investigating combinations of the market portfolio and the risk-free rate of interest. The actual return on any individual asset \( k \) can then be expressed as

\[
    r_k = r_f + \beta_k (r_M - r_f) + \text{cov} (r_k, r_M) \cdot \varepsilon_k
\]

where \( r_f \) is the risk-free rate of return, \( r_M \) is the return on the market portfolio

\[
    \beta_k = \frac{\text{cov} (r_k, r_M)}{\sigma^2_M}
\]

and \( \varepsilon_k \) is a random noise element.

Much attention has been focused in the investment markets, particularly in North America, on the determination of betas, as a measure of the systematic or non-diversifiable risk of particular assets.

5. ASSET RISK RELATIVE TO LIABILITIES

Portfolio selection models based purely on the trade-off of risk and return, where risk is measured in terms of variance of return, may be
useful tools for some investment managers. However, they fail to take into account the true nature of risk for those responsible for strategic investment decisions. For example, the directors of an insurance company, or the trustees of a pension fund, are concerned about the adequacy of the assets to meet the liabilities. Thus assets exhibiting no variability at all could be fundamentally unsatisfactory if the liability outflows are diverging from the income proceeds.

Wise (1984) considered this problem in relation to pension funds with inflation-linked liabilities, assuming a stochastic process for investment returns. He proposed that matching should be defined in terms of the ability to run off the existing liabilities with the proceeds from the existing portfolio of assets. He defined the ultimate surplus as the realizable market value of the assets remaining when all the liabilities have been extinguished.

Subsequent papers by Wilkie (1985) and Wise (1987) showed that the financial economists' two-dimensional portfolio selection model could be generalized into a three-dimensional model, where the three variables were the current market value or price ($P$) of the assets in the portfolio, the expected value ($E$) of the ultimate surplus and the variance ($V$) of the ultimate surplus. The investment manager's preoccupation with return in relation to price and the trade-off of risk (variance) and return gives rise to one solution for the portfolio optimization problem. The actuary's concern, on the other hand, to ensure a positive mean ultimate surplus and a reasonable probability that it will be positive, gives rise to another solution, from which the market value of the required portfolio can be deduced.

6. STOCHASTIC ASSET/LIABILITY MODELLING

The ideas put forward by Wise and Wilkie can be further generalized to apply to a fully stochastic pattern of liability outflows with stochastic investment returns. The potential complexities of the overall modelling situation limit the possibilities of deriving a neat analytical solution which can readily be computed. However, such problems can be addressed by means of computer simulation.

For any given portfolio of assets, and model of liability outflows, we can project the process forward using a transition equation, either on an annual basis or for some shorter interval.

The amount $A^+(t)$ available for new investment in year $t$ is given
by an equation of the form

\[ A^+(t) = B(t) + J(t) + A^M(t) - X(t) - E(t) - D(t) + Z(t) \]

where \( B(t) \) is the premium income or contributions in year \( t \),
\( J(t) \) is the investment income in year \( t \)
\( A^M(t) \) is the proceeds in year \( t \) from sold or matured assets, e.g. redeemable bonds
\( X(t) \) is the claim or benefit expenditure in year \( t \)
\( E(t) \) is the administrative and investment expenses in year \( t \)
\( D(t) \) is any dividend to shareholders or refund to contributors, e.g. the employer in the case of an occupational pension fund; and
\( Z(t) \) represents any other items, including perhaps taxation, special contributions (e.g. new capital in the case of a proprietary operation) or changes to debtor or creditor items.

In addition we must define a process for investing any cash flow surpluses, or disinvesting to meet any cash shortfalls. It is also necessary to model the actual values of the various categories of investment. Provision can be made in the simulation process to shift between one asset category and another, for example in order to rebalance the portfolio so as to retain a particular pattern of investments relative to the technical provisions or accrued liabilities.

Applying the cash flow process successively for \( t = 1, 2 \ldots T \), we obtain an estimate of the remaining assets at the end of year \( T \). Simulating the process a large number of times will generate a distribution \( A_T \) of assets at time \( T \). For each simulation the run-off of the liabilities can also be monitored, including any new liabilities created by additional premium income or contributions. An appropriate procedure can then be defined to set a value \( L_T \) on the outstanding liabilities at time \( T \). This enables the surplus \( R_T = (A_T - L_T) \) at time \( T \) to be obtained for each simulation, with a corresponding distribution.

We can then define appropriate optimization criteria, such as:
- maximize the expected value of surplus \( R_T \) at time \( T \)
- maximize the expected surplus \( R_T \) at time \( T \), subject to a specified maximum variance of \( R_T \)
- minimize the variance of \( R_T \) at time \( T \)
- maximize the expected surplus \( R_T \), subject to \( \text{Prob} \{ R_T < 0 \} \leq \varepsilon \).

If \( T' \) is such that \( L_t = 0 \ \forall t > T' \), but \( L_t > 0 \ \forall t < T' \), the above criteria relate to the distribution of surplus at the point at which all the liabilities have been paid off. The probability that \( A_T < 0 \) is defined as the probability of ruin.
Maximization would in principle be over all the available portfolios of assets which are of market value equal to the portfolio of assets currently held (at \( t = 0 \)). In practice it would not be possible to test all possible portfolios, down to the level of individual asset composition. Defining suitable stochastic models for individual assets may also be a problem. A realistic approach is to define a limited number of asset categories for which a suitable investment model is available and to optimize the strategic allocations between these investment categories. The choice of individual assets within each category would then be made on traditional investment criteria, or having regard to the portfolio selection techniques referred to earlier.

7. CASE STUDY

To illustrate these techniques we present some results from a particular case study. The study concerns the determination of an appropriate investment strategy for a defined benefit pension fund which is closed to new entrants and where no regular contributions are being made to the fund by either members or the (former) employers. Pensions in payment, now and in the future, are assumed to be increased each year in line with the consumer price index (the UK Retail Prices Index). Preserved pensions and retirement lump sums for those who have left service, and accrued liabilities to the date of closure of the fund for those remaining in service, are likewise assumed to be increased each year in line with the consumer price index up to retirement age.

At the date of the valuation the fund was invested in a wide range of investments, with 54\% in UK equities and 20\% in overseas equities. In order to illustrate the concept of the efficient frontier we make some assumptions about the mean and variance of the real rate of return on 6 major categories of asset. These assumptions are shown in Table 1.

<table>
<thead>
<tr>
<th>Asset category</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>UK equities</td>
<td>4.5</td>
<td>4.5</td>
</tr>
<tr>
<td>Overseas equities</td>
<td>5.0</td>
<td>7.0</td>
</tr>
<tr>
<td>Property</td>
<td>4.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Fixed-interest securities</td>
<td>3.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Index-linked securities</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Cash</td>
<td>2.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
We consider three alternative hypothetical investment strategies: 100% in UK equities, 50:50 in UK equities and index-linked government securities and 100% in index-linked government securities. Table 2 shows the distributions assumed for the portfolios denoted by A, B, C and D, and the respective means and variances of the projected real rates of return.

Table 2 - Distribution of assets assumed for each portfolio

<table>
<thead>
<tr>
<th>Asset category</th>
<th>Percentage by asset category in the portfolio denoted by</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>UK equities</td>
<td>54</td>
</tr>
<tr>
<td>Overseas equities</td>
<td>20</td>
</tr>
<tr>
<td>Property</td>
<td>6</td>
</tr>
<tr>
<td>Fixed-interest securities</td>
<td>12</td>
</tr>
<tr>
<td>Index-linked securities</td>
<td>2</td>
</tr>
<tr>
<td>Cash</td>
<td>6</td>
</tr>
<tr>
<td>Total assets</td>
<td>100</td>
</tr>
</tbody>
</table>

Yields (% a year)

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean real rate of return</td>
<td>4.3</td>
<td>4.5</td>
<td>3.7</td>
<td>3.0</td>
</tr>
<tr>
<td>Variance of real return</td>
<td>3.4</td>
<td>4.5</td>
<td>1.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Figure 1 shows the efficient frontier for combinations of the 6 categories of assets used in Table 1, and defined in terms of the original
portfolio selection model criterion of two-dimensional dominance, expressed here in terms of real rates of return. Thus the diagram shows the trade-off between expected real rate of return and the variance of the real rate of return. Points A, B, C and D represent the 4 portfolios of Table 2. None of these is efficient in the traditional sense.

Portfolio D reduces risk as compared to portfolio A but substantially reduces expected return. Portfolio B is quite efficient, at a relatively high level of risk. Portfolio C greatly reduces the risk, with not too serious a loss of expected return.

A valuation of the assets and liabilities has just been carried out by the actuary to the scheme. This assumed that the real return on the assets, net of pensions increases in line with the consumer price index, would average 2% a year for the remaining lifetime of the fund. In conjunction with a gross investment yield assumption of $8^{1/2}\%$ a year, this was equivalent to assuming an average rate of price inflation of $6^{1/2}\%$ a year. Assets were valued by assuming that they were notionally reinvested at the valuation date in a portfolio of UK equity shares corresponding to the Financial Times–Actuaries All Share Index. The dividends on this notional portfolio were valued at the nominal valuation rate of interest of $8^{1/2}\%$ a year, assuming annual growth in dividends of $3^{1/2}\%$ a year. This is equivalent to assuming that dividends will go up in future at 3% a year on average less than the consumer price index.

The above valuation basis conforms to the requirements of the Inland Revenue in the UK for determining the maximum size of fund which will remain eligible to be maintained without taxation of either income or capital gains. By common consent it is a fairly conservative basis. We shall assume that on this basis the assets were precisely 105% of the liabilities, the maximum permitted by the Inland Revenue if the tax-free status is to continue to apply to the whole fund.

Projections were made of the future liability outgo and investment income and asset values. For this purpose it was assumed that inflation would be at an average of 5% a year (rather than the $6^{1/2}\%$ assumed for the reserving basis). The liabilities were assumed to be defined by a deterministic model, allowing for appropriate “expected actual” mortality rates and other changes of status. The inflation rate was assumed to vary according to a stochastic model. Allowance was made for the not insignificant expenses of running off the portfolio, at 4.5% of the liability outgo each year, with a deduction of just over 1/4% a year of invested funds for investment expenses.

The investment income and asset values were projected from the ex-
isting portfolio of assets, assuming no rebalancing of the portfolio from year to year. Future investment income, market values and the consumer price index were assumed to be generated by a set of interlinked autoregressive models proposed by Wilkie (1986).

The Wilkie model was assumed to apply to overseas equities as well as UK equities. A model for UK property suggested by Wilkie and published in Daykin and Hey (1990) was adopted. A corresponding model was introduced for index-linked government securities and the Wilkie model for Consols was modified to allow for investment in redeemable fixed-interest securities.

Since the liabilities are almost all related to the consumer price index, investment in index-linked government securities would be expected to achieve a high level of immunization in terms of the adequacy of the assets to meet the liabilities. Equity investment, on the other hand, could be expected to offer a higher mean real rate of return. This pattern is illustrated in Figure 1. An asset/liability modelling exercise was carried out to explore these relationships further and to answer questions such as

- what risk would be taken by the trustees if they invested 100% in UK equities, i.e. what would the probability be that the liabilities could not be met completely?
- what loss of expected ultimate surplus would there be if the portfolio was invested 100% in index-linked government securities?
- what additional solvency margin (margin of assets over liabilities) would be necessary now, if investment was to be made 50% in UK equities and 50% in index-linked government securities, so as to reduce the probability that the assets would ultimately be insufficient to a given level?

The position was examined at the end of 15 years, on the basis of 1000 randomly generated simulations of the inflation and investment scenarios. The value of the outstanding liabilities was calculated at the end of 15 years, using the same conservative statutory surplus valuation basis as currently, and the market value of the assets was projected for each portfolio. The ratio of the asset values at the end of year 15 to the value of the residual liabilities \(A_{15}/L_{15}\) gives a measure of the degree of solvency at that time, which we refer to as the 15 year funding level.

For each of the portfolios investigated, the 1000 simulations can be ranked in terms of the ratio of assets to liabilities, from the least favourable outcome to the most favourable. This enables an impression
to be gained visually of the variability of possible outcomes, as compared to the median values of the projected 15 year funding level.

In Figure 2 the results are presented for the four portfolios A, B, C and D in terms of the 15 year funding level. The bold line in each column represents the median 15 year funding level for that portfolio. The cross-hatched portion above and below the mean represents the 50% probability band. In other words the bottom of this band represents the point at which there is a 25% probability that the outcome will be worse than this and a 75% probability that it will be better. At the top of this band there is a 75% probability that the outcome will be worse and a 25% probability that it will be better.

The diagonally hatched band represents the next 15% of the probability distribution of the 15 year funding level. Thus at the bottom of the lower diagonally hatched band there is a 10% probability that the outcome will be worse than this. At the top of the upper diagonally hatched band there is a 10% probability that the outcome will be even better.

Table 3 shows some key results from the analysis for the four portfolios. Since the distributions of ultimate surplus are skew, the mean funding level is significantly higher than the median shown in Figure 2. The simulations which produced a 15 year funding level of less than 100% do not necessarily represent ruins in the sense defined earlier. At this stage the business has still some years to run and the valuation
basis used for the test is a conservative one.

Table 3 - Results at 15 year point from asset/liability study

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Median funding level</th>
<th>Mean funding level</th>
<th>Standard deviation of funding level</th>
<th>Proportion of simulations with 15 year funding level below 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.42</td>
<td>1.56</td>
<td>0.80</td>
<td>25%</td>
</tr>
<tr>
<td>B</td>
<td>1.41</td>
<td>1.62</td>
<td>1.00</td>
<td>29%</td>
</tr>
<tr>
<td>C</td>
<td>1.30</td>
<td>1.39</td>
<td>0.48</td>
<td>20%</td>
</tr>
<tr>
<td>D</td>
<td>1.17</td>
<td>1.17</td>
<td>0.12</td>
<td>8%</td>
</tr>
</tbody>
</table>

The results shown in Figure 2 are consistent with intuition. The median funding level of portfolio B (100% equities) is high, but so is the spread around the median. Portfolio D has a lower median funding level, but the variation is considerably reduced. The median 15 year funding level is in all cases well above the initial 105% funding level. This arises because of the conservative assumptions underlying the funding level assessment, contrasted with the more realistic expected actual assumptions for the progress of the fund over the 15 year period.

8. Conclusion

The asset/liability modelling exercise described in this paper illustrates the use of stochastic investment and inflation models in a practical pension fund situation, in order to provide the trustees with information to assist them in making strategic investment decisions. Further analysis would be useful on the 15 year funding level using more realistic assumptions, and on the initial funding level necessary to achieve specific objectives at the 15 year point. In this particular case the trustees could greatly reduce the level of uncertainty regarding the ultimate outcome by investing in index-linked securities, but with a significant reduction in the median (and mean) 15 year funding level and the loss of the chance of substantial investment profits. The decision is likely to depend on their interpretation of their fiduciary responsibilities, the use to which any surplus is likely to be put (to increase benefits, to return to the sponsoring employer or to be paid out in tax charges) and the consequences of any shortfall.
BIBLIOGRAPHY
