Valuation of an American Put Catastrophe Insurance Futures Option: A Martingale Approach

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Summary

The goal of this paper is to develop an arbitrage-free valuation formula for an American put option on a catastrophe insurance futures contract. This contract (denoted CATS) was introduced in December 1992 by the Chicago Board of Trade. The option buyer's valuation problem is formulated as an optimal stopping problem within a continuous trading, arbitrage-free and complete financial market. The problem is then analyzed via Karatzas' unified equivalent martingale measure framework. His framework is a bit more elaborate than the Harrison-Kreps and Harrison-Pliska seminal models in which the key economic idea of the absence of arbitrage opportunities is linked to the probabilistic concept of a martingale. Specifically, Karatzas' framework enables market participants to consume as well as invest. This permits a unified approach to the problems of: option pricing, consumption and investment, and equilibrium in a financial market.

We extend Karatzas' framework to a futures option setting by replacing his risky stock with a CATS futures contract whose underlying asset is the loss-ratio index of the pool of insurers that comprise the index. This is a nontrivial extension since the loss-ratio index is a non-marketed "asset", i.e., a cashflow determined by the loss claims of insured victims of catastrophes. Thus, determining the market price of risk is difficult. In this setting the CATS futures option is a redundant asset and, thus, valued via the usual replication argument.

In addition to extending Karatzas' framework, we obtain two main financial results: 1) a representation of the arbitrage-free put value function as the expectation (under the risk-neutral probability measure) of the present value of the option payoff at the optimal exercise time; and, 2) a decomposition (via the Riesz Decomposition Theorem) of the American put value function into the corresponding European option price and the early exercise premium.

Finally, to demonstrate the method for obtaining explicit solutions for the value function and the optimal exercise (stopping) boundary, the stopping problem (for the conventional American put equity option) is reformulated (invoking the Feynman-Kac theorem) as a free-boundary (Stefan) problem. Several difficulties that hinder the numerical analysis of this problem are briefly discussed.

Evaluation d'une option américaine Put sur un contrat à terme d'assurance catastrophes : Méthode des martingales

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Résumé

Le but du présent article est d'établir une formule d'évaluation sans arbitrage pour une option américaine put sur un contrat à terme d'assurance catastrophes. Ce contrat (dénommé CATS) a été introduit en décembre 1992 par le Chicago Board of Trade. Le problème d'évaluation de l'acheteur de l'option est formulé en tant que problème d'arrêt optimal dans le cadre d'un marché financier à négociations continues, sans arbitrage, et complètes. Le problème est ensuite analysé par la méthode unifiée de Karatzas à équivalence de martingale. Cette méthode est un peu plus détaillée que les modèles de Harisson-Kreps et Harisson-Pliska dans lesquels l'idée économique clé de l'absence de possibilités d'arbitrage est liée au concept probabiliste de la martingale. Plus spécifiquement, le cadre de Karatzas permet aux intervenants sur le marché de consommer ainsi que d'investir. Ceci permet d'adopter une approche unifiée aux problèmes de l'évaluation des options, de la consommation et de l'investissement, et de l'équilibre dans un marché financier.

Nous étendons le cadre de Karatzas à celui des options sur contrat à terme en remplaçant son action à risque par un contrat à terme CATS dont l'instrument sous-jacent est l'indice de ratio de perte du pool d'assureurs compris dans l'indice. Il s'agit là d'une extension non triviale étant donné que l'indice de ratio de perte est un "bien" non commercialisé, c'est à dire un flux financier dépendant des demandes d'indemnité des assurés victimes de sinistres. Il est donc difficile de déterminer le prix du marché du risque. Dans ce contexte, l'option sur contrat à terme CATS est un bien redondant et elle est donc évaluée au moyen de la méthode habituelle de réplication.

Outre le fait d'étendre le cadre de Karatzas, nous obtenons deux grands résultats financiers : 1) une représentation de la fonction de la valeur du put sans arbitrage en tant que valeur attendue (par la mesure des probabilités à risque neutre) de la valeur actuelle du payoff de l'option au moment optimal d'exercice; et, 2) une décomposition (selon le théorème de décomposition de Riesz) de la fonction de la valeur du put américain en prix correspondant d'option européenne et en prime d'exercice précoce.

Enfin, pour obtenir des solutions explicites pour la fonction valeur et la limite optimale d'exercice (arrêt), le problème d'arrêt est reformulé (au moyen du théorème de Feynman-Kac) en tant que problème à limite libre (Stefan). Nous examinons aussi brièvement certaines des difficultés qui se présentent dans l'analyse numérique de ce problème.

Introduction

The objective of this paper is to develop an arbitrage-free valuation formula for an American put option on a catastrophe insurance futures contract. The Chicago Board of Trade introduced this contract (hereafter, called a CATS contract) in December 1992, hoping that property/casualty insurance companies would view the contract as a cheaper, flexible substitute for reinsurance. Thus, the underlying "asset" (unit of trading) tracks quarterly catastrophic losses on a pool of property/casualty insurance contracts written by 22 insurers. In particular, the value of the contract is $25,000 times the ratio of incurred cumulative losses (a random quantity) to quarterly premiums (known and constant throughout the listing period of the contract); the settlement value is capped at $50,000.

Of course, the viability of this contract is ultimately an empirical issue: will insurers go "long" an American Put[1/]? However, we are optimistic about demand for the contract, given the sharp decline in the capacity of the global reinsurance market and commensurate steep increase in price. And, reinsurance is currently the only (direct) hedging instrument for managing underwriting risk. For example, one of the major suppliers of reinsurance--Lloyd's of London--has lost $12.7 billion dollars over the 1987-1993 reporting period (New York Times, 1993). According to Hall (1992), at least 80 syndicates have disbanded with another 80 expected to fold in 1993. Thus, Lloyd's has experienced (over the last 6 years) a severe decline in its capital base from $17.2 billion to $13.1 billion. Moreover, Snyder (1993) has estimated that the costs of Hurricane Andrew will lead to increases in the cost of catastrophe reinsurance ranging from 35-300%, depending on the characteristics of the ceding company. Finally, our optimism is also based to some degree on Niehaus and Mann's (1992) theoretical result that, "insurers with low expected profits...are more likely to find short futures positions desirable."

The CATS contract is a very challenging contract to price for several reasons. First, cumulative catastrophic losses may not be lognormally distributed. This implies that standard futures options pricing models, e.g., Black (1976), Ramaswamy and Sundaresan (1985), and Whaley (1986) may be irrelevant. Second, establishing a relationship between the futures and the "spot" price is very difficult since the underlying "asset" (loss ratio index) is a non-tradeable cash flow. Moreover, the financial market may be incomplete since there does not appear to exist a traded asset which provides a perfect hedge against (or spans) the non-traded cashflow. Unfortunately, incorporating these three facets of the problem into our analysis is beyond the scope of this paper. Nonetheless, it is within this context that our results derive their significance.

We obtain three main results. First, we make a methodological contribution by extending Karatzas' (1988, 1989) unified, continuous-time option valuation framework to a futures option setting. This is done (in Sections II-IV) by replacing the risky stock with a CATS futures contract--a contingent claim on a non-marketed asset whose cash flows are modeled via cumulative changes in the resettlement (margin) account. In this assumed ideal, complete market setting, the CATS futures option is a redundant security. Thus, it is valued via the usual replication argument. Note, that Karatzas' framework is a bit more elaborate than the Harrison-Kreps (1979) and Harrison-Pliska (1981) seminal models. Here, market participants have the opportunity to consume as well as to invest. This permits a unified approach to the problems of option pricing, consumption/investment, and equilibrium in a financial market.

In this context our second result, which is financial, is a representation of the current (time t=0) arbitrage-free put value function as the expectation (under the risk-adjusted probability measure) of the present value of the option payoff at the optimal exercise time. Moreover, this value is shown (in Section V) to be equivalent to the minimal level of initial wealth which enables the put buyer to hedge (or replicate) the value of the put at any time. Note, that space limitations preclude the inclusion of proofs in this paper; please contact the author for details.

Our third result (in Section VI) is the early exercise premium representation of the put value function. This representation is obtained by decomposing (via the Riesz Decomposition theorem) the smallest supermartingale majorant to the expected put payoff function--the Snell envelope--into the sum of the corresponding European put futures option value and the early exercise premium.

The intuition of the decomposition is that the early exercise premium accumulates (over the intervals du) the flexibility value of a sequence of infinitesimal early exercise premiums. This flexibility value equals the discounted interest earnings on any credit made to the margin account for the interval du.

Since our characterization of the fair price and the optimal exercise policy of the option is in implicit form, Section VII contains a brief discussion of how to obtain an explicit solution--for a conventional equity put option--for the two unknowns of the option buyer's valuation problem. We conclude by relating this work to the literature (Section VIII) as well as suggesting (in Section IX) future research topics. Our first task, however, is to acquaint the reader with the CATS futures and futures option contracts.

I. CATS Futures and Futures Option Contracts

We briefly describe the Catastrophe Insurance (CATS) futures and futures-option contracts, focusing only on those attributes that are key determinants of contract value. A more detailed description, e.g., margin requirements, is contained in Chicago Board of Trade (1992). Note that the CBOT has adopted the insurance industry's definition of a property insurance catastrophe. That is, a catastrophe is an event which results in property/casualty insured losses of at least $5 million.

The motivation for developing the two types of contracts is to provide property insurers with the ability to hedge (on a quarterly basis) catastrophe underwriting risk; specifically, the level of insured victims' loss claims. The contract covers nine lines with five causes of losses.[2/] Thus, the contracts can be viewed as a risk-transferring contracts which ultimately may provide property and casualty insurers with a cheaper, flexible substitute for reinsurance.

A. CATS Futures

The CATS futures contract is traded on a quarterly basis and is cash settled. The underlying "asset" is a loss-ratio index on a pool of catastrophe insurance contracts. The loss-ratio index is defined as the ratio of quarterly losses (L) to quarterly premiums (P) of contracts included in the CATS pool. More specifically, quarterly losses are defined as:

$$L=\sum_{i=1}^{9}\sum_{j=1}^{51}\frac{\sum_{k=1}^{22}L_{ijk}}{\sum_{k=1}^{22}w_{ijk}}$$

where:

i= index of insurance lines included in index, i=1,2,...,9

j= index of states plus Washington, DC, j=1,2,...,51

k= index of insurers included in pool, k=1,2,...,22

L_{ijk} = total losses during the loss quarter reported by insurer k in state j for line i

w_{ijk} = weights that represent the proportion of premium income earned by insurer k in state j for line i, relative to the total premium income of all insurers in state j for line i.

(NOTE: the weights are held confidential by CBOT).

The denominator of the index is the amount of quarterly premiums collected for the pool of contracts. More specifically, quarterly premiums amount to 25% of the estimated (net) earned premiums that accrue to the policies in the pool during the calendar year. Quarterly premiums are computed prior to each quarter and their magnitude is announced by the CBOT prior to contract trading. Thus, quarterly premiums are known and constant throughout the listing (trading) period of the contract.

However, the numerator is a random quantity: the amount of <u>cumulative</u> claims generated (over a six month period for each contract) by the pool of catastrophe insurance contracts. Thus, a change in the price of a futures contract price is determined entirely by a change in the amount of unexpected catastrophic losses. Obviously, this is one type of underwriting risk that property insurers wish to hedge.

Note, that this accumulation of loss feature is the major difference between the CATS and a conventional futures contract. Usually, the value of a futures contract at any point in time is a function of the magnitude of the "price" of the underlying asset at that point in time; not the history of price movements. Finally, the value of each quarterly contract will be equal to $25,000 times the loss-ratio index value (L(t)). Final settlement value (denoted by F(T)), however, will equal: F(T)=MIN[$50,000, $25,000 x L(T)].

The final key feature is that there is no spot market for the underlying asset. That is, the loss-ratio index is neither a marketed financial security nor a homogenous commodity with a publicly observable spot price. Instead, it represents the cashflows of the insurers who comprise the pool of policies upon which the loss-ratio index is computed. The implications of these features will be drawn out in the sequel.

B. CATS Futures Option

The CATS futures option contracts are American style. Thus, the buyer of an American put has the right, but not the obligation, at any time prior to expiration, to exercise her option and, thereby, assume a short position in one CATS futures contract. The maturity month and strike price of the futures are set at the time the option was purchased. When the put holder exercises her option, she automatically acquires a short position in the corresponding CATS futures contract and receives a cash flow equal to the exercise price minus the futures price at the exercise date. Of course, the sign of the cash flow equals the sign of the difference between the two prices. Finally, the payoff for the buyer of an American puts CATS futures option is expressed as:

$$\text{Put Buyer's Payoff} = \text{Max}[0,\ q-F(t)], \quad \text{for all } t\varepsilon[0,T],$$

where q denotes the exercise price of the option.

For exercised American calls, the holder of the call automatically assumes a long position and receives a cash flow equal to the futures price at exercise minus the exercise price. The seller automatically becomes "short" the futures. Note, that the buyer of the futures option does not pay the exercise price when exercising the option (as is done in the case of options on shares of stock).

II. Financial Market Model

We now formulate a financial market model in which market participants (hereafter called agents) can continuously trade, over a finite time interval $[0,T]$. Each agent's economic behavior occurs in the following probabilistic setting.

A. Probabilistic Setting

We assume that futures contract price information is continuously revealed through time. All agents have no uncertainty about the present ($t=0$). However, they are uncertain about the future. Finally, at $t=T$ the true state of nature is revealed. This formulation is captured with the following model.

We take as primitive a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}, P)$. This space supports a Standard, one-dimensional Brownian motion (denoted by $W(t)$) over a finite horizon T. Let $\{\mathcal{F}(t)\}$ denote the augmentation under P of the natural filtration

(2.1) $$\mathcal{F}^W(t) = \sigma(W(s));\ 0 \leq s \leq t); \qquad t \in [0,T]$$

generated by the Brownian motion, where $\sigma(W(s);\ 0 \leq s \leq t)$ represents the smallest σ-field with respect to which the random variable $W(s)$ is measurable for every s. In this set-up, $\mathcal{F}(t)$ represents the information available to each agent at time t. $\{\mathcal{F}(t)\}$ satisfies the "usual conditions": $\{\mathcal{F}(t)\}$ is right-continuous, and $\mathcal{F}(0)$ contains the P-null events in $\mathcal{F}^W(T)$, with $\mathcal{F}=\mathcal{F}(T)$, the terminal σ-field. Finally, we make the substantive assumption that

A-2.1: Agents have homogenous beliefs regarding futures prices and information is symmetrically distributed among all participants, i.e., $\{\mathcal{F}(t)\}$ and P are common to all agents.

B. Futures Market

The futures market contains K CATS futures contracts, i.e., the current set of contracts cover K different listing periods, that are assumed to be continuously traded. In this section we specify the evolution of the prices of the futures contracts and the associated underlying "assets" which determine the settlement value of the futures contracts. These assets are the non-marketed cash flows comprising the loss-ratio index.

1. Underlying Asset: Loss-Ratio Index

To simplify matters, we will make the working assumption that the only type of uncertainty in the loss-ratio index is the level of loss claims paid to insured victims of catastrophes, given that catastrophes have occurred. Moreover, we invoke Shimko's (1992) arguments, to justify our interpretation of these discrete claims payments as a continuous cash flow stream. Hence, we assume

A-2.2: At any time t of the futures contract listing period, the amount of claims $S_i(t)$ paid to policyholders, covered by the pool of policies comprising the index underlying the futures contract (indexed by i), is described by a diffusion process with the stochastic differential representation: for all $t\varepsilon[0,T]$

(2.2) $$dS_i(t)=S_i[\mu_i(t) + \Sigma_{j=1,d}\sigma_{ij}dW_j(t)], \quad i=1,2,\ldots,K$$

where $W_j(t)$ is a Standard Brownian Motion under P and $S_i(0)>0$. We denote by $\mu(t)\varepsilon R^K$ and $\sigma\varepsilon R_+^{Kxd}$ the expected growth rate and instantaneous standard deviation (commonly called volatility) of the claims paid process, respectively. Note, that (2.2) embodies the assumption that

A-2.3: Aggregate loss claims paid over the finite interval [0,T] are lognormally distributed.

This may be viewed as a useful approximation. However, even if loss claims paid at any point in time evolve according to geometric Brownian motion, cumulative claims paid don't. The reason is that the sum of lognormally distributed random variables is not distributed lognormally.

2. CATS Futures Contract

We assume that the prices of the K futures contracts evolve according to a K-dimensional diffusion process, having the stochastic differential representation: for all $t\varepsilon[0,T]$

(2.3) $$dF_i(t) = F_i(t)[m_i(t)dt + \Sigma_{j=1,d}\sigma_{ij}d\Phi_j(t)], \quad i=1,2,\ldots,K$$

with $F_i(0)>0$. The "drift" or instantaneous expected growth rate of F(t) is denoted by $m(t)\varepsilon R^K$. The volatility of F is denoted by $\sigma\varepsilon R_+$, a Kxd-dimensional matrix.[3] The volatility measures the instantaneous intensity with which our d sources of uncertainty affect the "price" of the futures contract. Finally, we make several substantive assumptions:

A-2.4: The futures market is "ideal" since the futures contract is infinitely divisible and there are no: arbitrage opportunities, transaction costs, taxes or constraints on short-selling.

A-2.5: No convenience yield, dividends, or storage costs accrues to the holder of the underlying asset, i.e., the loss-ratio index.

C. Agent's Trading, Savings, Consumption and Wealth Processes

The agent is initially endowed with an amount of cash $x \geq 0$ that is either: invested in the futures contract, saved, or consumed. We model the agent's trading behavior in the futures market by introducing the notion of a trading strategy.[4]

1. Trading Strategy

DEF 2.1: A trading strategy, denoted by the K-dimensional process $\Theta(t)$ in R^K, is: progressively measurable with respect to $\{\mathcal{F}(t)\}$; assumes values in $[0,T]$; and, satisfies square-integrability.

We interpret $\Theta_i(t)$ as the number of futures contracts (in listing period i) held after trading at time t. By convention $\Theta_i(t)<0$ denotes a "short" position in the ith-listing period futures contract. Finally, we make the common assumption that

A-2.6: The agent is a price-taker. That is, the agent's trading decisions cannot affect equilibrium prices in the futures market.

2. Margin Account

Upon taking a futures position, the agent pays an initial margin fee and begins marking-to-market a margin account based on her position (or trading strategy) in the futures contract. At time t, the value of the margin account is denoted by $V^{\Theta}(t)$. As the futures price changes, the agent's futures position is either credited with gains or debited for losses. Of course, the credits are added to the margin account while the debits are deducted. Finally, the account earns interest at the constant, continuously compounding rate $r>0$. Thus, the margin account has the following continuous-time formulation

(2.4) $V^{\Theta}(t) = \int_0^t \exp[r(t-s)]\Theta(s)dF(s)$

This form has the following interpretation. V(0) is interpreted as the initial margin payment. Then, at time $s \neq t$, the margin account experiences an "incremental" change of magnitude $\Theta(s)dF(s)$. This magnitude is re-invested at the rate r. Thus, by time t, the account receives an increment of $\exp[r(t-s)\Theta(s)dF(s)$. For subsequent modeling purposes, (2.4) can be reexpressed in more convenient, equivalent form by applying Ito's Lemma and obtaining the stochastic differential equation

$$dV^{\Theta}(t) = rV^{\Theta}(t)dt + \Theta^{*}(t)dF(t)$$

(2.5) $$= (rV^{\Theta}(t) + \Theta^{*}(t)m(t))dt + \Theta^{*}(t)\sigma d\Phi(t),$$ where

* denotes the transpose operator. Finally, we assume that

A-2.7: The maintenance margin is equal to zero and other institutional details are ignored.

A-2.8: Losses, which are sufficiently large to cause $V^{\Theta}(t)<0$, for all $t \in [0,T]$, are covered by borrowing from the savings account (to be discussed shortly) at the interest rate r.

3. Savings Account

The non-risky savings account, denoted by B(t), represents the time value of money. It also serves as a unit of account (or numeraire) for the financial market. The value of B(t) evolves according to the following stochastic differential equation:

(2.6) $$dB(t) = rB(t)dt,$$

The solution of (2.6) is:

(2.7) $$B(t) = B(0)e^{rt}, \qquad \text{for all } t \in [0,T],$$

where $B(0) \equiv 1$, by convention.

In this model, B(t) determines the following discount factor:

(2.8) $$\beta(t) \equiv 1/B(t) = \exp(-rt), \qquad t \in [0,T].$$

Note that the savings account is similar to a risk (default)-free discount bond that earns interest at a constant, continuously compounded rate.

We assume that all money not invested in the futures contracts, plus any positive balance in the margin account, are deposited in the savings account. Thus, the amount held in the savings account is denoted by

(2.9) $$B(t)=X(t) + V^{\theta}(t),$$ where

$X(t)$ denotes the agent's wealth at time t. Remember, the maintenance margin is assumed to be zero. Thus, whenever $V^{\theta}(t)>0$, the agent (per standard practice) can withdraw this amount from the margin account and place it in the savings account where it earns interest at the constant, continuously compounded rate r. Also, when $V(t)<0$, we assume that the agent can borrow $|V(t)|$ at the constant rate r.

4. Cumulative Consumption Process

The agent's cumulative consumption behavior is defined as

DEF 2.2: A cumulative consumption process, $C=\{C(t),\mathcal{F}(t);\ 0\leq t\leq T\}$,

(2.10) $$C(t)=\int^{T}_{0}c(t)dt<\infty,$$

(where $c(t)$ is a consumption rate process), is nonnegative, progressively measurable with respect to $\{\mathcal{F}(t)\}$, and satisfies: i) $C(0,\omega)=0$; and, ii) the path $t\mapsto C(t,\omega)$ is nondecreasing and right-continuous for P-a.e., $\omega\varepsilon\Omega$.

5. Wealth Process

The agent's total wealth at time t, denoted by $X(t)$, consists of: the payoff from margin, $V^{\theta}(t)$, and savings accounts, $B(t)$, minus cumulative consumption through t, $C(t)$. $X(t)$ evolves according to the following stochastic differential representation:

(2.11) $$dX(t)= dV^{\theta}(t) + dB(t) - dC(t)$$
$$= rX(t)dt +[\Theta^{*}(t)m(t) + rV^{\theta}(t)]dt + \Theta^{*}(t)\sigma d\Phi(t) - dC(t).$$ The solution to this stochastic differential equation is

(2.12) $$\beta(t)X(t) = x - \int^{t}_{0}\beta(s)dC(s) + \int^{t}_{0}\beta(s)\{[\Theta^{*}(s)m(s)+\sigma d\Phi(s)]+rV^{\theta}(s)\}ds,$$ for all $t\varepsilon[0,T]$, where $X(0) = x\geq 0$, the agent's initial endowment. Finally, we assume that

A-2.7: The agent prefers more wealth to less, i.e., she is non-satiated in wealth.

III. Elimination of Arbitrage Strategies

So far, we have not placed any restrictions on the agent's choice of trading and consumption strategies. However, in our financial market, there are two types of strategies--collectively called arbitrage strategies--that an agent can use to make arbitrage (self-financing, risk-free) profits or "free lunches." [5/] An arbitrage strategy is defined as a trading strategy that enables an agent--who has no initial wealth--to achieve at terminal time T a positive amount of terminal wealth (with positive probability) and no possible loss (for sure). More formally, we have

DEF 3.1: We say that a trading strategy Θ is an <u>arbitrage strategy</u> when the wealth process $X(t)$, which corresponds to strategy Θ as well as zero-valued cumulative consumption and initial endowment, satisfies $P[X(t)>0]>0$ and $P[X(T)\geq 0]=1$.

The first "type" of arbitrage strategy involves the following sequence of actions, given that the futures and spot price of the underlying asset do not obey an arbitrage relationship. First, the agent acquires a short position in a futures contracts. Then, the agent uses borrowed funds to purchase one unit of the underlying asset. Finally, when the futures contract settles, deliver the asset and, in turn, use the receipts (from the "long") to repay the loan. Thus, at settlement the agent has made a riskless profit, without using her own money. The second type of arbitrage strategy is called a "doubling strategy." It is the continuous-time analogue of a roulette strategy which is self-financed via the agent's reliance on borrowed funds. Again, this type of strategy will (eventually) generate a win that, in turn, enables the agent to recoup all losses, assuming she can borrow an unbounded amount.

We want to exclude arbitrage strategies since a price system (in which rational agents prefer more to less) that contains arbitrage opportunities can never be an equilibrium price system. The reason is that non-satiated, rational agents will assume large (in fact, unbounded) positions in an arbitrage strategy. Hence, the financial market will not clear; obviously, an undesirable result.

A. <u>Short-Selling and Equivalent Martingale Measure</u>

The possibility of a short-selling arbitrage strategy in futures markets is widely known (Duffy, 1992). In such a strategy, by definition, the agent will be able to earn a positive return, without any initial investment. Of course, by increasing the amount of the contract sold short, the agent can increase her wealth without bound.

To eliminate this type of arbitrage strategy, we have to demonstrate that (with a reasonable alteration of our market model) a self-financing trading strategy can be found which yields a terminal wealth level whose current ($t=0$) discounted value is zero. This is achieved by transforming the discounted futures price process into a martingale; thus, making futures prices fair bets. With prices as fair bets, it is straightforward to show that the discounted capital gains process is a supermartingale. Thus, its terminal discounted value is zero. Hence, we can conclude that the discounted value of terminal wealth is equal to the agent's initial endowment. And, for a self-financing strategy the agent's initial endowment is zero, by definition.

The transformation entails normalizing the futures price (by dividing it by the price of the numeraire--the riskless asset) and then adjusting the expected returns so as to remove the risk premium on the futures contract. The latter step requires a change in the probability measure to an equivalent measure that is a martingale measure. Note, that this change of probability subsumes risk aversion since the risk premium on the futures contract is removed. Thus, the equivalent martingale measure is simply the continuous-time analogue of the familiar Cox and Ross (1976) risk-neutral probability.

We begin the transformation by defining the following process. It is interpreted, following Shimko (1992), as the local market price of assuming the claims level risk in dW, assuming a constant risk premium:

$$k(t)\equiv\sigma^{-1}m(t) \tag{3.1}$$

Note, that since σ is a non-square matrix with $K<d$, we interpret σ^{-1} as the right inverse. Associated with $k(t)$ is a martingale

$$\eta(t)\equiv\exp\left[-\int_0^t k^* d\Phi(s) \; -.5\int_0^t \|k(s)\|^2 ds\right], \tag{3.2}$$

where $\eta(t)$ is a lognormal random variable since $\Phi(s)\sim N(0,1)$. Hence, $\eta(t)$ is a strictly positive random variable with $E[\eta(t)]=1$ and finite moments. Next, we create a new probability measure, P_o, from the original probability measure P by defining

(3.3) $P_o(A) \equiv E[1_A \eta(t)]$, for all events $A \varepsilon \mathcal{F}(T)$,

where A is any distinguishable event at time T. 1_A is an indicator function of the event A, taking 1 if the true state of nature lies in A; otherwise, taking a value of zero. Thus, $\eta(t)$ is called the Radon-Nikodym derivative of P_o with respect to P and denoted

(3.4) $dP_o/dP = \eta(t)$

Note, that P and P_o are equivalent probability measures in the sense that they assign a probability of zero to the same events. That is, $P_o(A)>0$ whenever $P(A)>0$ and vice versa. Moreover, P_o is an equivalent martingale measure since the discounted price process $\beta(t)F(t)$ is a martingale with respect to P_o and dP_o/dP has finite variance.

To verify this claim, we appeal to Girsanov's (1960) theorem since it provides a recipe for adjusting probability assessments so that a given Ito process can be rewritten as an Ito process with almost arbitrary drift (Duffie, 1992). Girsanov's theorem implies that, under P_o, the process

(3.5) $\Phi_o \equiv \Phi(t) + \int^t_0 k^*(s)ds$, for all $t \varepsilon [0,T]$,

is a P_o-Standard Brownian Motion on $(\Omega, \mathcal{F}(T), P_o)$ that has the same standard filtration $\{\mathcal{F}(t)\}$. Thus, via Girsanov's theorem, we can reexpress the discounted futures price process as

(3.6) $\beta(t)F(t) = \exp[-\int^t_0 k^*(s)ds - .5\int^t_0 \| k(s) \|^2 ds + \sigma\Phi(t)]$, for all $0 \leq s \leq t \leq T$. Thus, we can quickly conclude that the discounted futures price process $(\beta(t)F(t))$ is a martingale under P_o, i.e.,

(3.7) $E_o[\beta(t)F(t) | \mathcal{F}(s)] = \beta(s)F(s)$.

Note, that when substituting P_o for P, we change only the drift term of the discounted futures price process; σ is unaffected. This can be easily demonstrated by applying Ito's lemma to (3.6) and obtaining

(3.8) $d(\beta(t)F(t)) = \sigma\beta(t)F(t)d\Phi_o(t)$

Finally, to aid the reader's understanding of this process, note that the equivalent martingale measure, P_o, is simply the familiar risk-neutral probability of Cox and Ross (1976). This is easily seen by noting that, after replacing P with P_o, the undiscounted futures price process (2.3) must evolve according to

(3.9) $dF_i(t) = F_i(t)\Sigma_{j=1,d}\sigma_{i,j}d\Phi_{o,j}(t)$, $i=1,2,\ldots,K$.

Observe that the instantaneous expected rate of return (drift) of F is equal to the riskless rate, as in Cox and Ross (1976). Hence, after substituting (3.8) into the wealth equation (2.12), we obtain

(3.10) $ß(t)X(t) = x + \int^t_0 ß(s)\Theta^*(s)\sigma d\Phi_o(s) - \int^t_0 ß(s)dC(s)$

The first integral on the right-hand side is the discounted gains process M(t). It is a zero-mean, P_o-local martingale and a supermartingale. Thus, $E_o[M(T)]\leq E_o[M(0)]=0$. But, since $M(T)\geq 0$, this relationship implies that $M(T)=0$. Thus, eliminating arbitrage strategies.

In sum, the discounted, risk-adjusted futures price process becomes (under P_o) an Ito integral with no local drift and, therefore, a fair gamble over brief time intervals. This strategy for eliminating the short-selling type of arbitrage trading strategy is encapsulated in the following theorem:

THEOREM 3.1: If the market model contains no short-selling arbitrage trading strategies, then there exists an $\mathcal{F}$-adapted, real-valued process k(t) such that

(3.11) $k(t)\equiv\sigma^{-1}m(t)$

Conversely, if such a k(t) exists and:

i) is bounded, i.e., $\int^T_0 \| k(t)\|^2 dt<\infty$ as well as satisfies,

ii) $\eta(t)\equiv\exp[-\int^t_0 k^* d\Phi(s) \;-.5\int^t_0 \| k(s)\|^2 ds]$, which is a P_o-martingale. Then, the market model contains no short-selling arbitrage trading opportunities.

Unfortunately, the agent can still find it feasible to execute a doubling strategy (as might be used by a roulette player on a fair wheel, i.e., the green slots 0 and 00 are omitted). So, we have to exclude self-financing, doubling strategies.

B. <u>Doubling Strategies and Equivalent Martingale Measures</u>

A doubling strategy is analogous to that of a roulette player who (with borrowed funds) doubles her/his bet every time she/he loses. By waiting (in a discrete-time, infinite horizon setting) for the eventual win, all losses are recovered for sure. Thus, this strategy creates guaranteed profits for a player who plays only with borrowed funds. For example, suppose that the agent is playing a coin flipping game in which she wins \$1 for every \$1 she bets when the coin comes up heads. Also, she loses \$1 for every \$1 that she bets when the coin comes up tails. The agent's betting strategy is to double her bet when she loses. Thus, if she loses the first n coin flips and then wins, her net winnings (W) will be:

(3.12) $$W = \Sigma_{i=0,n-1} 2^i + 2^n = \$1.$$

As Dybvig and Huang (1989) indicate, there are three equivalent ways to eliminate arbitrage opportunities in continuous-time, finite horizon financial market models. We follow their lead and invoke a nonnegative wealth constraint (for all $t \in [0,T]$) in order to eliminate doubling strategies that generate an arbitrage profit. The motivation for the wealth constraint is the realization that an arbitrage strategy must have a corresponding wealth process that is unbounded from below. The nonnegative wealth constraint is:

(3.13) $$X(t) \geq 0, \text{ for all } t \in [0,T]$$

The main advantage of this approach, relative to other approaches, e.g. the integrability condition of Harrison and Pliska (1981), is that the nonnegative wealth constraint on trading strategies is rather easy to motivate from an economic perspective. In particular, nonnegative wealth can be thought of as a limit on the agent's ability to borrow or obtain credit. Moreover, a nonnegative final wealth constraint, $X(T) \geq 0$, forces the agent to repay all money that was borrowed. Finally, this constraint eliminates the possibility that the agent can declare bankruptcy at some $t<T$.

In sum, we have eliminated two types of arbitrage strategies from the financial market model by assuming the existence of a equivalent martingale measure and invoking a nonnegative wealth constraint on the agent's trading, saving, and consumption behavior. Trading and consumption strategies, which result in nonnegative wealth (for all $t \in [0,T]$), are thus called admissible, per the following definition.

DEF 3.2: A pair (Θ, C) of portfolio and consumption processes is said to be admissible for the initial endowment $x>0$ if the wealth process of (3.10) satisfies: (3.13) $X(t) \geq 0, \quad 0 \leq t \leq T$, a.s.

Finally, note that in the sequel, we shall call the market model "standard," if it contains no arbitrage strategies.

IV. Market Completeness

As we shall soon observe, the futures option valuation problem can be greatly simplified if all the risk represented by the underlying Brownian motion W can be hedged by investment in the set of available futures contracts and the risk-free savings account. The standard market model is said to be "complete" if and only if all risks can be completely hedged by a dynamic trading strategy involving the futures contract and the risk-free savings account. In this section, we characterize market completeness via the squareness and non-singularity of the matrix-valued volatility process. A corollary of the theorem is the more familiar result of Harrison and Pliska (1983): the market is complete if and only if the Yield-Equivalent Martingale Measure (P_0) is unique.

1. Attainability

Consider a generic contingent claim which we shall represent as a non-negative random variable D. Let D be related to the initial endowment (x) in the following way:

(4.1) $x \equiv E_0(\beta(t)D) < \infty.$

In this context, the concept of "attainability" is defined as follows

DEF 4.1: We say that the contingent claim D is attainable if, for some admissible trading strategy $\Theta(t)$, we have $X^{x,\Theta}(T)=D$, a.s.

The idea of attainability is that the investor can replicate the value of the contingent claim from an initial endowment of wealth x by using x to invest in and adroitly manage (over time) the trading strategy $\Theta(t)$, while consuming according to the process C(t) as the agent's wealth evolves over the horizon [0,T].

2. Completeness

The notion of attainability motivates the following definition.

DEF 4.2: The financial market is said to be complete if every contingent claim is attainable. Otherwise, M is called incomplete.

The following theorem characterizes market completeness in terms of properties of the volatility matrix; it is a key result.

THEOREM 4.1: Completeness and Volatility Matrix. A standard market is complete if and only if σ is a square, non-singular matrix.

Outline of PROOF: 1) Sufficiency. The objective of this argument is to demonstrate that every contingent claim is attainable. Begin by supposing that the filtration F is generated by the process $W_o(t)$, a Brownian motion under P_o. Then, for any contingent claim D, the bounded (finite second-moment), nonnegative martingale under P_o (and consistent with the filtration $\mathcal{F}$) is

$$(4.2) \qquad M(t)\equiv E_o[ß(T)D|\mathcal{F}(t)].$$

It can be represented, via the martingale representation theorem (Kunita and Watanabe, 1967), as a stochastic integral with respect to $\{W_o(t)\}$:

$$(4.3) \qquad M(t) = M(0) + \int^t_0 \Psi'(s)d\Phi_o(s), \qquad t\varepsilon[0,T]$$

where $M(0)=x$ and $\Psi\equiv ß(s)\Theta(s)\sigma$ is a measurable, $\mathcal{F}$-adapted (thus predictable) process which satisfies:

i) $\Psi\ \varepsilon\ \text{Range}(\sigma^*(t))$, $\qquad t\varepsilon[0,T]$, a.s.,

ii) $\int^T_0\|\Psi(t)\|^2dt<\infty$.

Then, after several steps, one can demonstrate that $E_o[M(T)]=E_o[ß(t)D]-x=0=E_o[M(0)]$. Thus, the attainability condition is satisfied.

2) Existence. The objective of this argument is to demonstrate, via contradiction, that the kernel (null space) of the matrix σ has zero value. This implies that σ is square (K=d) and non-singular.■

Typically, market completeness is characterized according to the following corollary.

Corollary 4.2: Completeness and Uniqueness of EMM. A standard market is complete if and only if P_o is the unique equivalent martingale measure.

PROOF: Since σ is square and non-singular, we know that equation (3.1) has a unique solution: $k(t)\equiv\sigma^{-1}m(t)$. Hence, P_o is unique since P_o is completely determined by its Radon-Nikodym derivative $\eta(t)$. And $\eta(t)$, in turn, is solely determined by k. ■

These results imply that the agent can employ a dynamic trading strategy (using only the futures contract and the savings account) to reconstruct the contingent claim from the initial endowment of wealth x. More specifically, the agent can translate, via an uncountably infinite number of trading opportunities, a finite number of traded futures contracts into an infinite-dimensional attainable final wealth space.

In sum, we have constructed an arbitrage-free financial market that can act as a "platform" for the arbitrage-free valuation of contingent claims. Specifically, we have demonstrated that a reasonable price system should contain futures price processes that are martingales, after a normalization (relative to the riskless asset numeraire) and a change of probability measure. In particular, the change of probability measure is equivalent to adjusting the expected returns on futures contracts to remove risk premia. (Babbs and Selby, 1992). Consequently, we have transformed the normalized price process of the futures contract into a martingale, thereby making investment in the futures contracts a "fair bet." Let us now proceed to the major objective of this paper: determining the arbitrage-free price of the CATS insurance futures option contract.

V. Valuation

To reiterate, the agent's valuation problem is: what is a "fair" or arbitrage-free price to pay today (t=0), or at any future time $t\varepsilon[0,T]$, for a CATS future option contract? The fair price is a useful benchmark since the agent can use it to ascertain mispricing in the futures option market and, thereby, determine the effectiveness of either hedging or speculative positions.

A. Valuation via Replication of Cash Flows

The central idea underlying the arbitrage-free approach to the pricing of traded (and non-traded) "assets" is that: two cash flows which (in the absence of arbitrage) have the same present value must have the same market price. An immediate consequence of no arbitrage is the Law of One price (Dybvig and Ross, 1989).

Hence, we begin by supposing that the agent can use her initial endowment of wealth x to: i) purchase a portfolio of futures contracts, ii) dynamically manage the portfolio, savings account and consumption so that wealth is never negative and, iii) replicate the payoff stream of the option. Observe that if, instead of investing her wealth to purchase the replicating portfolio, the agent decided (at time t=0) to purchase the CATS futures option contract, she should have paid no more than x for it. The reason is that she could have replicated the cash flows of the CATS futures option contract by simply investing an amount x in the portfolio. Thus, by the Law of One Price, the "fair" or arbitrage-free price of the CATS buyer is the smallest amount of initial endowment $x \geq 0$ that enables the buyer to dynamically replicate the cash flows of the contract via appropriate use of her endowment. To sharpen the argument, we employ the following definitions.

B. Definitions

DEF 5.1: An **American Insurance Futures Put Option**--denoted by CATS(T,Y(t))--is a contingent claim with the following properties:

i) a finite expiration date $T \varepsilon (0,\infty)$, ii) the option exercise price is $q \geq 0$, iii) the payoff process of the claim, denoted by $Y(t)=(q-F(\bullet))^+$, is a continuous, non-negative, $\{\mathcal{F}(t)\}$-adapted process with finite expectation, i.e., $E_0[\sup_{0 \leq t \leq T}(\beta(t)Y(t))]<\infty$, and

iv) the option is exercisable at any stopping time $\tau \varepsilon \mathfrak{I}_{(0,T)}$ of the filtration $\{\mathcal{F}(t)\}$, with associated payoff on exercise of $Y(\tau)$.

Note, that by restricting the buyer's exercise policy to a stopping time that belongs to the set of stopping times taking values in [0,T], i.e., the set $\mathfrak{I}_{(0,T)}$, we are requiring that the exercise decision be determined by the information accumulated to date; not by the future levels of aggregate claims. Thus, (iv) excludes the possibility of insider trading and/or clairvoyance.

The agent replicates the cash flows via a hedging strategy which is defined as follows.

DEF 5.2: For the finite horizon T>0, let the agent possess: i) an initial endowment of wealth $x \geq 0$; ii) an admissible (for x) trading and consumption pair $(\Theta,C) \varepsilon A(T,x)$; and, iii) a corresponding wealth process X(t). We say that (Θ,C) is a **hedging strategy** against the CATS(T, Y(t)), and denote it by $(\Theta,C) \varepsilon H(T,x)$, if the following requirements hold: i) $X^{x,\Theta,C}(t) \geq Y(t)$ $0 \leq t \leq T$, ii) $X^{x,\Theta,C}(T)=Y(T)$, and

iii) $C(t)=\int_t^T dV(u)1_{(F(u)<F(u)^*)}du$.

To construct such a hedging strategy, the agent needs some minimal endowment of initial wealth. That amount is, in turn, the fair price of the CATS contract, as indicated by the following definition.

DEF 5.3: The **fair price** today (t=0) for the CATS(T, Y(t)) is the number

(5.1) $v(0)\equiv\inf(x>0|\exists(\Theta,C)\varepsilon H(T,x)$ such that $X^{x,\Theta,C}(t)\geq Y(t),\ 0\leq t\leq T)$.

B. Rational Exercise Problem

Begin by supposing that the agent has selected a particular exercise time $\tau'\varepsilon\mathfrak{I}_{0,T}$ in the definition of the CATS contract. Since the American option offers the buyer more flexibility (with respect to the timing of the exercise decision) than a corresponding European option, common sense dictates that the fair value of the American option (at this particular exercise time) should be no smaller than the fair value, denoted by u(0), of the corresponding European option. Therefore, after replacing the expiration date (T) of the European option with the particular exercise time, and interpreting the American CATS option payoff Y as a European option payoff, we would expect that the current fair price of the American option (exercised at some hypothetical time τ') would equal

(5.2) $$v_{\tau'}(0)=E_0[\beta(\tau)Y(\tau)].$$

Now, let us drop the supposition that a particular exercise time has been chosen. Then, recall that our agent is presumed to be rational in the sense that she prefers more wealth to less (non-satiation in wealth). In this realistic context, the agent presumably would want to choose the exercise time which maximizes her current (t=0) American option payoff. This intuition is formally captured by the following conjectured relationship:

(5.3) $$v(0) = \sup_{\tau\varepsilon\mathfrak{I}_{0,T}} u_\tau(0) = \sup_{\tau\varepsilon\mathfrak{I}_{0,T}} E_0[\beta(\tau)Y(\tau)].$$

From an analytical perspective, the preceding discussion indicates that the option buyer's valuation problem is to choose an option exercise policy (stopping time) that maximizes the current arbitrage-free value of the CATS contract. Hence, the value of the option is given by the supremum (over all exercise policies) of the conditional expectation. So, the rational exercise problem is:

(5.4) $$z^*(0)=\sup_{\tau\in\mathfrak{I}_{0,T}} u_\tau(0)=\sup_{\tau\in\mathfrak{I}_{0,T}} E_0[\beta(\tau)Y(\tau)],$$ where

$z^*(0)$ is today's optimal discounted reward of the stopping problem.

Sufficient regularity conditions for the existence of a solution to this optimal stopping problem are specified in the following existence theorem.

THEOREM 5.1: Existence of Rational Exercise Policy. Suppose that the following regularity conditions hold: i) the American CATS put option payoff process $Y(t)$ is a non-negative, continuous process, ii) $E[Y^q_*]<\infty$ for some $q>2$, where $Y_*=\sup Y(t)$ over all $t\varepsilon[0,T]$, and

iii) the risk-free rate r is bounded. Then, there exists a rational exercise policy τ^* that solves the rational exercise problem. Also, the optimal payoff $Y(\tau^*)$ has a finite expectation under P_0.

In our market model the American CATS put options future contract satisfies condition (i) by definition. Condition (ii) is satisfied since the payoff of the CATS put option is bounded by its exercise price. Finally, condition (iii) is satisfied since r is constant in our market model.

VI. Results

This section contains the main financial results: a characterization of the fair price and option holder's optimal exercise policy as well as the early exercise premium representation of the put option value function.

A. Lower Bound of Fair Price

We begin with an intermediate result. That is, the maximum reward associated with the optimal solution of the rational exercise problem is a lower bound on the fair price of the option.

LEMMA 6.1: Lower Bound of Fair Price. Suppose that: i) $x\geq 0$ is a number for which there exists a hedging strategy $(\Theta,C)\in H(T,x)$, and ii) $z(t) \equiv \sup_{t\varepsilon\mathfrak{I}_{0,T}} EQ(\tau)$, where $Q(t)\equiv\beta(t)Y(t)$, $0\leq t\leq T$. Then, the current $(t=0)$ fair price for the CATS is greater than or equal to the discounted (to time zero) expected reward $z(0)$, i.e., $v(0)\geq z(0)$.

B. Existence of Hedging Strategy

Our next result is a verification of the existence of a hedging strategy, whose initial wealth is the number z(0), against the CATS contract.

THEOREM 6.1: Existence of Hedging Strategy. Suppose that we have: i) a standard, complete financial market with constant, deterministic: interest rate (r>0), expected instantaneous growth rate r for a futures contract, and volatility σ, ii) a CATS futures option contract that is a redundant financial security at time τ, given any exercise policy τ, and iii) a nonnegative level of initial wealth for the buyer of the contingent claim, i.e., $x \geq 0$. Then, there exists a hedging strategy $(\Theta, C) \in H(T, z(0))$ such that:

i) the corresponding unique, continuous wealth process X(t), with initial wealth x=z(0), satisfies

$$(6.1)\ X(t) \equiv X_{(t)}^{v(0),\Theta,C} = \frac{1}{\beta(t)}\ ess\ sup_{\tau \in \mathcal{L}_{t,T}}\ E_o[\beta(\tau)Y(\tau) \mid \mathcal{F}(t)]\ a.s.\ for\ every\ t \in [0,T]$$

ii) $C(t) = \int^T_t dV(u) 1_{\{F(u)<F(u)^*\}} du + \int^t_0 \exp(rs)\, dA_s$.

iii) the consumption process C is absolutely continuous, nondecreasing and constant (or flat) off $\{t \in [0,T] \mid X(t) < Y(t)\}$, i.e.,

(6.2) $\int^T_0 1_{\{X(t)<Y(t)\}} dC(t) = 0$, a.s. holds.

One major implication of this existence theorem is that the wealth process X(t) hedges against the reward of the CATS contract as follows: i) $X(t) \geq (q-F(t))^+$, $t \in [0,T]$ a.s., ii) $X(T) = (q-F(T))^+$. **Remark.** One can demonstrate that the wealth (valuation) process and the corresponding trading strategy $\Theta(\cdot)$ are the unique processes corresponding to the fair price.

C. Characterization of Fair Price

Our main result is a characterization of the current (t=0) fair price of the CATS futures put option as the expectation of the present value of the payoff (received at the time of optimal exercise) of the CATS contract.

THEOREM 6.2: Fair Price. Retain the hypotheses of Theorem 6.1. Let X(t) be defined by (6.1). Then, the infimum in (5.1) is attained and the current (t=0) fair price of the CATS put option is:

$$(6.3)\qquad v(0) = \sup_{\tau \in \mathcal{L}_{0,T}} E_o[\beta(\tau) Y(\tau)].$$

D. Characterization of Optimal Exercise Policy

Drawing upon standard results in the theory of optimal stopping (see Fakeev, 1970 and El Karoui, 1981), the optimal exercise (stopping) time τ_t for the interval [t,T] is the time at which the smallest supermartingale majorant to the expected payoff of the option--the Snell envelope J(t)--first equates (drops to) the discounted payoff of the option, i.e.,

(6.4) $$\tau_t^* = \inf\{s \in [t,T];\ J(s)=Q(s)\},\quad \text{where}$$

(6.5) $$J(t)=\operatorname*{ess\,sup}_{\tau \in \mathcal{Y}_{t,T}} E_0[\beta(\tau)Y(\tau)\,|\,\mathcal{F}(t)],\qquad t \in [0,T]\quad \text{a.s.}$$

We now have an implicit solution for the option buyer's rational exercise problem as well as an "implicit" expression for the current fair (arbitrage-free) value of the CATS futures option contract. Fortunately, more explicit characterizations can be obtained by a combination of analytical and numerical methods.

E. Early Exercise Representation

A common, more explicit characterization of the fair price is the "early exercise premium" representation of the put value function v(F(t),t). Following standard practice in the American options literature, we decompose the American put value function into the sum of the corresponding European put value function, p(F(t), t), and the early exercise premium, e(F(t),t).

THEOREM 6.3: Early Exercise Premium Representation. For initial conditions (F(t),t) the early exercise premium representation of the CATS put option is:

(6.4) $$v(F(t),t)=p(F(t),t) + e(F(t),t),\quad \text{where}$$

(6.5) $$p(F(t),t)=E_0[\exp(-r(T-t))[q-F(T)]^+],$$

(6.6) $$e(F(t),t)=E_0[\textstyle\int_t^T \exp(-r(u-t))[dV(u)]1_{(F(u)<F(u)^*)}du],\ \text{and}$$

$$dV(t)=\text{variation margin - initial margin payment}$$

$$=q - F(t) - V(0)$$

$$dV(u\neq t)=rV(u) + dF(u).$$

1. Intuition of the Decomposition

The underlying idea of the decomposition is relatively easy to grasp. First, observe that the optimal exercise (stopping) time implies that F^* is the critical value of the futures contract at or below which the decision to exercise is optimal for every $t\varepsilon[0,T)$. Thus, at any time $t\varepsilon[0,T)$, whenever $F_u>F_u^*$ the option to exercise--during a specific interval du--has no value. However, when $F_u \leq F_u^*$ the option to exercise is worth the present value (at time t) of the accumulation (over the intervals du) of the value of flexibility associated with being able to exercise ("stop") at any time over the life of the option. This flexibility value, over one interval du, equals the discounted sum of the interest earnings on the margin account and the payment/receipt of variation margin. Hence, $e(F(t),t)$ is the sum over the intervals du of the sequence of infinitesimal early exercise premiums. In contrast, the European put option value measures the payoff of the option, assuming that this payoff is realized at the termination date T.[5/] Finally, we suggest how one can obtain an explicit solution for the optimal exercise (stopping) boundary $F^*(t)$.

VII. Towards An Explicit Solution

To give the reader some idea of the complexities encountered in a numerical approach to finding the value function and exercise boundary, we briefly review numerical efforts for the conventional American put equity option. Specifically, we characterize the American put value function and optimal exercise boundary as solutions to a type of free boundary problem called a Stefan problem. In this setting the solutions consist of a closed-form valuation formula for the put value function and a nonlinear integral equation (Volterra-2nd kind) for the optimal exercise boundary $F^*(t)$.

A. Free-Boundary (Stefan) Problem

As McKean (1965) and van Moerbeke (1976) initially demonstrated, the rational exercise optimal stopping problem is represented by the following free boundary problem.

Proposition 7.1: Free Boundary Representation. The American put value function and the optimal exercise boundary $F^*(t)$ jointly solve, on the continuation region ($\mathbf{C}$) of the stopping problem, the following free boundary problem consisting of the second-order partial differential equation and the associated boundary conditions:

(7.1) $\mathfrak{L}[\exp{-rt} \text{ times } v(F(t),t)]=0$, for $(F,t)\varepsilon\mathbf{C}$, where

$\mathcal{L}$ is a second-order partial differential operator, i.e.,

(7.2) $\mathcal{L} \equiv 0.5\sigma^2 F^2 [\partial^2 v/\partial F^2] + rF\partial v/\partial F + \partial v/\partial t.$

<u>Boundary Conditions</u>:

(7.3) $\lim_{t \to T} v(F,t) = (q - F(t))^+,$ $q \in R_+$

(7.4) $\lim_{F(t) \to +\infty} v(F,t) = 0,$ $t \in [0,T]$

(7.5) $v(F,t) \geq (q-F)^+,$ $(F,t) \in R_+ \times [0,T]$

(7.6) $\lim_{F \downarrow F^*(t)} v(F,t) = (q-F)$ $t \in [0,T]$

(7.7) $\lim_{F \downarrow F^*(t)} \partial v(F,t)/\partial F = -1$ $t \in [0,T]$

PROOF: Apply Feynman-Kac theorem.

The boundary (Dirichlet and optimality) conditions are interpreted as follows. Equation (7.3) indicates that, at expiration, the American put becomes a European put option, if the put has not been exercised early. Equation (7.4) indicates that, as the value of the underlying asset approaches infinity, the value of the put tends to zero. In other words, this condition reflects the fact that if the futures price increases without limit, it will never fall back (below the exercise boundary) within a finite time period. Equation (7.5) is the hedging condition. Equation (7.6), the value matching constraint, specifies the payoff received by the put holder at exercise.

Unfortunately, conditions (7.3)-(7.6) do not enable us to solve for the optimal exercise boundary $F^*(\bullet)$. Thus, we need to add (7.7), the "high-contact" (also called "smooth fit") condition. The role of (7.7) is to provide sufficient differentiability of the value function so that the option seller can continuously adjust the hedge portfolio across the exercise boundary without incurring "costs" for transitions through the boundary. Finally, the pair of conditions (7.6) and (7.7) are commonly referred to as the "smooth pasting" conditions. Their role is to guarantee that early exercise of the put option will be optimal and self-financing, when the (endogenous) early exercise boundary is attained by the price process of the underlying asset.

B. Numerical Difficulties: Exercise Boundary

Unfortunately, obtaining closed-form solutions for the optimal exercise boundary appears to be impossible at this point in time. The reason is that one has to solve a Volterra integral equation of the second kind; and an analytic solution has not yet been found. Moreover, the solution of the integral equation via numerical techniques is hampered in two ways. First, the integral contains the slope of the exercise boundary and this slope must be approximated via a finite difference algorithm. Second, the slope becomes infinite at maturity, thus creating a singularity.[7/]

VIII. Relationship to the Literature

Our main results--characterization of fair price and early exercise representation--appear to be novel for the American put CATS futures option contract. More generally, within the context of the arbitrage-free, continuous-time valuation of futures and futures option contracts, there do not appear to be any papers employing the unified, continuous-time methodology developed by Karatzas. The closest work, perhaps, is that of Duffie and Stanton (1992). They extend the Cox-Ingersoll-Ross (CIR) (1981) valuation methodology to the problem of valuing continuously resettled (i.e., marked-to-market) futures and futures options contracts. The latter are sold on the LIFFE exchange in London. However, they only value European futures options in a Markov diffusion setting. Thus, they don't employ the equivalent martingale measure technique. Related to Duffie and Stanton is the paper by Duffie and Jackson (1990) which addresses the optimal futures hedging problem in a continuous-time framework. These results were extended to an incomplete market setting by Duffie and Richardson (1991) and, then, subsequently generalized by Schweizer (1992).

Turning to the general problem of valuing a "plain vanilla" American put option, we observe that equivalent results have been obtained by others using a variety of approaches. However, common to all approaches is McKean's (1965) seminal formulation of the American call valuation problem (in Samuelson's (1965) equilibrium framework) as an optimal stopping problem and its equivalent free-boundary formulation. From a methodological perspective, the paper by Jacka

(1991) is closest to this one. He also employs martingale techniques to obtain the early exercise decomposition of the American put value function.

One of the most recent, comprehensive solutions is the work of Carr-Jarrow-Myneni (denoted as C-J-M, 1992). They adapt McKean's (1965) approach to the American put valuation problem and obtain a decomposition of the American put option into the early exercise representation. Moreover, they develop a relatively efficient numerical procedure--involving only one-dimensional normal distribution functions--for approximating the value of an American put option. Thus, the C-J-M work has a wide range of applicability. However, the potential of the C-J-M approach to unify the problems of option pricing, consumption/investment, and financial market equilibrium remains to be seen.

A related strand in the literature has focused on the development of analytic (quadratic) approximations of the American put option and futures option. For details consult, MacMillan (1986) as well as Barone-Adesi and Whaley (1987, 1988). Unfortunately, these approximations cannot be made arbitrarily accurate. One can remedy this deficiency by using the "compound option" approach developed by Geske and Johnson (1984). However, the approach is numerically cumbersome, requiring the use of multi-variate normal distribution functions. In contrast, the C-J-M approach enables one to determine exactly the magnitude of the early exercise premium. Finally, Kim (1990) uses a regression-based methodology to obtain the early exercise premium decomposition as a limit of the Geske-Johnson (1984) discrete formula.

IX. Future Research

In this paper martingale techniques are used to derive an arbitrage-free valuation formula (in implicit form) for an American CATS futures put option. We label this result an approximation in light of our questionable assumption that cumulative losses are lognormally distributed (after being risk-adjusted).

To improve this result, the following research agenda might be pursued. First, resolve the empirical issue: How close is the approximation derived here to market prices as well as prices obtained from other approximation models, e.g., a binomial model? Secondly, relax several of the substantive assumptions of the model. For example, as pointed out earlier, the assumption that losses are lognormally distributed is inconsistent with the assumption that cumulative losses are also lognormally distributed. A more appropriate assumption might be to represent cumulative losses as an arithmetic average.

Finally, a theoretically challenging task is to relax the assumption of market completeness. The problem is that when financial markets are incomplete, the presence of American options (with nominal strike prices) may lead to the creation of inefficient equilibria (Magill and Shafer, 1991). More fundamentally, the option holder can, via the choice of the early exercise policy, affect the span of the markets and, thereby, affect the pricing of securities.

* * *

NOTES:

1. During the first week of October 1993, open interest in the CATS futures option contract totalled more than 1600 contracts. By open interest, we mean the number of existing contracts that have not yet been offset by an opposite transaction.

2. The nine applicable insurance lines are: homeowners, commercial multiple peril and inland marine, earthquake, fire, allied, auto physical damage (private and commercial), and farmowners. The five causes of loss are: wind, hail, earthquake, riot, and flood. Note that only the auto and marine lines cover all five causes of loss. The remaining lines cover at least three causes of loss.

3. The volatility of F is set equal to S based on arguments contained in Hull (1992), Chapter 11. Also, for a specification of the futures price process based on the compound Poisson and lognormal actuarial models of aggregate insurance claims processes, see Cox and Schwebach (1992).

4. The results of this paper are independent of the agent's motivation--hedging, speculation, or arbitrage--for taking a position in a CATS contract.

5. The term "free lunch" was coined by Harrison and Kreps (1979). It refers to a sequence of admissible trading strategies whose initial costs go to a non-positive real number and whose end of horizon ($t=T$) value goes to a non-zero positive random variable.

6. For an explicit representation of the European CATS futures put option value function $p(F(t),t)$, see Cox and Schwebach (1992).

7. For details about these difficulties, consult Carr, et al. (1992).

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