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COMBINING ACTUARIAL AND FINANCIAL RISK : STOCHASTIC CORPORATE MODEL AND ITS CONSEQUENCES FOR PREMIUM CALCULATION

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RESUME

Les risques actuariels - coûts stochastiques des indemnisations - et les risques financiers - rentabilité stochastique des investissements - sont combinés en un modèle d'entreprise monopériodique de l'activité d'une compagnie d'assurances. Le modèle autorise explicitement des dépendances entre les risques actuariels et les risques financiers, telles que, par exemple, d'éventuelles corrélations ou une dépendance du capital investi disponible à l'égard du chiffre d'affaires.

Le problbme du calcul de la prime (brute) est envisage sur la base de ce modèle d'entreprise, d'un point de vue actuariel et d'un point de vue financier. Les primes brutes sont calculées selon une probabilité restreinte de ruine, ainsi que sur la base du Modèle de fixation du prix d'actifs immobilisés.

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ABSTRACT

Actuarial risks (stochastic claim costs) and financial risks (stochastic investment returns) are combined in a one-period corporate model of insurance business. The model explicitely allows for dependencies between actuarial and financial risks, as e.g. possible correlations or the dependence of the investment capital at disposal from the premium income.

On the basis of the corporate model the problem of (gross) premium calculation again is considered from an actuarial and from a financial perspective. Gross premiums are calculated according to a restricted probability of ruin and alternatively on the basis of the Capital Asset Pricing Model.

KEYWORDS

Corporate model, actuarial risk, investment risk, gross premium calculation, CAPM

1. INTRODUCTION

The present paper tries to combine the only partial analytical views of either a pure capital market theoretical analysis of the investment sector of an insurance company or either a pure risk theoretical analysis of the company's claims process.

The basis for the analysis is an analytically tractable one-period stochastic corporate model of insurance business. The model explicitely considers the stochastic nature of the claim costs and the investment returns as well as the following dependencies between the investment and claims sector:

1) The amount of investment capital at disposal (and therefore implicitely the absolute income from investments) depends on the reserves generated by insurance business
2) Correlations between underwriting profits of different lines of insurance and investment profits from different investment activities, which may be of significant nature, cf. e.g. KAHANE/NYE (1975, pp. 86/87) or KAHANE (1977, p. 1064).

Because of its one-periodic nature, the model primarily is of importance for (composite) property-liability stock insurance companies. By reason of their importance multiperiod investment effects of loss reserves, i.e. the reserved part of the premiums of one accounting year is at investment disposal for several years, have not been neglected but have been transformed to a one-period basis.

The model constructed is much in the spirit of the models developed in a series of portfolio optimization approaches to the problem of the determination of a simultaneously optimal composition of the insurance and investment portfolios of an insurance company, cf. e.g. KAHANE/NYE (1975), KAHANE (1977, 1978, 1980), CUMMINS/NYE (1981) and SMIES-LOK (1984), but is differently evaluated for a different problem.

Not only in the construction of the model both the actuarial and the financial views are followed up, but also in the evaluation of the model.

The first evaluation is an extension of a classical actuarial approach, namely the restriction of the (one-period) ruin probability. On the basis of a distributional assumption (Normal-Approximation) the applied control criterion is to keep very low the probability that the one-year result eliminates the security capital of the company, a criterion which in fact is the theoretical basis for solvency considerations. Applying this kind of evaluation to the problem of premium calculation, one obtains a calculatory minimum gross premium (not only a risk premium!) for the entire collective that guarantees the security level of the company. The second evaluation is based on results from capital market theory. On the basis of the Capital Assel Pricing Model (CAPM) fair company returns, i.e. company returns consistent with a capital market equilibrium, and by that fair market premiums can be determined, thus generalizing some of the results of KAHANE (1979) and URRUTIA (1986). The resulting premium can be considered as the maximum gross premium a company should earn. Indeed the CAPM has been used as a consistent theoretical basis for insurance rates regulation, cf. FAIRLEY (1979) or CUMMINS/HARRINGTON (1984).

An important remark is that a minimum premium calculated on the basis of a restricted ruin probability not necessarily is smaller than a maximum premium calculated on the basis of the CAPM. This is because the CAPM does not value the probability of ruin of the companies, whose market value (equivalently: fair return) is determined This, however, means that the importance of CAPM-based premiums is very restricted regardless their attractiveness as equilibrium premiums.

2. CONSTRUCTION OF A STOCHASTIC ONE-PERIOD MODEL OF INSURANCE BUSINESS

During one accounting period the main costs and proceeds are aggregated to the one-period result of the company according to the following scheme :

(1)

compagny result G =

premium proceeds π

- aggregated claim cost S
  (for claims occurred in the period)

+ asset proceeds I
  (after investment cost)

- operating cost K.
  (claim independent)

Or in short :

(2) $$G = \pi - S + I - K$$

It is assumed that claimdependent operating costs (as e.g., claim regulation costs) are subsumed under S to facilitate the analysis. As a consequence K can be modelled as a deterministic quantity, the same holds for π.

S and I, however, are stochastic quantities. We factorize I into the product of the (stochastic) one-period return on investment R and the (average) investment capital A at disposal for one period :

(3) $$I = A \times R .$$

The investment capital is assumed to be of the form :

(4) $$A = U + B + L .$$

A few remarks are in order on the components of the investment capital. The quantity U denotes the security capital of the company. We assume that the security capital is that part of the equity capital, which is not invested in operating assets, i.e. is at disposal for capital investment. The remaining components of the investment capital are the main insurance specific liability reserves, the uneamed premium reserve B and the loss reserve L. As we are primarily interested in the (periodized) result of a fixed accounting period we only consider reserves set for the claims of that period. Reserves set for past periods are not taken into account. In addition not the accounting valuation of the liability reserves gives the relevant quantities but we have to choose a valuation which adequately reflects the mean amount at disposal for a one-year capital investment ("cash flow valuation"). Special attention has to be drawn to the loss reserve, as parts of the loss reserve set for one accounting year are at disposal to a decreasing extent for several periods. The corresponding compound interest effect has to be transformed to a one-period basis.

The exact specification of the premium reserve depends on the way premiums are paid and on the way premiums are needed to cover claims and operating expenses. If premiums are paid at the beginning of the accounting period and are earned linearly over the period, the average amount in the unearned premium reserve is one half of the premium proceeds π, less the immediate expenses (acquisition expenses) paid. Therefore we assume

$$(5) \qquad B = \frac{1}{2}(1-\alpha)\,\pi,$$

where a is that fraction of the premium proceeds which is paid out for immediate expe es.

The exact (cash flow) specification of the loss reserve depends on the (stochastic) characteristics of the loss payment process. Incorporation of a stochastic development of the loss reserve into the model would, however, make the model analytically intractable. Therefore we have to use a deterministic approximation of the real process. A simple estimation of the interest effect of the loss reserve is obtained, if we assume that a constant fraction m $(0 < m < 1)$ of the premium receipts are reserved and the average settlement period is k. In the stationary state, attained after k periods, an amount of $km\pi$ will be permanently in the loss reserve, i.e. we obtain the approximation

$$(6) \qquad L = km\pi\,.$$

On the basis of the preceding analysis of the quantification of the liability reserves the following generalization is reasonable. Whe introduce a coefficient $h > 0$, which represents the average fraction, regularly exceeding an amount of one, of the premium proceeds π being at disposal f a capital investment for one period, i.e. $B + L = h\pi$.

Summing up, we obtain from (4) :

(7) $A = A(\pi, U) = U + h\pi$

and from (7), (3) and (2):

(8) $G = \pi - S - K + (U + h\pi) R$

Equation (8) is a reasonable one-period model for the total result of the insurance business, involving explicitely the stochasticity of the claims expenses S and the investment return R and the dependence of the investment capital on the security capital and the premium receipts. It will be the main basis of the following analysis.

A more refined corporate model is obtained, if we allow for several insurance branches $i = 1,\dots,n$ and for several capital investment opportunities $j = 1,\dots,m$.

Let S_i and π_i denote the accumulated claims of insurance branch i resp. the accumulated premium receipts of this branch. In addition it is reasonable that every branch differs with respect to the amount of liability reserves generated, which can be quantified by using different capital generating coefficients $h_i > 0$. The security capital U however has to be considered undividedly for the entire business of the insurance company. The total investment capital A therefore can be modelled as

(9) $$A = U + \sum_{i=1}^{n} h_i \pi_i$$

Let K_i $(i = 1,\dots,n)$ denote the accumulated (claim independent) operating expenses of branch i.

Let R_j and a_j denote the one-year return of the j-th asset resp. the fraction $(0 \leq a_j \leq 1, \sum a_j = 1)$ of A invested in this asset.

Altogether we obtain the following expression for the total result of the company:

(10) $$G = \sum_{i=1}^{n} (\pi_i - S_i - K_i) + \sum_{j=1}^{m} a_j \left(U + \sum_{i=1}^{n} h_i \pi_i\right) R_j .$$

3. IMPLICATIONS OF A FIXED SECURITY LEVEL FOR THE GROSS PREMIUM

3.1 General analysis

In this chapter we analyse the implications of a fixed security level of the insurance company for several basic risk theoretic problems. The basis of the analysis will be the following stability criterion, which as a strong intuitive appeal:

(11) $P(G < -U) \leq \varepsilon$

The probability that the one period result of the company completely eliminates the security capital shall be bounded by a small quantity $\varepsilon > 0$.

There are several ways of evaluating the stability criterion (11) depending on the way the loss probability $P(G < -U)$ is quantified.

A first natural approach is to use the Normal Approximation for the result G of the company, which can be justified by the Central Limit Theorem and the fact that we consider the result based on the entire portfolio of the company. The use of the Normal Approximation essentially simplifies the analysis of the problems considered. Results based on the more advanced approach using the Normal Power-Approximation will be reported elsewhere.

The following general relation is useful for the further analysis. Let $F\varepsilon$ be the $(1 - \varepsilon)$-quantile of a random variable X, we then have:

(12) $P(X > c) \leq \varepsilon \leftrightarrow c \geq F_\varepsilon$

The $(1 - \varepsilon)$-quantile of the normal distribution with mean $E(X)$ and variance $\sigma^2(X)$ is

(13) $F_\varepsilon = E(X) + N\varepsilon\, \sigma(X)$,

where $N\varepsilon$ is the $(1 - \varepsilon)$-quantile of the standard normal distribution.

Using the Normal Approximation for G and applying (12) and (13) to (11), we obtain

(14) $P(G < -U) \leq \varepsilon \div P(-G > U) \leq \varepsilon \leftrightarrow U \geq E(-G) + N\varepsilon\, \sigma(-G) \leftrightarrow E(G) \geq N\varepsilon\, \sigma(G) - U.$

To guarantee that the one year result of the company does not completely eliminate the security capital with a very high probability $(\geq 1 - \varepsilon)$ the expected one year result has to be controlled in a way that it exceeds a lower bound depending on:

1) the extent of stochastic variation of the result (in (14) quantified by $N\varepsilon\, \sigma(G)$)

2) the security capital U of the company.

The evaluate (14) expressions for the moments of G as functions of the main components of G have to be calculated. From (8) we obtain

(15) $E(G) = \pi - E(S) - K + (U + h\alpha\pi)\, E(R)$

and

(16) $\quad \mathrm{Var}(G) = \mathrm{Var}(S) + K + (U + h\,\pi)\,E(R)$

An important point is the correlation a between S and R. Using the Normal Approximation to G, i.e., arguing on the basis of equation (14), a negative correlation increases (!) σ (G) and the minimum level of E(G), a positive correlation decreases (!) a (G) and the minimum level of E(G).

3.2. Premium calculation

We consider the following problem. Given the security capital U_0, the stochastic characteristics of the accumulated claims S and the investment return R, the accumulated claim independent operating expenses K and the security level ε of the company. Then what is the minimum gross premium π that guarantees the security level of the company, i.e. the premium that gives:

(17) $\quad P(G < -U_0) = \varepsilon$

A few remarks are in order. First we do not ask for the relevant risk premium (the compensation for the pure claim costs) as it is usually done in risk theory but we ask for the calculatory gross premium necessary to maintain a given security level of the company. The so-defined gross premium not only includes the risk premium and a loading for expenses but also involves considerations to what extent investment income can be included in premium calculation. Second, we do not ask for the adequate premium of a single policy or of special collectives of risks, we want to calculate the necessary (minimum) total premium income for the entire insured collective of the company, given the characteristics of the accumulated costs and proceeds of the period, as well as the amount of security capital at disposal.

Using the Normal-Approximation, we obtain the following (implicit) equation for π on the basis of (14), (15) and (16):

$$(18) \qquad \pi = \frac{1}{1+hE(R)}\Big[E(S) + K - U_0\,(1 + E(R)) + N_\varepsilon \sqrt{\mathrm{Var}(S) + (U_0 + h\pi)^2\,\mathrm{Var}(R) - 2\,(U_0 + h\pi)\;\rho(S,R)\,\sigma(S)\,\sigma(R)}\;\Big]$$

Already the most simple-case of a normally distributed company result leads to complications if we include the stochastic dependencies between S and R as well as the dependency between the premium income and the investment capital into the analysis.

Equation (18) leads to a quadratic equation for π and is (conceptually) easily solved. However, the result leads to a formula which is unpleasant and rather difficult to interpret

To gaine some intuition we in the following look for larger, but more simple, lower bounds for π in special situations. Stability criterion **(11) remains** valid, **if** the right hand side of equation **(18)** is increased. **In** case of $\rho(S,R) \geq 0$, we eliminate the **correlation** term and obtain as larger lower bound

(19) $$\pi(1+hE(R)) \geq E(S) + K - U_0(1+E(R)) + N_\varepsilon \sqrt{Var(S) + (U_0+h\pi)^2 Var(R)} \,.$$

Using $(a,b > 0)$ $\sqrt{a+b} < \sqrt{a} + \sqrt{b}$ we obtain

(20)

$$\pi(1 + hE(R)) \geq E(S) + K - U_0(1 + E(R)) + N_\varepsilon [\sigma(S) + (U_0 + h\pi) s(R)]$$

as a **larger** lower bound. Solving (20) for n we **finally** obtain

(21)

$$\pi \geq \{1 + h[E(R) - N_\varepsilon \sigma(R)]\}^{-1} x$$

$$\{E(S) + N_\varepsilon \sigma(S) + K - U_0 1 + E(R) - N_\varepsilon \sigma(R)]\}$$

as a **lower** bound for the premium income, that guarantees **the** security level of the company. The lower bound given by **(21)** very nicely and intuitively quantifies the different influencing factors on the necessary premium **income.** The investment return induced by premium income is quantified by means of the discounting factor $1 + h[E(R) - N\varepsilon\, \sigma(R)]$. The **investment** return earned by the **security** capital and the **security** capital itself lead to subtraction factors. However, not the full expected return on investment is subtracted. the expected return has to be corrected by a term measuring the variation of the investment return.

4. IMPLICATIONS OF THE CAPM FOR THE GROSS PREMIUM

In its basic **form** the CAPM states that the expected return of a security in capital market equilibrium is given by

(22) $$E(R) = r_0 + \beta [E(R_M) - r_0$$

(23) $$\beta = \frac{Cov(R, R_M)}{Var(R_M)}$$

R_M denotes the return of the market portfolio, which is composed of all securities in the market, in the present case all stocks in the market.

r_0 denotes the riskless interest-rate and ß the beta-coefficient of the security, which is the basic measure of risk within the CAPM-framework.

To apply the CAPM to the problem of the determination of a gross premium consistent with a capital market equilibrium we have to transform the general expression (10) for the total result of an insurance company with n insurance lines and m possible types of invesunent.

Let

$$(24) \qquad R_i = \frac{\Pi_i - S_i - K_i}{\Pi_i}$$

be the return (in percent of premium) of the i-th insurance activity and a_j be the fraction of the total investment capital which is invested in investment type j:

$$(25) \qquad A_j = a_j (U + \Sigma\, h_i\, \Pi_i) .$$

From (10) we now obtain the following expression for the result of the company:

$$(26) \qquad G = \sum_{i=1}^{n} R_i\, \Pi_i + \sum_{j=1}^{m} R_j\, A_j$$

The corresponding market return on equity capital is given by the division of the total result of the company through the market value of the equity capital MEC at the beginning of the period:

$$(27) \qquad R_{MEC} = \sum_{i=1}^{n} R_i \frac{\Pi_i}{MEC} + \sum_{j=1}^{m} R_j \frac{A_j}{MEC}$$

From the basic equation (22) of the CAPM we obtain the equilibrium condition for the expected return on equity:

$$(28) \qquad E(R_{MEC}) = r_0 + \frac{E(R_M)-r_0}{MEC} \left\{ \sum_{i=1}^{n} \frac{Cov(R_i, R_M)}{Var(R_M)} \Pi_i + \sum_{j=1}^{m} \frac{Cov(R_j, R_M)}{Var(R_M)} A_j \right\}$$

Before **evaluating** (28) with **respect** to the problem of a fair market premium a separate useful analysis is performed.

Denoting

$$(29)\quad \mu_i = E(S_i),\ R_i^* = 1 - \frac{S_i}{E(S_i)}$$

we obtain from (24) :

$$(30)\quad \pi_i \operatorname{Cov}(R_i, R_M) = \operatorname{Cov}(\pi_i - S_i - K_i, R_M)$$

$$= -\operatorname{Cov}(S_i, R_M) = E(S_i) \operatorname{Cov}\left(-\frac{S_i}{E(S_i)}, R_M\right)$$

$$= \mu_i \operatorname{Cov}(R_i^*, R_M)$$

The reason that we prefer the last term instead of Cov (S_i, R_M) is that **all** the **appearing covariances** have to be **estimated** before the results can be applied **in** practice. **Therefore** they should be approximately equal for all the insurance companies in the sample. Because of different collective sizes for different companies in estimation for

$$\operatorname{Cov}\left(\frac{S_i}{\mu_i}, R_M\right)$$

will be much more stable and reliable than for $\operatorname{Cov}(S_i, R_M)$.

Now as from (8) we also have

$$(31)\quad E(R_{MEC}) = \frac{\pi - E(S) - K + (U + h\pi)E(R)}{MEC}$$

equating (26) **with (28),** solving for π, paying attention to (30) and **using** the two sets of **beta-factors** for the different insurance lines **resp.** for the different investment types

$$(32)\quad \beta_i^* = \frac{\operatorname{Cov}(R_i^*, R_M)}{\operatorname{Var}(R_M)},\ \beta_j = \frac{\operatorname{Cov}(R_j, R_M)}{\operatorname{Var}(R_M)},$$

we finally obtain.

$$(33)\quad \pi = \frac{1}{1+hE(R)} \Big\{ E(S) - UE(R) + K + r_0 MEC + \left[E(R_M) - r_0 \right] \Big(\sum_{i=1}^{n} \mu_i \beta_i^* + \sum_{j=1}^{m} A_j \beta_j \Big) \Big\}$$

This is the expression for the fair gross premium of an insurance company with n different insurance lines and m different types of investment within the framework of the CAPM.

In case the market portfolio M does not only contain all stocks in the stock market (including the stocks of the considered insurance company) but in addition all securities in a market consisting of the m types of possible investment assets, the valuation formula (33) can be considerably simplified following the lines of KAHANE (1979), pp. 233/234). For now for each return R_j a CAPM equilibrium equation of the type (22) is valid. i.e.:

$$E(R_j) = r_0 + [E(R_M) - r_0] \frac{Cov(R_j, R_M)}{Var(R_M)} \tag{34}$$

This implies

$$\sum_{j=1}^{m} \frac{A_j}{MEC} E(R_j) = \frac{r_0}{MEC} \sum_{j=1}^{m} A_j + \frac{E(R_M)-r_0}{Var(R_M)} \sum_{j=1}^{m} \frac{A_j}{MEC} Cov(R_j, R_M) \tag{35}$$

and from (27) we obtain

$$E(R_{MEC}) = \sum_{i=1}^{n} \frac{\pi_i}{MEC} E(R_i) + \frac{r_0}{MEC} \sum_{j=1}^{m} A_j + \frac{E(R_M)-r_0}{Var(R_M)} \sum_{j=1}^{m} \frac{A_j}{MEC} Cov(R_j, R_M) \tag{36}$$

Subtracting (28) from (36) we have

$$0 = \sum_{i=1}^{n} \frac{\pi_i}{MEC} E(R_i) + \frac{r_0}{MEC} \sum_{j=1}^{m} A_j - r_0 - \frac{E(R_M)-r_0}{MEC} \sum_{i=1}^{n} \frac{Cov(R_i, R_M)}{Var(R_M)} \pi_i \tag{37}$$

From

$$\sum_{i=1}^{n} \frac{\pi_i}{MEC} E(R_i) = \frac{\pi - E(S) - K}{MEC} \tag{38}$$

and (30) this gives

$$\pi - E(S) - K + r_0 A = r_0 MEC + [E(R_M) - r_0] \sum_{i=1}^{n} \mu_i \beta_i^* .$$

From (7) and solving for π we finally obtain for the fair premium:

$$(39) \qquad \pi = \frac{1}{1+hr_0} \{ r_0 (MEC-U) + E(S) + K + [E(R_M) - r_0] \sum_{i=1}^{n} \mu_i \beta_i^* \}$$

In comparison with (33), expression (39) does not contain anymore explicitely risks from the investment sector. This is because these risks are accounted for by the risk premium element which is imbedded in the expected return on each risky asset under capital market equilibrium.

However, while (33) only is based on the assumption of the validity of the CAPM-valuation for stock markets, which has been investigated extensively, cf. e.g. HARRINGTON (1983), (39) is based on the assumption that the CAPM-valuation is valid for a market consisting of stocks, bonds, real estate, mortgages etc., which still is an open question empirically.

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