Martingale Approach to Pricing Perpetual American Options

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Summary

The method of Esscher transforms is a tool for valuing options on a stock, if the logarithm of the stock price is governed by a stochastic process with stationary and independent increments. The price of a derivative security is calculated as the expectation, with respect to the risk-neutral Esscher measure, of the discounted payoffs. Applying the optional sampling theorem we derive a simple, yet general formula for the price of a perpetual American put option on a stock whose downward movements are skip-free. Similarly, we obtain a formula for the price of a perpetual American call option on a stock whose upward movements are skip-free. We also study a family of stochastic processes for modeling such stock prices. This family includes the Wiener process, gamma process and inverse Gaussian process, and their combinations. Under the classical assumption that the stock price is a geometric Brownian motion, the general perpetual American contingent claim is analysed. The martingale approach avoids the use of differential equations and provides additional insight. We also explain the relationship between Samuelson's high contact condition and the first order condition for optimality.

Evaluation des options américaines perpétuelles par la méthode des martingales

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Résumé

La méthode des transformées d'Esscher est un instrument pour déterminer la valeur d'une option, si le logarithme du prix de l'action sous-jacente est un processus à accroissements indépendants et stationnaires. Le prix d'une option est basé sur l'espérance mathématique (selon la mesure-martingale d'Esscher) de la valeur escomptée du payoff. En applicant le théorème d'arrêt, on obtient une formule simple et générale pour le prix d'une option put américaine perpétuelle, sous l'hypothèse que les trajectoires du prix de l'action sous-jacente sont semi-continues vers le bas. D'une façon analogue, on obtient une formule pour le prix d'une option call américaine perpétuelle, sous l'hypothèse que les trajectoires sont continues vers le haut. On étudie la famille des processus stochastiques ayant ces propriétés. Cette famille comprend le processus de Wiener, le processus gamma, le processus Gaussien inverse, et leur combinaisons. Sous l'hypothèse classique que le logarithme du prix de l'action est un processus de Wiener, on analyse la valeur d'une option américaine générale perpétuelle. En utilisant des martingales, on évite des équations différentielles et on obtient une meilleure compréhension des résultats. En plus, on explique la relation entre la condition de continuité de Samuelson ("high contact condition") et la condition d'optimalité de premier ordre.

1. Introduction

The option-pricing theory of Black and Scholes (1973) is perhaps the most important development in the theory of financial economics in the past two decades. A fundamental insight in advancing the theory is the concept of *risk-neutral valuation* introduced by Cox and Ross (1976). Further elaboration on this idea was given by Harrison and Kreps (1979), Harrison and Pliska (1981) and others under the terminology of *equivalent martingale measure*. It is now understood that the absence of arbitrage is "essentially" equivalent to the existence of an equivalent martingale measure, and some authors (Dybvig and Ross, 1987; Schachermayer, 1992) call this the *Fundamental Theorem of Asset Pricing*.

Under the assumption that the logarithm of the stock price is governed by a stochastic process with stationary and independent increments, one may determine such an equivalent martingale measure by a time-honored technique in actuarial science – the Esscher transform (Esscher, 1932). An Esscher transform induces an equivalent probability measure on such a stock-price process. The risk-neutral Esscher parameter (which is unique when it exists) is determined so that the stock price, discounted by the risk-free interest rate less the dividend yield, becomes a martingale under the new probability measure. The price of a derivative security is simply calculated as the expectation, with respect to this equivalent martingale measure, of the (optimal) payoffs discounted by the risk-free interest rate.

The pricing of American options with a finite expiration date has been a challenging problem in the field of financial economics. A main difficulty is the determination of the optimal exercise boundary. Some papers on American options in the past decade are Bensoussan (1984), MacMillan (1986), Barone-Adesi and Whaley (1987), Omberg (1987), Karatzas (1988), Jaillet, Lamberton and Lapeyre (1990), Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992), Myneni (1992), Chesney, Elliot and Gibson (1993), Lamberton (1993), Hull and White (1993), and Tilley (1993). In this paper we study the pricing of American options without expiration date by means of the Esscher transform and the optional sampling theorem. This is a more tractable problem because the optimal exercise boundary of a perpetual American option is a fixed stock value independent of time. We derive a simple, yet general formula for the price of a perpetual American put option on a stock whose downward movements are skip-free. Similarly, we obtain a formula for the price of a perpetual American call option on a stock whose upward movements are skip-free. We also study a family of stochastic processes for modeling such stock-price movements. This family includes the Wiener process, gamma process and inverse Gaussian process, and combinations of such processes.

Under the classical assumption that the stock price is a geometric Brownian motion, the general perpetual American contingent claim is analysed. The martingale approach avoids the use of differential equations and provides additional insight. We also explain the relationship between Samuelson's high contact condition and the first order condition for optimality.

2. The Risk-Neutral Esscher Transform

Let S(t) be the price of a stock at time t, $t \geq 0$. We assume that the process, $\{X(t)\}_{t\geq 0}$, defined by

$$S(t) = S(0)e^{X(t)}, \quad t \geq 0, \tag{2.1}$$

is one with stationary and independent increments. Let

$$F(x, t) = \Pr[X(t) \leq x], \quad t \geq 0, \tag{2.2}$$

be the distribution of the random variable X(t), and

$$M(z, t) = E[e^{zX(t)}], \quad t \geq 0, \tag{2.3}$$

its moment generating function. Under a mild continuity condition (Breiman, 1968, Section 14.4),

$$M(z, t) = [M(z, 1)]^t, \quad t \geq 0. \tag{2.4}$$

While the Esscher transform of a random variable is a well-established concept, in this paper we consider the Esscher transform of a stochastic process which satisfies (2.4). The Esscher transform (parameter h) of $\{X(t)\}_{t\geq 0}$ is again a process with stationary and independent increments; the modified distribution of X(t) is now

$$\begin{aligned} F(x, t; h) &= \Pr[X(t) \leq x; h] \\ &= \frac{\int_{-\infty}^{x} e^{hy}\, dF(y, t)}{\int_{-\infty}^{\infty} e^{hy}\, dF(y, t)} \\ &= \frac{1}{M(h, t)} \int_{-\infty}^{x} e^{hy}\, dF(y, t). \end{aligned}$$

The corresponding moment generating function is

$$M(z, t; h) = \frac{M(z + h, t)}{M(h, t)}. \tag{2.5}$$

It follows from (2.4) that

$$\begin{aligned} M(z, t; h) &= \left[\frac{M(z + h, 1)}{M(h, 1)}\right]^t \\ &= [M(z, 1; h)]^t. \end{aligned} \tag{2.6}$$

Because the exponential function is positive, the old and new measures have the same null sets, i.e., they are equivalent probability measures. The appropriate parameter $h = h^*$ is determined according to the principle of risk-neutral valuation (Cox and Ross, 1976), or, using the terminology of Harrison and Kreps (1979) and Harrison and Pliska (1981), we seek $h = h^*$ to obtain an equivalent martingale measure.

In this paper we assume that the risk-free force of interest is constant, and it is denoted by δ. Furthermore, we assume that the stock pays a continuous stream of dividends, at a rate proportional to its price, i.e., there is a nonnegative constant ρ such that the dividend paid between time t and t + dt is

$$S(t)\rho dt.$$

The parameter $h = h^*$ is chosen so that the process, $\{e^{-(\delta - \rho)t}S(t)\}_{t \geq 0}$, is a martingale with respect to the probability measure corresponding to h^*. In particular,

$$S(0) = E[e^{-(\delta - \rho)t}S(t); h^*]; \tag{2.7}$$

hence, by (2.1) and (2.6),

$$\begin{aligned} e^{(\delta - \rho)t} &= E[e^{X(t)}; h^*] \\ &= [M(1, 1; h^*)]^t, \end{aligned}$$

or

$$\ln[M(1, 1; h^*)] = \delta - \rho. \tag{2.8}$$

We call the equivalent martingale measure with respect to the Esscher parameter h^* the *risk-neutral Esscher measure*. The price of a derivative security, whose payments depend on $\{S(t)\}$, is calculated as a discounted expected value, where the expectation is taken with respect to the risk-neutral Esscher measure.

Under some regularity conditions, equation (2.8) has a unique solution. To see this, consider the function

$$g(h) = \ln[M(1, 1; h)] = \ln[M(1 + h, 1)] - \ln[M(h, 1)].$$

The formula

$$\frac{d}{dh}E[X(1); h] = \mathrm{Var}[X(1); h]$$

shows that $E[X(1); h]$ is an increasing function in h. Hence

$$g'(h) = E[X(1); 1 + h] - E[X(1); h]$$

is positive, showing that $g(h)$ is an increasing function. This proves the uniqueness of the solution of equation (2.8), which is

$$g(h) = \delta - \rho.$$

To discuss the existence of the solution, let M and m denote the right and left end point of the (essential) range of $X(1)$, respectively. (M may be $+\infty$ and m may be $-\infty$.) We may assume

$$m + \rho < \delta < M + \rho,$$

or

$$m < \delta - \rho < M,$$

because otherwise arbitrage would be possible. Let (a, b) denote the interval of values of h for which g(h) exists. Under some regularity conditions,

$$\lim_{h \downarrow a} g(h) = m, \qquad \lim_{h \uparrow b} g(h) = M,$$

in which case (2.8) does have a solution. It should be noted that, although the risk-neutral Esscher measure is unique when it exists, there may be other equivalent martingale measures; see Delbaen and Haezendonck (1989) for a study on equivalent martingale measures of compound Poisson processes.

The price of a derivative security is taken as the expectation of its discounted (optimal) payoffs with respect to the risk-neutral Esscher measure. For example, consider a *European call option* on the stock with exercise price K and exercise date t, $t > 0$. Let $I(\cdot)$ denote the indicator function and $\kappa = \ln[K/S(0)]$. The price of the option (at time 0) is

$$\begin{aligned} &e^{-\delta t}E[(S(t) - K)I(S(t) > K); h^*] \\ &= e^{-\delta t}E[S(t)I(S(t) > K); h^*] - e^{-\delta t}KE[I(S(t) > K); h^*]. \qquad (2.9) \end{aligned}$$

The second expectation in the right-hand side of (2.9) is

$$\Pr[(S(t) > K); h^*] = 1 - F(\kappa, t; h^*).$$

To evaluate the first expectation in the right-hand side of (2.9), note that, for each measurable function $g(\cdot)$,

$$\begin{aligned} E[g(S(t)); h] &= \frac{E[g(S(t))\, e^{hX(t)}]}{E[e^{hX(t)}]} \\ &= \frac{E[g(S(t))\, S(t)^h]}{E[S(t)^h]}. \qquad (2.10) \end{aligned}$$

Applying (2.10) yields the following result.

Lemma. Let h and k be two real numbers. Assume that the Esscher transforms of parameters h and h + k exist. Then, for each measurable function $\psi(\cdot)$,

$$E[S(t)^k \psi(S(t)); h] = E[S(t)^k; h]E[\psi(S(t)); h + k]. \tag{2.11}$$

Applying the Lemma [with $k = 1$, $\psi(x) = I(x > K)$ and $h = h^*$] and (2.7), we obtain

$$\begin{aligned} E[S(t)I(S(t) > K); h^*] &= E[S(t); h^*]E[I(S(t) > K); h^* + 1] \\ &= S(0)e^{(\delta - \rho)t}\Pr[(S(t) > K); h^* + 1]. \end{aligned}$$

Thus the price of the European call option is

$$S(0)e^{-\rho t}[1 - F(\kappa, t; h^* + 1)] - Ke^{-\delta t}[1 - F(\kappa, t; h^*)]. \tag{2.12}$$

If $\{X(t)\}$ is a Wiener process with variance per unit time σ^2, then (2.12) (with (4.2) below) yields the expression

$$S(0)e^{-\rho t}\Phi(\frac{-\kappa + (\delta - \rho + \sigma^2/2)t}{\sigma\sqrt{t}}) - Ke^{-\delta t}\Phi(\frac{-\kappa + (\delta - \rho - \sigma^2/2)t}{\sigma\sqrt{t}}), \tag{2.13}$$

where $\Phi(\cdot)$ denotes the standardized normal distribution function. For $\rho = 0$ this is the celebrated *Black-Scholes formula*. Formula (2.13) is the same as formula (53) in Smith (1976).

Remarks: We assume that the stock pays dividends at a constant proportional rate ρ. If all dividends are reinvested in the stock, then each share of the stock at time 0 grows to $e^{\rho t}$ shares at time t; this gives an interpretation for formula (2.7),

$$S(0) = E[e^{-\delta t}S(t)e^{\rho t}; h^*].$$

On the other hand, we can also consider the situation where none of the dividends are reinvested in the stock, leading to the intuitive formula:

$$S(0) = E[\int_0^t e^{-\delta u}S(u)\rho du + e^{-\delta t}S(t); h^*]. \tag{2.14}$$

To prove (2.14), we interchange the order of expectation and integration on the right-hand side and apply the formula

$$E[e^{-\delta u}S(u); h^*] = e^{-\rho u}S(0);$$

thus

$$\begin{aligned} \text{R.H.S.} &= S(0)\left(\int_0^t e^{-\rho u}\rho du + e^{-\rho t}\right) \\ &= S(0) \\ &= \text{L.H.S.} \end{aligned}$$

3. Pricing Perpetual American Options

In this section, by applying the optional sampling theorem, we derive pricing formulas for perpetual American put and call options on a stock. As discussed in the last section, the stock-price process is assumed to be one satisfying (2.4), and dividends are paid continuously at a constant proportional rate ρ. When pricing a perpetual American put option, we assume that the downward movements of the stock price are skip-free. Similarly, when pricing a perpetual American call option, we assume that the upward movements of the stock price are skip-free. Under these opportune assumptions attractive formulas can be obtained.

First, we consider a *perpetual American put option* with exercise price K. We temporarily assume that $K < S(0)$, so that an immediate exercise of the option can be excluded. The owner of this option exercises it according to some strategy: a stopping time T. Then, at time T, he will get

$$(K - S(T))_+,$$

where $x_+ = \text{Max}(x, 0)$. Thus the value (at time 0) associated with the strategy is

$$E[e^{-\delta T}(K - S(T))_+; h^*]. \tag{3.1}$$

We can limit ourselves to strategies of the form

$$T_L = \inf\{t \mid S(t) \le L\}, \tag{3.2}$$

where $L \le K$; the option is exercised the first time when the price of the stock falls below or equals the level L. The price of the option is the maximal value of

$$E[e^{-\delta T_L}(K - S(T_L))_+; h^*]. \tag{3.3}$$

With the assumption that the stock-price process, $\{S(t)\}_{t\geq 0}$, is skip-free downwards, the stock price is equal to L at the time when the option is exercised, i.e.,

$$L = S(T_L) = S(0)e^{X(T_L)}. \tag{3.4}$$

For simplicity denote the current stock price S(0) by S and expression (3.3) by V(S, L). Since $L \leq K$,

$$V(S, L) = (K - L)E[e^{-\delta T_L}; h^*]. \tag{3.5}$$

The expectation in (3.5) is a Laplace transform of T_L and can be calculated by the following classical argument.

Consider the stochastic process $\{e^{-\delta t + \theta X(t)}\}_{t\geq 0}$. For $t \leq T_L$, it is a bounded martingale with respect to the risk-neutral Esscher measure if the coefficient θ is the negative solution of the equation

$$E[e^{-\delta t + \theta X(t)}; h^*] = 1,$$

or

$$M(\theta, 1; h^*) = e^{\delta}. \tag{3.6}$$

Equation (3.6) has two (real) solutions; one is negative and the other is greater than one. To see this, consider the function

$$\phi(\theta) = M(\theta, 1; h^*) = E[e^{\theta X(1)}; h^*].$$

Since

$$\phi''(\theta) = E[X(1)^2 e^{\theta X(1)}; h^*] > 0,$$

the function $\phi(\theta)$ is convex. Consequently, equation (3.6),

$$\phi(\theta) = e^{\delta},$$

has at most two solutions. We note that

$$\phi(0) = 1 < e^{\delta}$$

and, because of (2.8),

$$\phi(1) = e^{\delta - \rho} < e^{\delta}.$$

Since

$$\Pr[X(1) < 0] > 0$$

and

$$\Pr[X(1) > 0] > 0,$$

$\phi(\theta) \to +\infty$ for $\theta \to -\infty$ and for $\theta \to +\infty$. Thus equation (3.6) has two solutions, $\theta_0 < 0$ and $\theta_1 > 1$.

By the optional sampling theorem, we have

$$E[e^{-\delta T_L + \theta_0 X(T_L)}; h^*] = 1,$$

which, because of (3.4), becomes

$$E[e^{-\delta T_L}; h^*] = \left(\frac{L}{S}\right)^{-\theta_0}. \tag{3.7}$$

Applying (3.7) to (3.5) yields

$$V(S, L) = (K - L)\left(\frac{L}{S}\right)^{-\theta_0}. \tag{3.8}$$

For a given current stock price S, we seek the maximal value of (3.8) by varying the option-exercise boundary L. Let V_L denote the partial derivative of V with respect to L. Solving the equation

$$V_L(S, L) = 0$$

yields the optimal exercise boundary

$$L = \tilde{L} = \frac{-\theta_0}{1 - \theta_0} K, \tag{3.9}$$

which is independent of S. Thus the maximal value is

$$V(S, \tilde{L}) = \frac{K}{1 - \theta_0}\left[\frac{-K\theta_0}{S(1 - \theta_0)}\right]^{-\theta_0}.$$

This is the price of the perpetual American put option provided that $S \geq \tilde{L}$. For $S < \tilde{L}$, the option is exercised immediately and the price is simply $K - S$. Hence the option price is

$$\begin{cases} \dfrac{K}{1-\theta_0}\left[\dfrac{-K\theta_0}{S(1-\theta_0)}\right]^{-\theta_0} & \text{if } S \geq \tilde{L} \\ K - S & \text{if } S < \tilde{L} \end{cases} \tag{3.10}$$

Next we study the pricing of a *perpetual American call option* with exercise price K, and we temporarily assume that $K > S$. For $M \geq K$, let

$$T_M = \inf\{t \mid S(t) \geq M\} \tag{3.11}$$

and

$$W(S, M) = E[e^{-\delta T_M}(S(T_M) - K)_+; h^*]. \tag{3.12}$$

With the assumption that the stock-price process, $\{S(t)\}_{t\geq 0}$, is skip-free upwards, the stock price is equal to M at the time when the option is exercised, i.e., $S(T_M) = M$. Since $M \geq K$, formula (3.12) becomes

$$W(S, M) = (M - K)E[e^{-\delta T_M}; h^*]. \tag{3.13}$$

The expectation in (3.13) is evaluated in the same way as above, except that we now use θ_1, the positive root of (3.6), to make sure that $\{\exp[-\delta t + \theta_1 X(t)]\}$ is a bounded martingale (with respect to the risk-neutral Esscher measure) for $t \leq T_M$. The resulting formula is

$$E[e^{-\delta T_M}; h^*] = \left(\frac{S}{M}\right)^{\theta_1}. \tag{3.14}$$

For a given current stock price S, the maximal value of

$$W(S, M) = (M - K)\left(\frac{S}{M}\right)^{\theta_1} \tag{3.15}$$

is attained at

$$M = \tilde{M} = \frac{\theta_1}{\theta_1 - 1}K, \tag{3.16}$$

and

$$W(S, \tilde{M}) = \frac{K}{\theta_1 - 1}\left[\frac{S(\theta_1 - 1)}{K\theta_1}\right]^{\theta_1}. \tag{3.17}$$

This gives the price of the perpetual American call option provided $S \le \tilde{M}$. For $S > \tilde{M}$, the option is exercised immediately and the price is simply $S - K$. Thus the option price is

$$\begin{cases} \dfrac{K}{\theta_1 - 1}\left[\dfrac{S(\theta_1 - 1)}{K\theta_1}\right]^{\theta_1} & \text{if } S \le \tilde{M} \\ S - K & \text{if } S > \tilde{M} \end{cases} \tag{3.18}$$

Remarks. As the dividend yield ρ tends to 0, the coefficient θ_1 tends to 1, the optimal exercise boundary $\tilde{M}$ tends to ∞, and the perpetual American call option price tends to S, the current stock price. It is a well-known paradox that an American call option on a non-dividend-paying stock is never optimally exercised before its maturity date (Merton, 1973). This paradox may be verified by means of Jensen's inequality:

$$\begin{aligned} E[e^{-\delta T}(S(T) - K)_+; h^*] &\ge (E[e^{-\delta T}S(T); h^*] - e^{-\delta T}K)_+ \\ &= (S(0) - e^{-\delta T}K)_+ \\ &\ge (S(0) - K)_+. \end{aligned}$$

For $T > 0$, the last inequality is strict if the option is currently in the money, i.e., if $S(0) > K$.

That the perpetual American call option on a non-dividend-paying stock should have the same price as the stock may also be explained as follows: If $\rho = 0$, then $\theta_1 = 1$ and (3.15) reduces to

$$W(S, M) = (1 - \frac{K}{M})S, \qquad M \ge K. \tag{3.19}$$

Since this is a strictly increasing function of M, its supremum is not attained for a finite value of M.

To avoid this anomaly (to which Ingersoll (1987, p. 373) refers as the problem of "infinities") we might modify the payoff of the call option as

$$[(S(T) - K)_+]^\alpha, \qquad 0 < \alpha < 1.$$

Then

$$W(S, M) = (M - K)^\alpha \frac{S}{M},$$

which is maximal for

$$\tilde{M} = \frac{K}{1 - \alpha}.$$

3.1. The High Contact Condition

Each of (3.10) and (3.18), as a function of the current stock price S, has a continuous first derivative, because

$$\begin{aligned} V(\tilde{L}, \tilde{L}) &= K - \tilde{L}, \\ V_S(\tilde{L}, \tilde{L}) &= -1, \\ W(\tilde{M}, \tilde{M}) &= \tilde{M} - K, \end{aligned} \tag{3.1.1}$$

and

$$W_S(\tilde{M}, \tilde{M}) = 1. \tag{3.1.2}$$

Formulas (3.1.1) and (3.1.2) are special cases of the so-called *high contact condition* (Samuelson, 1965); in the literature about optimal stopping problems (Sirjaev, 1973, p. 113; Shirayayev, 1978, p. 160) the term is *smooth pasting condition.*

Merton (1973, p. 171, footnote 60; 1990, p. 296, footnote 47) has derived the high contact condition as a first order condition necessary for optimality. (Merton's proof is reformulated on page 189 of Brekke and Øksendal (1991).) Under some weak conditions, the converse is also true – a solution proposal to an optimal stopping problem satisfying the high contact condition is in fact an optimal solution to the problem; a recent paper on this is Brekke and Øksendal (1991). It is easy to check that condition (3.1.1) does determine the optimal exercise boundary $\tilde{L}$, while (3.1.2) determines $\tilde{M}$.

We now derive a formula explaining how the high contact condition (3.1.1) and optimality for $V(S, \cdot)$ are related. Let

$$\lambda(S, L) = E[e^{-\delta T_L}; h^*]. \tag{3.1.3}$$

For $0 < x < S - L$,

$$\lambda(S, L) = \lambda(S, L + x)\lambda(L + x, L) \tag{3.1.4}$$

(cf. Lemma 7.1 on page 243 of Karlin and Taylor (1981)). Differentiating (3.1.4) with respect to x and setting $x = 0$ yields

$$0 = \lambda_L(S, L) + \lambda(S, L)\lambda_S(L, L). \tag{3.1.5}$$

Now, let

$$\pi(x) = (K - x)_+ \tag{3.1.6}$$

denote the payoff function. Then (3.5) becomes

$$V(S, L) = \pi(L)\lambda(S, L). \tag{3.1.7}$$

Differentiating (3.1.7) with respect to L and applying (3.1.5) yields

$$\begin{aligned} V_L(S, L) &= \pi'(L)\lambda(S, L) + \pi(L)\lambda_L(S, L) \\ &= \pi'(L)\lambda(S, L) - \pi(L)\lambda(S, L)\lambda_S(L, L) \\ &= \lambda(S, L)[\pi'(L) - V_S(L, L)]. \end{aligned} \tag{3.1.8}$$

(Formula (3.1.8) can also be derived using (3.8).) Since $\lambda(S, L)$ is positive, $V_L(S, L) = 0$ if and only if

$$V_S(L, L) = \pi'(L). \tag{3.1.9}$$

Equation (3.1.8) shows explicitly that the optimal exercise boundary $\tilde{L}$ does not depend on the current stock price S. We note that (3.1.7), (3.1.8) and (3.1.9) are valid for payoff functions $\pi(\cdot)$ more general than (3.1.6).

Similarly, one can derive the formula

$$W_M(S, M) = \mu(S, M)[\pi'(M) - W_S(M, M)], \tag{3.1.10}$$

with

$$\mu(S, M) = E[e^{-\delta T_M}; h^*].$$

4. Logarithm of the Stock Price as a Wiener Process

The stochastic process with stationary and independent increments and sample paths which are both skip-free upwards and downwards (i.e., continuous) is the Wiener process. In this section we assume that $\{X(t)\}_{t\geq 0}$ is a Wiener process; this is the classical geometric Brownian motion model for stock-price movements (Samuelson, 1965; Black and Scholes, 1973). Let μ and σ^2 denote, respectively, the mean and variance of $\{X(t)\}$ per unit time.

Since

$$M(z, t) = \exp[(\mu z + {}^1\!/_2\sigma^2 z^2)t],$$

it follows from (2.5) that

$$\ln[M(z, t; h)] = [(\mu + h\sigma^2)z + {}^1\!/_2\sigma^2 z^2]t.$$

This shows that the transformed process has modified mean per unit time $\mu + h\sigma^2$ and unchanged variance per unit time σ^2. From (2.8) we get

$$(\mu + h^*\sigma^2) + {}^1\!/_2\sigma^2 = \delta - \rho. \tag{4.1}$$

Thus to evaluate a derivative security we use a Wiener process with mean per unit time

$$\mu + h^*\sigma^2 = \delta - \rho - {}^1\!/_2\sigma^2. \tag{4.2}$$

From (3.6) we obtain

$$(\delta - \rho - {}^1\!/_2\sigma^2)\theta + {}^1\!/_2\sigma^2\theta^2 = \delta,$$

or

$$\sigma^2\theta^2 + (2\delta - 2\rho - \sigma^2)\theta - 2\delta = 0. \tag{4.3}$$

The roots of this quadratic equation are

$$\theta_0 = \frac{-(2\delta - 2\rho - \sigma^2) - \sqrt{(2\delta - 2\rho - \sigma^2)^2 + 8\sigma^2\delta}}{2\sigma^2} \tag{4.4}$$

and

$$\theta_1 = \frac{-(2\delta - 2\rho - \sigma^2) + \sqrt{(2\delta - 2\rho - \sigma^2)^2 + 8\sigma^2\delta}}{2\sigma^2}. \tag{4.5}$$

Formula (4.5) should be attributed to McKean (1965, Section 3) who studied the pricing of perpetual warrants; of course, he did not solve the problem in the context of the risk-neutral measure. With zero dividend yield ($\rho = 0$), formula (4.4) becomes

$$\theta_0 = \frac{-2\delta}{\sigma^2}, \tag{4.6}$$

which was first given by Merton (1973, Section 8; 1990, Section 8.8), who evaluated the perpetual American put option by adopting McKean's (1965) technique. Discussions on pricing perpetual American options can also be found in the books by Karlin and Taylor (1975, p. 365), Ingersoll (1987, p. 375) and Lamberton and Lapeyre (1991, p. 82), and in the recent articles by Karatzas (1988, p. 59, e.g. 6.7), Kim (1990) and Jacka (1991, Proposition 2.3). (In formula (9) of Kim (1990), the denominator $1 - \beta$ should be $\beta - 1$.)

In the finance literature, the formulas for pricing perpetual American options are usually derived as follows. Let D denote the value of a derivative security. It follows from the hedging argument first given by Black and Scholes (1973) that D satisfies the partial differential equation,

$$\tfrac{1}{2}\sigma^2 S^2 D_{SS} + (\delta - \rho) S D_S - \delta D + D_t = 0, \tag{4.7}$$

subject to the appropriate boundary conditions. In the case of a perpetual option, we have $D_t = 0$ and (4.7) becomes a homogeneous, linear, second-order ordinary differential equation in S,

$$\tfrac{1}{2}\sigma^2 S^2 D_{SS} + (\delta - \rho) S D_S - \delta D = 0. \tag{4.8}$$

The function, $D = S^\theta$, is a solution of (4.8) if the number θ satisfies the quadratic equation,

$$\tfrac{1}{2}\sigma^2 \theta(\theta - 1) + (\delta - \rho)\theta - \delta = 0, \tag{4.9}$$

which is the same as (4.3). Then any solution of (4.8) is of the form

$$D = c_0 S^{\theta_0} + c_1 S^{\theta_1}, \tag{4.10}$$

where c_0 and c_1 are constants.

In this paper we use the martingale approach and avoid differential equations. Additional insight for (4.10) is provided in the following; see (4.1.16) below.

4.1. Perpetual Contingent Claims

In the section we consider the pricing of perpetual contingent claims with payoff functions such as

$$\pi(x) = a_1(K_1 - x)_+ + a_2(x - K_2)_+. \tag{4.1.1}$$

For $a_1 = a_2 = 1$, the contingent claim may be called a *perpetual American strangle* if $K_1 < K_2$, and called a *perpetual American straddle* if $K_1 = K_2$. The assumption on $\{X(t)\}$ remains that it is a Wiener process.

Let $S = S(0)$ denote the current stock price. We consider exercise strategies arising from stopping times of the form

$$T_{L,M} = \inf\{t \mid S(t) = L \text{ or } S(t) = M\},$$

where $0 \le L \le S \le M$. The value of the contingent claim according to such a strategy is

$$V(S, L, M) = E[\pi(S(T_{L,M}))e^{-\delta T_{L,M}}; h^*]. \tag{4.1.2}$$

Put

$$\lambda(S, L, M) = E[I(S(T_{L,M}) = L)e^{-\delta T_{L,M}}; h^*] \tag{4.1.3}$$

and

$$\mu(S, L, M) = E[I(S(T_{L,M}) = M)e^{-\delta T_{L,M}}; h^*]. \tag{4.1.4}$$

Then

$$V(S, L, M) = \pi(L)\lambda(S, L, M) + \pi(M)\mu(S, L, M). \tag{4.1.5}$$

For $\theta = \theta_0$ and $\theta = \theta_1$ (the roots of equation (4.3)), the process $\{e^{-\delta t + \theta X(t)}\}$ is a bounded martingale (with respect to the risk-neutral measure) for $t \le T_{L,M}$. Applying the optional sampling theorem to these two martingales yields the equations

$$\lambda(S, L, M)\left(\frac{L}{S}\right)^{\theta_0} + \mu(S, L, M)\left(\frac{M}{S}\right)^{\theta_0} = 1 \tag{4.1.6}$$

and

$$\lambda(S, L, M)\left(\frac{L}{S}\right)^{\theta_1} + \mu(S, L, M)\left(\frac{M}{S}\right)^{\theta_1} = 1, \tag{4.1.7}$$

respectively, from which we obtain

$$\lambda(S, L, M) = \frac{M^{\theta_1} S^{\theta_0} - M^{\theta_0} S^{\theta_1}}{M^{\theta_1} L^{\theta_0} - M^{\theta_0} L^{\theta_1}} \tag{4.1.8}$$

and

$$\mu(S, L, M) = \frac{S^{\theta_1} L^{\theta_0} - S^{\theta_0} L^{\theta_1}}{M^{\theta_1} L^{\theta_0} - M^{\theta_0} L^{\theta_1}}. \tag{4.1.9}$$

Note that

$$\lim_{M \to \infty} \lambda(S, L, M) = \left(\frac{S}{L}\right)^{\theta_0} = \left(\frac{L}{S}\right)^{-\theta_0}, \tag{4.1.10}$$

which confirms (3.7), and

$$\lim_{L \downarrow 0} \mu(S, L, M) = \left(\frac{S}{M}\right)^{\theta_1}, \tag{4.1.11}$$

which is (3.14).

The remaining problem is to optimize V(S, L, M), considered as a function of the exercise boundaries L and M. The first order conditions are

$$V_L(S, L, M) = 0$$

and

$$V_M(S, L, M) = 0.$$

These conditions do not depend on S (as long as S is between L and M). At first this seems surprising, but it follows immediately from the formulas

$$V_L(S, L, M) = \lambda(S, L, M)[\pi'(L) - V_S(L, L, M)] \tag{4.1.12}$$

and

$$V_M(S, L, M) = \mu(S, L, M)[\pi'(M) - V_S(M, L, M)], \tag{4.1.13}$$

which generalize (3.1.8) and (3.1.10), respectively. Thus the first order conditions become

$$V_S(\widetilde{L}, \widetilde{L}, \widetilde{M}) = \pi'(\widetilde{L}) \tag{4.1.14}$$

and

$$V_S(\widetilde{M}, \widetilde{L}, \widetilde{M}) = \pi'(\widetilde{M}), \tag{4.1.15}$$

which are the high contact conditions. The optimal exercise boundaries $\widetilde{L}$ and $\widetilde{M}$ are determined by solving (4.1.14) and (4.1.15) simultaneously. For $\widetilde{L} \leq S \leq \widetilde{M}$, the price of the perpetual contingent claim is

$$\begin{aligned} V(S, \widetilde{L}, \widetilde{M}) &= \pi(\widetilde{L})\lambda(S, \widetilde{L}, \widetilde{M}) + \pi(\widetilde{M})\mu(S, \widetilde{L}, \widetilde{M}) \\ &= (S^{\theta_0} \quad S^{\theta_1}) \begin{pmatrix} \widetilde{L}^{\theta_0} & \widetilde{L}^{\theta_1} \\ \widetilde{M}^{\theta_0} & \widetilde{M}^{\theta_1} \end{pmatrix}^{-1} \begin{pmatrix} \pi(\widetilde{L}) \\ \pi(\widetilde{M}) \end{pmatrix}. \end{aligned} \tag{4.1.16}$$

To prove (4.1.12), consider the identities

$$\lambda(S, L, M) = \lambda(S, L + x, M)\lambda(L + x, L, M)$$

and

$$\mu(S, L, M) = \mu(S, L + x, M) + \lambda(S, L + x, M)\mu(L + x, L, M),$$

where $0 < x < S - L$. Differentiating these equations with respect to x and setting $x = 0$ yields

$$0 = \lambda_L(S, L, M) + \lambda(S, L, M)\lambda_S(L, L, M) \tag{4.1.17}$$

and

$$0 = \mu_L(S, L, M) + \lambda(S, L, M)\mu_S(L, L, M), \tag{4.1.18}$$

respectively. Differentiating (4.1.5) with respect L and applying (4.1.17) and (4.1.18), we have

$$\begin{aligned} V_L(S, L, M) &= \pi'(L)\lambda(S, L, M) + \pi(L)\lambda_L(S, L, M) + \pi(M)\mu_L(S, L, M) \\ &= \lambda(S, L, M)[\pi'(L) - \pi(L)\lambda_S(L, L, M) - \pi(M)\mu_S(L, L, M)] \\ &= \lambda(S, L, M)[\pi'(L) - V_S(L, L, M)], \end{aligned}$$

which is (4.1.12). The proof of (4.1.13) is similar.

We remark that, for certain payoff functions, the optimal non-exercise interval $(\widetilde{L}, \widetilde{M})$ is not unique. Indeed, there might be several disjoint optimal non-exercise intervals.

4.2. Perpetual Down-and-out Option

In this section we consider the pricing of a perpetual "down-and-out" American call option with exercise price K. The option contract becomes null and unexercisable, if the stock price declines to the knock-out price L, $L < K$. When this occurs, a rebate or refund of amount R is given. For $M \geq K$, it follows from (4.1.5) that the value of the strategy to exercise the call option when the stock price increases to M for the first time is

$$V(S, L, M) = R\lambda(S, L, M) + (M - K)\mu(S, L, M), \quad L \leq S \leq M. \tag{4.2.1}$$

Note that the lower exercise boundary L is fixed, and the problem is to maximize V as a function of the upper exercise boundary M.

We now consider the special case where the stock pays no dividends (hence $\theta_1 = 1$ and $\theta_0 = -2\delta/\sigma^2$). We shall show that the maximal value of (4.2.1) is obtained for $M \to \infty$ and that it is

$$\begin{aligned} V(S, L, \infty) &= S + (R - L)\left(\frac{L}{S}\right)^{-\theta_0} \\ &= S + (R - L)\left(\frac{L}{S}\right)^{\frac{2\delta}{\sigma^2}} . \end{aligned} \tag{4.2.2}$$

This result can also be found in Merton (1973, (57); 1990, (8.57)) and Ingersoll (1987, p. 372, (39)).

For the proof we first observe that $\lambda(S, L, M)$ is an increasing function of M and hence the first term on the right-hand side of (4.2.1) is bounded by

$$R\lambda(S, L, \infty) = R\left(\frac{L}{S}\right)^{-\theta_0} .$$

The second term on the right-hand side of (4.2.1) may be estimated as follows:

$$
\begin{aligned}
(M-K)\mu(S, L, M) &= (M-K)\frac{SL^{\theta_0} - S^{\theta_0}L}{ML^{\theta_0} - M^{\theta_0}L} \\
&= \frac{M-K}{M - \left(\frac{L}{M}\right)^{-\theta_0} L}\left[S - L\left(\frac{L}{S}\right)^{-\theta_0}\right] \\
&< S - L\left(\frac{L}{S}\right)^{-\theta_0} \\
&= \lim_{M \to \infty} (M-K)\mu(S, L, M).
\end{aligned}
$$

In a similar way perpeptual up-and-out American put options can be priced.

5. Semi-Continuous Sample Paths

In the rest of this paper, we consider the assumption that the sample paths of $\{S(t)\}$, or equivalently, those of $\{X(t)\}$, are skip-free downwards. (This assumption was used in deriving (3.10).) Then the following decomposition holds:

$$X(t) = Y(t) + v^2W(t) - ct, \; t \geq 0. \tag{5.1}$$

Here, $\{Y(t)\}$ is either a compound Poisson process with positive increments or the limit of such processes; $\{W(t)\}$ is an independent standardized Wiener process (with zero drift and unit variance per unit time); the last term, ct, represents a deterministic drift. The cumulant generating function of the random variable $X(t)$ is of the form

$$\ln[M(z, t)] = t\{\int_0^\infty (e^{zx} - 1)[-dQ(x)] + v^2z^2/2 - cz\}, \tag{5.2}$$

where $Q(x)$ is some nonnegative and nonincreasing function with $Q(\infty) = 0$. Note that, for each positive number ε, the integral

$$\int_\varepsilon^\infty (e^{zx} - 1)[-dQ(x)],$$

as a function in z, is the cumulant generating function of a compound Poisson distribution with Poisson parameter

$$\lambda(\varepsilon) = Q(\varepsilon),$$

and jump amount distribution

$$P(x; \varepsilon) = \frac{Q(\varepsilon) - Q(x)}{Q(\varepsilon)}, \qquad x \geq \varepsilon.$$

For notational simplicity, we assume that

$$-dQ(x) = q(x)\, dx$$

for some nonnegative function $q(x)$. Let μ and σ^2 denote, respectively, the mean and variance of $\{X(t)\}$ per unit time. Then

$$\mu t = E[X(t)] = [\int_0^\infty x\, q(x)\, dx - c]t, \tag{5.3}$$

$$\sigma^2 t = \mathrm{Var}[X(t)] = [\int_0^\infty x^2\, q(x)\, dx + v^2]t, \tag{5.4}$$

and

$$E[(X(t) - \mu t)^3] = t\int_0^\infty x^3\, q(x)\, dx. \tag{5.5}$$

In general, for $n \geq 3$, the n-th cumulant of $X(t)$ is given by

$$t\int_0^\infty x^n\, q(x)\, dx.$$

It follows from (2.5) and (5.2) that

$$\begin{aligned} \ln[M(z, t; h)] &= \ln[M(z + h, t)] - \ln[M(h, t)] \\ &= t\{\int_0^\infty (e^{zx} - 1)e^{hx} q(x)dx + v^2z^2/2 - (c - v^2h)z\}. \end{aligned} \tag{5.6}$$

Thus the Esscher transform (parameter h) of a process defined by (5.1) is of the same type, with the following modifications:

$$q(x) \to e^{hx}q(x), \tag{5.7}$$

$$v^2 \to v^2 \qquad \text{(unchanged)}, \tag{5.8}$$

$$c \to c - v^2h. \tag{5.9}$$

Furthermore, it follows from (5.6) that (2.8) and (3.6) can be written as

$$\int_0^\infty (e^x - 1)\, e^{h^*x}\, q(x)\, dx + v^2h^* = c + \delta - \rho - \frac{v^2}{2} \tag{5.10}$$

and

$$\int_0^\infty (e^{\theta x} - 1)\, e^{h^*x}\, q(x)\, dx + \frac{v^2\theta^2}{2} - (c - v^2h^*)\theta = \delta, \tag{5.11}$$

respectively.

6. A Particular Family

For the model defined by (2.1) and (5.1), we now assume that $v = 0$, i.e.,

$$S(t) = S(0)e^{Y(t) - ct},$$

and that

$$q(x) = ax^{\alpha - 1}e^{-bx}, \quad x > 0, \tag{6.1}$$

where $a > 0$, $\alpha > -1$, and $b > 0$ are three parameters. In the context of risk theory, Dufresne, Gerber and Shiu (1991) have considered such a $q(x)$ function.

According to (5.7), for $h < b$, the Esscher transform of a process defined by (6.1) is a member of the same family, with b replaced by

$$b(h) = b - h. \tag{6.2}$$

The moment generating function of $Y(t)$ is

$$\exp[t\int_0^\infty (e^{zx} - 1)\, q(x)\, dx] = \exp[at\int_0^\infty (e^{zx} - 1)\, x^{\alpha - 1}\, e^{-bx}\, dx]$$

$$= \begin{cases} \left(\dfrac{b}{b-z}\right)^{at} & \text{if } \alpha = 0 \\ \exp\left(\dfrac{a\Gamma(\alpha)}{b^\alpha}\left[\left(\dfrac{b}{b-z}\right)^\alpha - 1\right]t\right) & \text{if } \alpha \neq 0 \end{cases} \tag{6.3}$$

Thus, for $\alpha = 0$, $\{Y(t)\}_{t\geq 0}$ is a gamma process; for $\alpha > 0$, it is a compound Poisson process with Poisson parameter

$$\lambda(a, \alpha, b) = \frac{a\,\Gamma(\alpha)}{b^{\alpha}},$$

and gamma jump density

$$p(x; \alpha, b) = \frac{b^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} e^{-bx}, \quad x > 0.$$

For $-1 < \alpha < 0$, the most prominent case is $\alpha = -^1/_2$, where $\{Y(t)\}_{t\geq 0}$ is an inverse Gaussian process and the density function of Y(t) is

$$\frac{at}{x^{3/2}} \exp[\frac{-(\sqrt{b}x - \sqrt{\pi}at)^2}{x}], \qquad x > 0.$$

The condition for

$$b^* = b(h^*) = b - h^*$$

becomes

$$\frac{b^*}{b^* - 1} = e^{\frac{c + \delta - \rho}{a}} \qquad \text{if } \alpha = 0 \tag{6.4}$$

and

$$\frac{1}{(b^* - 1)^{\alpha}} - \frac{1}{b^{*\alpha}} = \frac{c + \delta - \rho}{a\Gamma(\alpha)} \qquad \text{if } \alpha \neq 0. \tag{6.5}$$

Solving (6.4) yields

$$b^* = \frac{1}{1 - e^{-(c + \delta - \rho)/a}}, \tag{6.6}$$

which, with $\rho = 0$, is formula (3.1.7) in Gerber and Shiu (1993). In general, equation (6.5) does not yield a closed-form solution for b*. However, if $\alpha = 1$ (exponential jump amounts), one finds

$$b^* = \frac{1 + \sqrt{1 + \frac{4a}{c + \delta - \rho}}}{2}. \tag{6.7}$$

A discussion on the case where $\alpha = -^1/_2$ can be found in Gerber and Shiu (1993).

For each fixed α, we might determine the parameters, a, b and c, by the method of moments. Thus we assume that we know μ, σ and the third central moment of X(1), which we write as $\gamma\sigma^3$ (γ being the coefficient of skewness). Matching the first three moments (by means of formulas (5.3), (5.4) and (5.5)) yields the equations:

$$\mu = \int_0^\infty x\, q(x)\, dx - c = \frac{a\,\Gamma(\alpha+1)}{b^{\alpha+1}} - c,$$

$$\sigma^2 = \int_0^\infty x^2\, q(x)\, dx = \frac{a\,\Gamma(\alpha+2)}{b^{\alpha+2}},$$

and

$$\gamma\sigma^3 = \int_0^\infty x^3\, q(x)\, dx = \frac{a\,\Gamma(\alpha+3)}{b^{\alpha+3}}.$$

From these equations we obtain

$$b = \frac{\alpha+2}{\gamma\sigma}$$

(to be replaced by b* for the evaluation of a derivative security),

$$a = \frac{(\alpha+2)^{\alpha+2}}{\Gamma(\alpha+2)\,\gamma^{\alpha+2}\,\sigma^{\alpha}}, \tag{6.8}$$

and

$$c = \frac{\alpha+2}{\alpha+1}\frac{\sigma}{\gamma} - \mu. \tag{6.9}$$

These formulas generalize (and explain!) the formulas in Sections V.2 and V.3 of Gerber and Shiu (1993). We note that Heston (1993) has independently introduced the gamma process for modeling stock-price movements; his formula (10a) is the same as formula (4.1.7) of Gerber and Shiu (1993).

6.1. Formulas for the Negative Root

With the assumptions $\nu = 0$ and

$$q(x) = ax^{\alpha-1}e^{-bx}, \quad x > 0,$$

equation (5.11) becomes

$$a\int_0^\infty (e^{\theta x} - 1)\, e^{h^* x}\, x^{\alpha-1} e^{-bx}\, dx - c\theta = \delta,$$

or

$$\int_0^\infty (e^{\theta x} - 1)x^{\alpha-1} e^{-b^* x}\, dx = \frac{\delta + c\theta}{a}. \tag{6.1.1}$$

The value of the integral in the left-hand side of (6.1.1) can be read off from (6.3).

If $\alpha = 0$, then (6.1.1) becomes

$$\frac{b^*}{b^* - \theta} = e^{\frac{\delta + c\theta}{a}}. \tag{6.1.2}$$

Substituting b* in (6.1.2) with formula (6.6) yields

$$e^{-c\theta/a} + \theta[e^{\delta/a} - e^{-(c-\rho)/a}] = e^{\delta/a}. \tag{6.1.3}$$

By (6.8) and (6.9),

$$\frac{1}{a} = \frac{\gamma^2}{4}$$

and

$$\frac{c}{a} = \frac{\gamma}{2}\left(\sigma - \frac{\mu\gamma}{2}\right).$$

For example, assume that $\delta = 0.1$, $\rho = 0$, $\mu = 0.1$, $\sigma = 0.2$ and $\gamma = 1$. Then (6.1.3) becomes

$$e^{-3\theta/40} + \theta[e^{1/40} - e^{-3/40}] = e^{1/40},$$

from which we obtain

$$\theta_0 = -7.559609675.$$

Note that, in the Wiener process model, $\theta_0 = -5$ by formula (4.6).

If $\alpha \neq 0$ and $\alpha > -1$, then (6.1.1) becomes

$$\frac{1}{(b^* - \theta)^{\alpha}} - \frac{1}{b^{*\alpha}} = \frac{\delta + c\theta}{a\Gamma(\alpha)}, \tag{6.1.4}$$

where b* is defined by (6.5). In the special case where $\alpha = 1$, (6.1.4) simplifies as

$$\frac{1}{b^* - \theta} - \frac{1}{b^*} = \frac{\delta + c\theta}{a}, \tag{6.1.5}$$

which is a quadratic equation in θ, where b* is given by (6.7),

$$a = \frac{27}{2\gamma^3\sigma}$$

and

$$c = \frac{3}{2}\frac{\sigma}{\gamma} - \mu.$$

Now, consider the zero dividend case ($\rho = 0$), then the positive root of (6.1.5) is $\theta_1 = 1$, and the negative root is

$$\theta_0 = -\delta b^*/c;$$

using the same numerical values as above, $\delta = 0.1$, $\mu = 0.1$, $\sigma = 0.2$ and $\gamma = 1$, we obtain

$$b^* = \frac{1 + \sqrt{901}}{2}$$

and

$$\theta_0 = -7.75416551.$$

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