NORMAL AND LOGNORMAL SHORTFALL-RISK

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ABSTRACT

Shortfall-risk - the probability that a specified miminum return level will not be exceeded is an important measure of risk that is more consistent with the investors' perception of risk than the traditional measure of risk, the variance of returns.

The present paper treats the calculation and the control of the shortfall-risk in the case of normal and lognormal returns over one and over many periods.

1. INTRODUCTION

Shortfall-risk (downside-risk) - the probability that a special minimum return level (target return, benchmark return) will not be exceeded - is a measure of risk which has recently attracted considerable interest in modern investment management, cf. e.g. Leibowitz/Henriksson (1989), Leibowitz/Kogelman (1991a, 1991b), Leibowitz/Krasker (1988), Leibowitz/Langetieg (1990) and Sortino/Van Der Meer (1991). In contrast to the traditional measure of risk in portfolio theory, the variance of returns, shortfall-risk is more consistent with the investors' intuitive perception of risk in that it focusses more on the real economical risk of an investor, whereas the variance or returns is rather a measure of the volatility of financial assets. "Upward" volatility results in investment chances, only "downward" volatility will result in investment risks, so that asymmetric measures of risk, cf. Harlow (1991), are attractive alternatives to the variance. Shortfall-risk is the most elementary asymmetrical risk measure.

This paper concentrates on the calculation and the control of shortfall-risk in the case of normal and lognormal returns over one and over many periods.

Especially for life insurance companies the control of the shortfall risk of investment returns over short and over long horizons should develop into an important tool in investment and risk management.

2. One-Period Shortfall-Risk

Let R denote the one-period return of an asset. The expected value resp. the variance of the return will be denoted by

[1] $$\mu = E(R), \qquad \sigma^2 = \mathrm{Var}(R).$$

We will assume that R is normally distributed

[2] $$R \sim N(\mu, \sigma^2)$$

or alternatively that $1 + R$ is lognormally distributed with parameters m and v^2, i.e.

[3] $$\ln(1+R) \sim N(m, v^2).$$

To ensure that [1] still is valid, we have to choose the parameters m and v^2 as follows (the relevant properties of the lognormal distribution used in this paper are given in the appendix):

[4] $$v^2 = \ln\left[1 + \left(\frac{\sigma}{1+\mu}\right)^2\right]$$
$$m = \ln(1+\mu) - \frac{1}{2}v^2.$$

To quantify the shortfall-risk we have to specify a desired minimum return M, the shortfall-risk then is given by

[5] $$SR(M) = P(R \leq M).$$

A graphical illustration of $SR(M)$ is given in figure 1.

To control the shortfall-risk $SR(M)$ we have to specify a (small) probability ε and to set up the shortfall-constraint

[6] $$P(R \leq M) \leq \varepsilon.$$

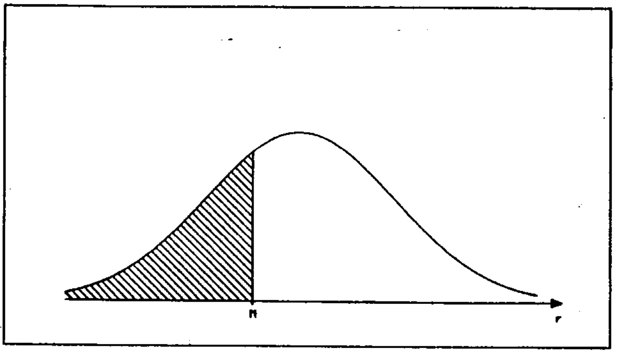

Fig. 1 Shortfall-Risk $SR(M)$

Let F_ε denote the $(1-\varepsilon)$-quantile of the distribution F of R, i.e. $P(R > F_\varepsilon) = \varepsilon$, then the probability constraint [6] is equivalent to the "deterministic" constraint

[7] $$M \leq F_{1-\varepsilon}\,.$$

Let N_ε denote the $(1-\varepsilon)$-quantile of the standard normal distribution, then the $(1-\varepsilon)$-quantiles $N_\varepsilon(\mu,\sigma)$ of a normal distribution resp. $LN_\varepsilon(m,v)$ of a lognormal distribution are given by

[8] $$\begin{aligned} N_\varepsilon(\mu,\sigma) &= \mu + N_\varepsilon\sigma \\ LN_\varepsilon(m,v) &= \exp[m + N_\varepsilon v]\,. \end{aligned}$$

As we have $N_{1-\varepsilon} = -N_\varepsilon$ for a standard normal distribution, the relevant shortfall-constraint for a normal distribution is given by

[9] $$\mu \geq M + N_\varepsilon\sigma\,.$$

In a (σ,μ)-diagram the shortfall-constraint [6] can be visualized as follows. The straight line $\mu = M + N_\varepsilon\sigma$ divides the (σ,μ)-plane into two seperate sectors. The sector above the line (including the line itself) contains all (σ,μ)-positions with controlled shortfall-risk. Figures 2 and 3 give two examples for the corresponding lines in the case of normal distribution.

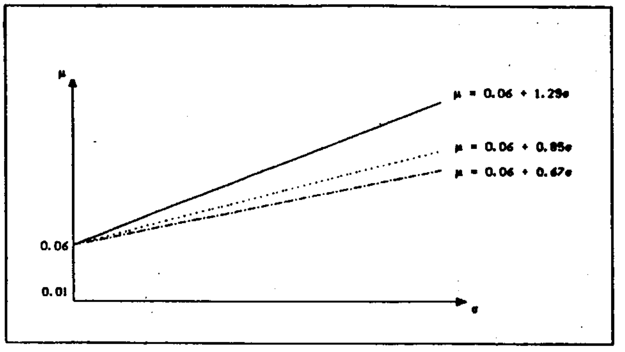

Fig. 2 Shortfall-Risk in the normal case: $M = 0.06; \varepsilon = 0.25, 0.2, 0.1.$

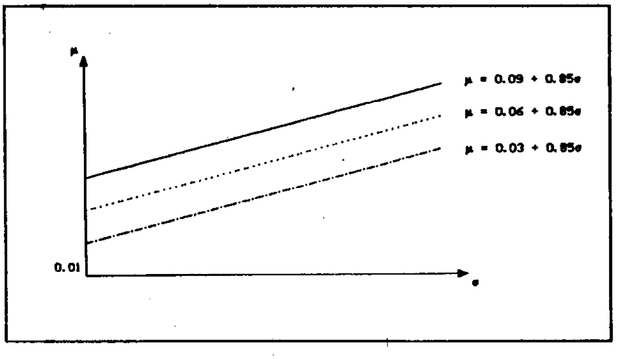

Fig. 3 Shortfall-Risk in the normal case: $\varepsilon = 0.2; M = 0.03\ 0.06, 0.09.$

The case of the lognormal distribution is much more complex. First we have from [7] in connection with [8]

$$[10] \qquad m \geq ln(1 + M) + N_\varepsilon v \,,$$

but we are interested in the possibility of visualization and therefore we need a constraint involving σ and μ. From the appendix we obtain the adequate transformation as follows. Let $G_\varepsilon(v)$ be defined by

[11] $$G_\varepsilon(v) = \frac{1 - \exp[-N_\varepsilon v - 1/2v^2]}{[\exp(v^2) - 1]^{1/2}}\,.$$

Then [10] is equivalent to [12]

[12] $$\mu \geq M + G_\varepsilon(v)\sigma\,.$$

As $G_\varepsilon(v) = G_\varepsilon(\mu,\sigma)$ it becomes clear that [12] is only an implicit inequality.

The following two figures are the counterpart to the figures 2 and 3 for the lognormal case.

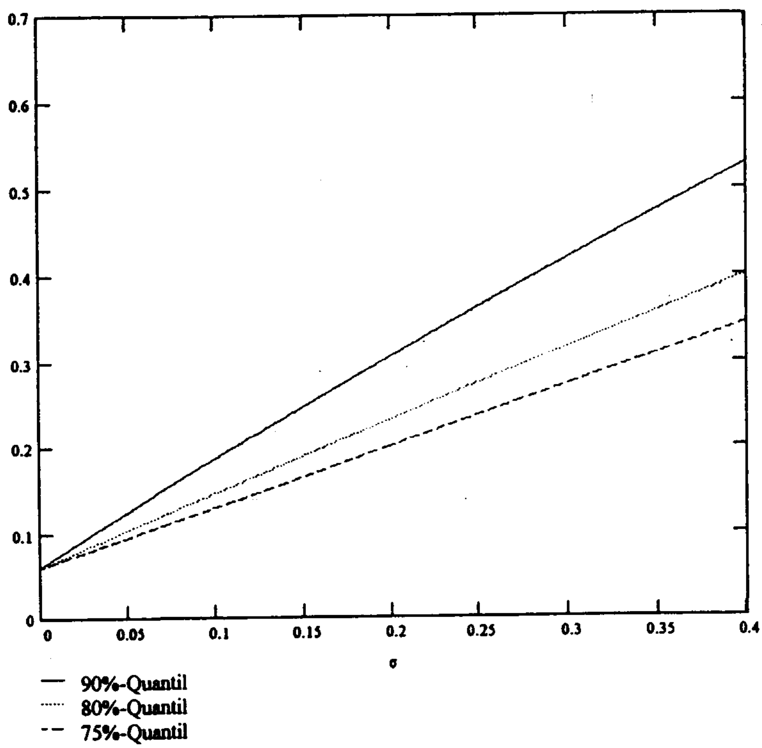

Fig. 4 Shortfall-risk in the lognormal case: $M = 0.06$; $\varepsilon = 0.25, 0.2, 0.1$.

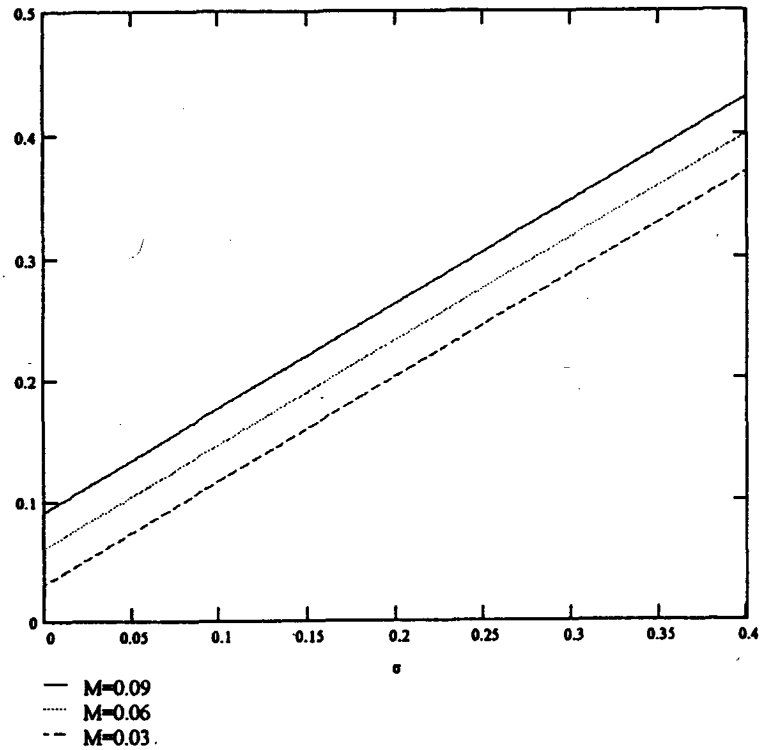

Fig. 5 Shortfall-risk in the lognormal case: $\varepsilon = 0.2$; $M = 0.03, 0.06, 0.09$.

Finally Figure 6 contains a comparison of the normal and the lognormal situation in a special case.

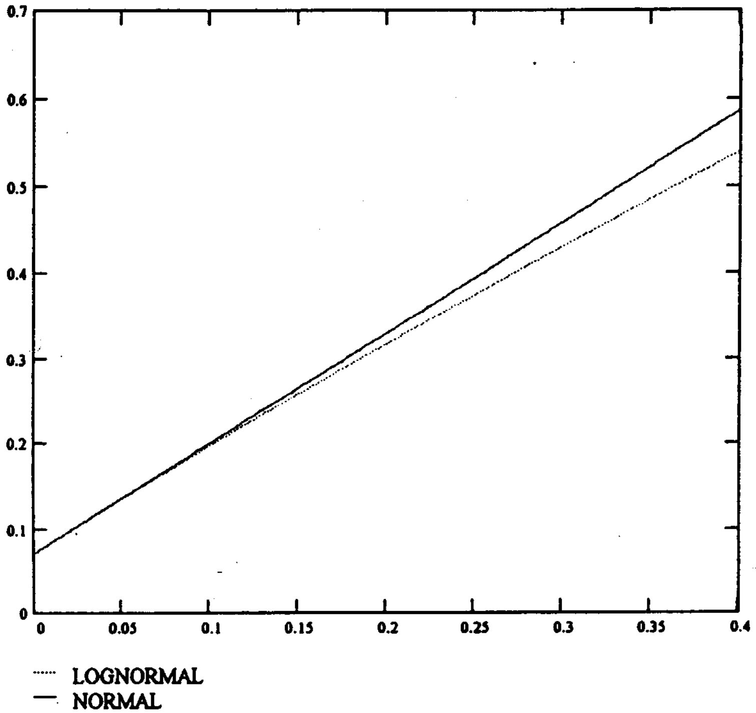

Fig. 6 Normal case vs. lognormal case: $M = 0.07$, $\varepsilon = 0.10$.

3. Multi-Period Shortfall-Risk

In the multi-period case there are two possibilities to annualize a series of successive, say T, one-period return $R_1, \ldots, R_T$.

First the arithmetically annualized return $R_A(T)$ is given by

$$R_A(T) = \frac{1}{T}(R_1 + \cdots + R_T). \quad [13]$$

The geometrically annualized return $R_G(T)$ is given by

$$R_G(T) = \sqrt[T]{\prod_{t=1}^{T}(1 + R_t)} - 1. \quad [14]$$

The geometrical annualization clearly is the correct one, but the arithmetical annualization is often applied in practice because of its simplicity.

In the following we will make different distributional assumptions which have their reason in the corresponding properties of the normal resp. lognormal distribution. When working with $R_A(T)$ we will assume $R_t \sim N(\mu, \sigma^2)$, which gives

[15] $$R_A(T) \sim N\left(\mu, \frac{\sigma^2}{T}\right),$$

i.e $E(R_A) = \mu$ and $\sigma(R_A) = \sigma/\sqrt{T}$.

The dispersion of the (arithmetically) annualized return is monotonically decreasing with increasing time horizon.

When working with $R_G(T)$, we will assume $1n(1+R_t) \sim N(m, v^2)$, which gives

[16] $$1 + R_G(T) \sim LN\left(m, \frac{v^2}{T}\right).$$

We obtain:

[17]
$$E(R_G) = \exp\left(m + \frac{1}{2}v^2\right) - 1 = \mu$$
$$\text{Var}(R_G) = (1+\mu)^2 \left[\exp\left(\frac{v^2}{T}\right) - 1\right] = (1+\mu)^2 \left[\sqrt[T]{1 + \left(\frac{\sigma}{1+\mu}\right)^2} - 1\right].$$

Again we have the property of a monotonically decreasing (geometrically) annualized return.

Figure 7 gives an illustration of this effect in the lognormal case.

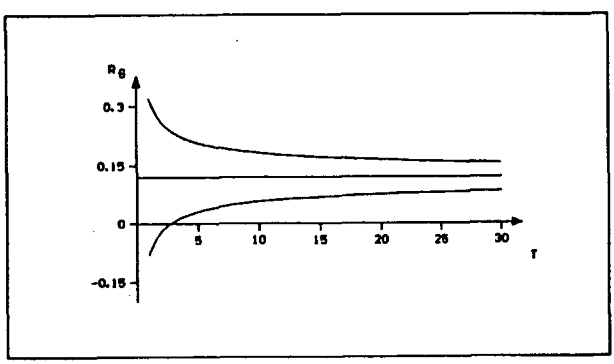

Fig. 7 One-sigma-interval of the geometrically annualized return.

Controlling the shortfall-risk of the arithmetically annualized return $R_A(T)$ results in the shortfall-constraint

[18] $$P(R_A(T) \leq M) \leq \varepsilon\,,$$

which is equivalent to the constraint

[19] $$\mu \geq M + N_\varepsilon \frac{\sigma}{\sqrt{T}}\,.$$

Compared to the corresponding one-period constraint [9] this shows that the time horizon is taken into consideration in dividing the second term on the right side of inequality [9] by $\sqrt{T}$. Figure 8 illustrates this effect graphically.

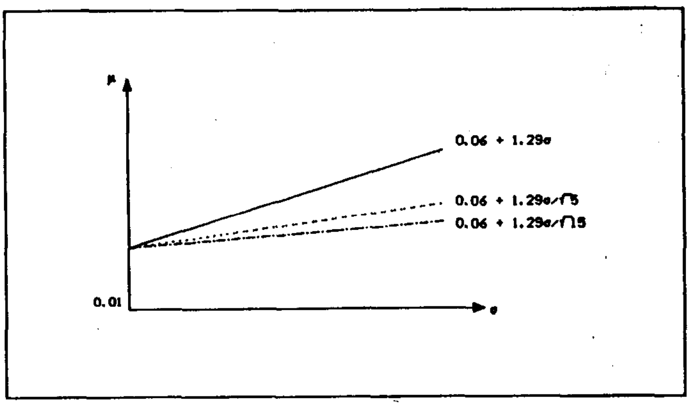

Fig. 8 Multi-period shortfall risk (arithmetically annualized return): $M = 0.06$; $\varepsilon = 0.10$; $T = 1, 5, 15$.

In case of a geometrically annualized return $R_G(T)$ the control of the shortfall-risk results in the shortfall-constraint

[20] $$P(R_G(T) \leq M) \leq \varepsilon\,,$$

which is equivalent to the (implicit) constraint

[21] $$\mu \geq M + G_\varepsilon(v/\sqrt{T})\sigma/\sqrt{T}\,.$$

Figure 9 is the counterpart to figure 8 in the lognormal case.

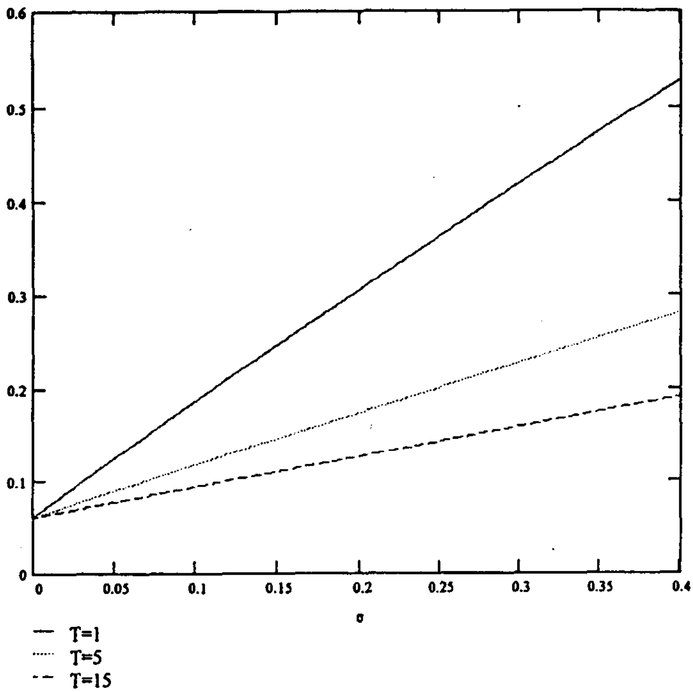

Fig. 9 Multi-period shortfall-risk (geometrically annualized return):
$M = 0.06$; $\varepsilon = 0.10$; $T = 1, 5, 15$.

The shortfall-constraints [18] resp. [20] only allow the shortfall risk to be controlled over the total horizon of time. Only the annualized return is controlled, the desired minimum return will be realized with high "on the average" probability . To be able to control the shortfall-risk in addition over sub-intervals of time one could proceed as follows. Specify the periods of time for which control of the return is desired, e.g. $1 \leq t_1 < \ldots < t_N \leq T$, the desired minimum returns $M_1, \ldots, M_N$ and the desired control levels $\varepsilon_1, \ldots, \varepsilon_N$. An appropriate control criterion in the case of arithmetically annualized returns could be

[22]
$$\begin{gathered} P\big(R_A(t_1) \leq M_1\big) \leq \varepsilon_1 \\ \vdots \\ P\big(R_A(t_N) \leq M_N\big) \leq \varepsilon_N \end{gathered}$$

i.e. a collection of controls of the type [18]. The case of geometrically annualized returns can be treated in complete analogy. The following figure illustrates the following constraints:

(1) $P\big(R_A(3) \leq 0.03\big) \leq 0.10$

(2) $P\big(R_A(10) \leq 0.05\big) \leq 0.10$

(3) $P\big(R_A(15) \leq 0.07\big) \leq 0.10.$

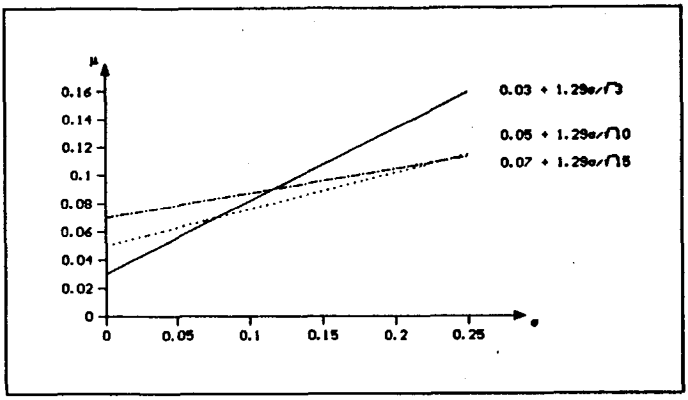

Fig. 10 Shortfall-constraints over different time horizons.

4. Implications for Portfolio Selection

In portfolio selection one has to identify the optimal portfolio composition for a given investor with given preferences. In the framework of the Markowitz approach, the preferences only depend on $E(R)$ and $\sigma(R)$ and the optimal portfolio has to be selected from the efficient frontier.

This selection of the optimal portfolio on the efficient frontier can be easily performed on the basis of the multi-period shortfall-constraints as follows. Solve the optimization problem

$$E\big[R_A(T)\big] = E(R) \to \max$$

subject to the constraints

$$\text{[23]} \qquad \begin{aligned} &P\big[R_A(t_1) \leq M_1\big] \leq \varepsilon_1 \\ &\qquad\vdots \\ &P\big[R_A(t_N) \leq M_N\big] \leq \varepsilon_N\,. \end{aligned}$$

Because of [19] in case of the normal distribution [23] is equivalent to a linear programming problem. It is obvious that the resulting portfolio must be efficient in the Markowitz sense, so one could as well first identify the efficient frontier and then choose that portfolio on the efficient frontier fulfilling the constraints [22] and possessing the highest expected value. Figure 11 illustrates this graphically.

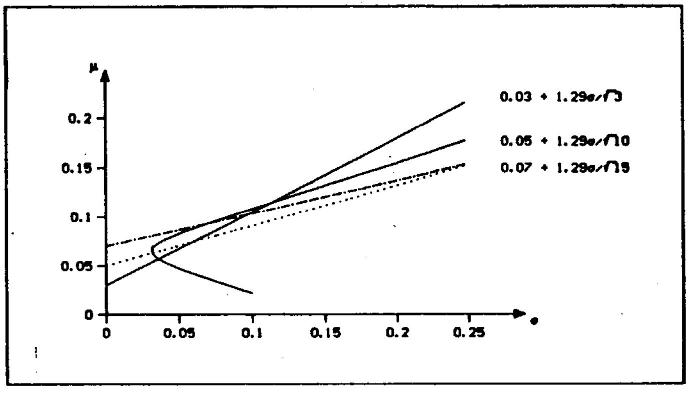

Fig. 11 Portfolio selection under shortfall-constraints.

5. Appendix: Properties of the lognormal distribution

A random variable $X(>0)$ is lognormally distributed, if $1nX$ is normally distributed:

$$\begin{aligned} X \sim LN(\mu, \sigma^2) &\iff 1nX \sim N(\mu, \sigma^2) \\ &\iff \frac{1nX - \mu}{\sigma} \sim N(0,1)\,. \end{aligned}$$

The density function of the lognormal distribution is given by

$$fx(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma}} \exp\left[\dfrac{(1nx-\mu)^2}{2\sigma^2}\right] & x > 0 \\ 0 & x \leq 0\,. \end{cases}$$

The first two moments of the lognormal distribution are given by

$$\begin{aligned} E(X) &= \exp\left(\mu + \frac{1}{2}\sigma^2\right) \\ \mathrm{Var}(X) &= \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right] = E(X)^2\left[\exp(\sigma^2) - 1\right]. \end{aligned}$$

These moments uniquely determine the parameters of the lognormal distribution:

$$\sigma^2 = ln\left[1 + \frac{\mathrm{Var}(X)}{E(X)^2}\right]$$
$$\mu = lnE(X) - \frac{1}{2}\sigma^2 =$$
$$= 2lnE(X) - \frac{1}{2}\ln\left[E(X)^2 + \mathrm{Var}(X)\right].$$

For n stochastically independent lognormal distributions we have

$$X_i \sim LN(\mu_i, \sigma_i^2) \qquad i = 1, \ldots, n$$
$$\Longrightarrow \prod_{i=1}^{n} X_i \sim LN\left(\sum_{i=1}^{n} \mu_i \;,\; \sum_{i=1}^{n} \sigma_i^2\right),$$

in case of identical distributions therefore

$$\prod_{i=1}^{n} X_i \sim LN(n\mu, n\sigma^2).$$

For the geometric mean we obtain therefore

$$\sqrt[n]{\prod_{i=1}^{n} X_i} \sim LN\left(\mu, \frac{\sigma^2}{n}\right).$$

Let $LN_\varepsilon(\mu, \sigma)$ denote the $(1-\varepsilon)$-quantile of a $LN(\mu, \sigma^2)$-distribution and N_ε the $(1-\varepsilon)$-quantile of the standard normal distribution, then we have

$$LN_\varepsilon(\mu, \sigma) = \exp[\mu + N_\varepsilon \sigma].$$

From Johnson/Kotz (1970, p. 117) we have

$$\frac{LN_{1-\varepsilon} - E(X)}{\sigma(X)} = \frac{\exp\left[N_{1-\varepsilon}\sigma - \frac{1}{2}\sigma^2\right] - 1}{\left[\exp(\sigma^2) - 1\right]^{1/2}}.$$

Because of $N_{1-\varepsilon} = -N_\varepsilon$ and introducing the function

$$G(\sigma) = \frac{1 - \exp\left[-N_\varepsilon\sigma - \frac{1}{2}\sigma^2\right]}{\left[\exp(\sigma^2) - 1\right]^{1/2}}$$

this is equivalent to

$$\begin{aligned} LN_{1-\varepsilon} &= E(X) - G(\sigma)\,\sigma(X) = \\ &= E(X) - G\big(E(X),\ \sigma(X)\big)\sigma(X), \end{aligned}$$

as the parameter σ is itself a function of $E(X)$ and $\sigma(X)$.

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