ON THE VALUATION OF LOAN GUARANTEES UNDER STOCHASTIC INTEREST RATES

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Summary

We extend the loan guarantees literature by modelling, under stochastic interest rates, private financial guarantees when the guarantor potentially defaults. By performing numerical simulations under plausible parameters values, we characterize the differential impact of the incorporation of stochasticity of interest rates on the valuation of both public and private guarantees.

The authors thank William Beranek, David Laughton, William Megginson, Jim Musumeci, Barry Schachter, Louis Scott, Joseph Sinkey, Arthur Snow and Pierre Yourougou for their comments on an infant version of this paper. We are indebted to Peter Carr for suggesting the derivation technique used in the paper. All errors are those of the authors. Financial support from the Social Sciences and Humanities Research of Canada and the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged.

Evaluation des garanties d'emprunts dans le cas de taux d'intérêt stochastiques

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Résumé

Nous développons ici la littérature consacrée aux garanties des prêts par une modélisation, dans des conditions de taux d'intérêt stochastiques, des garanties financières privées lorsque le garant fait potentiellement défaut. En effectuant des simulations numériques avec des valeurs de paramètres plausibles, nous caractérisons l'impact différentiel de l'inclusion de la stochasticité des taux d'intérêt sur l'évaluation des garanties publiques et privées.

Les auteurs remercient William Beranek, David Laughton, William Megginson, Jim Musumeci, Barry Schachter, Louis Scott, Joseph Sinkey, Arthur Snow et Pierre Yourougou de leurs commentaires sur la version préliminaire du présent document. C'est à Peter Carr que nous devons d'avoir suggéré la technique de dérivation appliquée ici. Toutes les erreurs nous sont attribuables. Nous exprimons également notre gratitude au Conseil de recherches en sciences humaines du Canada et au Conseil de recherches en sciences naturelles et en génie du Canada pour leur appui financier.

ON THE VALUATION OF LOAN GUARANTEES UNDER STOCHASTIC INTEREST RATES

Financial guarantees have become increasingly widespread with the development of securitization of various types of loans and the growth of off-balance sheet guarantees by commercial banks and insurance companies (see Hirtle [1987]). A financial guarantee is a commitment by a third party to make payment in the event of a default in a financial contract. Typically a parent company, a bank or an insurance company and often different levels of the government, stand as the third party.

One traditional reason for the popularity of financial guarantees is that they constitute off-balance sheet items. For instance, while loan guarantees by the government represent taxpayers contingent liabilities, they are still not included in the budget (*e.g.*, Baldwin, Lessard, and Mason [1983], Selby, Franks, and Karki [1988]). Bank standby letters of credit likewise are recorded off-balance sheet. Nevertheless, bank regulators recently started to monitor off-balance sheet liabilities and require banks to maintain sufficient capital to support them. The burgeoning demand of municipal bond insurance (see Quigley and Rubinfeld [1991]) and other financial guarantee insurance (*e.g.*, surety bonds, commercial paper insurance, etc.) from insurance companies have also forced regulators to devise safeguards to ensure that losses resulting from financial guarantees do not affect the insurer's other insurance businesses.

In a financial engineering perspective, Merton [1990], Merton and Bodie [1992] show that the purchase of any loan is equivalent in both a functional and valuation sense to the purchase of a pure default-free loan and the simultaneous issue of a default-free guarantee of that loan. They conclude that the analysis of guarantees has relevance to the evaluation of virtually all financial contracts, whether or not guarantees are explicit. Clearly, the valuation of loan guarantees is of interest to all economic agents involved in financial contracting.

While assuming a nonstochastic interest rate financial economists have focussed their studies on the valuation of loan guarantees by the federal government or its affiliated agencies, which may be considered riskless or default-free guarantors (*e.g.*, Merton's [1977], Jones and Mason [1980], Sosin[1980], Chen, Chen, and Sears [1986], Selby, Franks, and Karki [1988]). Recently, after reviewing option-pricing and the valuation of loan guarantees, Lai [1992] uses a discrete-time framework to analyze guarantees by a risky guarantor, but still in a nonstochastic interest-rate environment.

Loan guarantees are not only subject to credit risk but also, as financial claims, to interest-rate risk which to our knowledge has not been taken into account in existing models. The ensuing question is whether the explicit incorporation of stochastic interest rates gives rise to economically meaningful effects on the valuation of loan guarantees. The answer to this question is by no means obvious. In the related risk-adjusted deposit insurance literature, Ronn and Verma [1986] show that the incorporation of stochastic processes for the riskless rate of interest does not materially affect the valuation of such insurance. On the other hand, McCulloch [1985] and Pennacchi [1987] find that the volatility of interest rates do affect the value of deposit insurance. Following the work of Merton [1973], Jones and Mason [1980] conjecture that since stochastic interest rates could be treated as an increase in total risk, guarantee values computed under nonstochastic interest rates are low estimates of the "exact" values.

To investigate the impact of the stochasticity of interest rate in the valuation of loan guarantees, we develop a general model which explicitely accounts for both credit risk and interest-rate risk using Merton's [1973] interest rate process.

Our numerical simulations under plausible parameters values demonstrate that a) the incorporation of a stochastic interest-rate regime does affect significantly the value of loan guarantees and that b) the elasticity of the value of guarantees with respect to the volatility of interest rate is larger for public guarantee than for private guarantee. We are able to verify the

Jones and Mason's [1980] conjecture about the underestimation of loan guarantees when they are computed with deterministic interest rates. The rest of the paper is organized as follows.

In Section I, we derive our model of vulnerable loan guarantees under Merton's [1973] interest-rate process. We position our model in relation to the loan guarantees literature in Section II; in particular, we show that existing models with deterministic interest rates are special cases of our extended model. We present and discuss our simulation results in Section III . Section IV concludes the paper.

I. A SIMPLIFIED MODEL OF VULNERABLE LOAN GUARANTEES

A. Assumptions

Under the standard framework of continuous-time finance, we assume that all assets are traded in perfect and informationally symmetric markets (no taxes, no transaction and backruptcy costs, perfect divisibility, etc.), that continuous trading is allowed, and that the return on the guarantee and guarantor's assets and the interest rate follow continous-time diffusion processes.[1] We assume there is no violation of the absolute priority rule and ignore all potential agency problems inherent to financial contracting (for a discussion of agency problems, see Campbell [1988], Smith [1980]). We assume that the capital structure of the guaranteed firm consists solely of equity and the single issue of the debt being valued. More complex bond characteristics such as call and sinking funds features are not considered.

More specifically we make the following assumptions for our valuation model.

[1] Following Black and Scholes [1973], we adopt the continuous-time approach which leads to a valuation relationship independent of investors preferences. Even if some underlying assets and insurance contracts (*e.g.*, standby letters of credit and surety bonds) are not traded continously and publicly, it is only needed that capital markets are sufficiently complete to assure the existence of assets for replication of the underlying assets.

(A.1) Bond price dynamics

As in Merton [1973], Schwartz [1982], Carr [1987], Chance [1990] among others, let $Q(\tau)$ be the price of a default-free unit discount bond with the same time to maturity, τ, as the debt to be valued. Assume that $Q(\tau)$ satisfies

$$dQ/Q = \alpha_Q(\tau)dt + \sigma_Q(\tau)dz_Q(t;\tau) \tag{1}$$

where α_Q is the instantaneous expected return on the bond, σ_Q is the instantaneous standard deviation, deterministic function of time, t, and $dz_Q(t;\tau)$ is a Gauss-Wiener process for maturity τ. We also denote r, as the instantaneous riskless rate of interest.[2,3]

(A.2) Dynamics of the guarantor and guarantee's firm value

Let W be the value of the guarantor firm and V be the value of the firm issuing the debt to be guaranteed. The continuous paths these values follow are described by the stochastic differential equations

$$dW/W = \alpha_W dt + \sigma_W dz_W \tag{2}$$

and

$$dV/V = \alpha_V dt + \sigma_V dz_V, \tag{3}$$

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[2] Following the work of Merton [1973], Vasicek [1977], Dothan [1978], a substantial body of literature has emerged for the characterization of the term structure of interest rates in arbitrage or equilibrium settings, (see for instance, Cox, Ingersoll and Ross [1985], Brennan and Schwartz [1979, 1982], Courtadon [1982], Ball and Torous [1983], Schaefer and Schwartz [1984, 1987], Longstaff, and Schwartz [1992], Ho and Lee [1986], Heath, Jarrow and Morton [1992]). See Chan, Karolyi, Longstaff and Sanders [1992] for an empirical comparison of a variety of competing continuous-time models of the short term riskless rate. In spite of its drawback, the lognormality assumption for bond prices, which can imply negative interest rates, enables us to obtain tractable model of vulnerable financial guarantees under stochastic interest rates. In her analysis of the feasibility of arbitrage-based option pricing when stochastic bond prices are involved, Cheng [1991] recognizes that ***the simple log-normal bond price process is likely to satisfy the bond price specification***, even though it implies negative interest rates with positive probability.

[3] As in Merton (1973, footnote 43), we assume the short-rate r follows a Gauss-Wiener process according to $dr = \alpha_r\, dt + \sigma_r\, dz$. This stochastic process is simply a Brownian motion with drift α_r and volatility σ_r, (see Chan, Karolyi, Longstaff, and Sanders [1992] for an empirical specification). As will be evidenced later, we have constrained the exercise price to not exceed the face value of the loan at maturity and we do not allow the interest rate to be negative. Furthermore for our baseline parameters values, problems related to pinning the bond price to a fixed value at maturity would only occur with excessive interest rate volatility.

where α_W and α_V are the instantaneous returns on the assets, and σ_W and σ_V are the deterministic instantaneous standard deviations of respectively the guarantor and the insured firm. The Gaus-Wiener processes dz_Q, dz_W, and dz_V are such that their correlation, ρ, are given by $dz_W.dz_Q = \rho_{WQ}\,dt$; $dz_V.dz_Q = \rho_{VQ}\,dt$; and $dz_V.dz_W = \rho_{VW}\,dt$.

(A.3) No dividends or coupons

There are no payouts from either the firm or its guarantor to shareholders and bondholders before the maturity date of the debt.

To calculate the value of the guarantee, *G*, we first compute the value of the guaranteed debt, *Bg*, from which we substract the value of the debt without guarantee, *B*.

B. Value of the guaranteed debt

Consider a pure discount (zero coupon) debt, *Bg*, with promised principal *F*. Under the assumptions A.1 to A.3 and perfect capital markets, Merton's [1973] standard hedging arguments can be used to derive the following partial differential equation (PDE) governing the value of a guaranteed debt, *Bg*

$$\frac{1}{2}\sigma_W^2 W^2 Bg_{WW} + \frac{1}{2}\sigma_V^2 V^2 Bg_{VV} + \frac{1}{2}\sigma_Q^2 Q^2 Bg_{QQ}$$
$$+\sigma_{VW} VWBg_{VW} + \sigma_{WQ} WQBg_{WQ} + \sigma_{VQ} VQBg_{VQ} - Bg_{\tau} = 0, \qquad (4)$$

where subscripts for *Bg* denote partial derivatives, and

$$\sigma_{WQ} = \rho_{WQ}\sigma_W\sigma_Q\,;\ \sigma_{VQ} = \rho_{VQ}\sigma_V\sigma_Q\,;\ \sigma_{VW} = \rho_{VW}\sigma_V\sigma_W.$$

Equation (4) is a special form of the Fundamental Differential Equation for contingent claims and could be obtained in the general equilibrium framework of Cox, Ingersoll, and Ross [1985a] with appropriate assumptions about preferences and technologies. We also note that the value of the debt is independent of the expected return on the firm and its guarantor. It depends only on the current values of the firm and the guarantor of the debt.

The boundary condition at maturity for Bg is

$$Bg = Min(V+W, F). \tag{5}$$

A solution to this PDE (4) subject to (5) can be obtained using Cox-Ross [1976] generalized "risk-neutralized" approach, where the value $\Omega(x_1, x_2, ..x_n)$ of a n-asset contingent claim which pays off $\Omega_T(\cdot)$ at time T is given by [4]

$$\Omega = \hat{E}\left[\exp(-\bar{r}(T-t))\Omega_T\right] = \int \cdot \int \cdot \int \Omega_T(\cdot)L(\cdot)\,dx_1 dx_2 \cdots dx_n,$$

where $\hat{E}$ is the expectation operator in a risk neutral world, $\bar{r}$ the average risk-free rate between t and T, and $L(\cdot)$ the n-joint probability density function. Under our assumptions $\bar{r}$ is α_Q and $L(\cdot) = L(V, W, Q)$ joint lognormal.

The value of the insured debt Bg can be decomposed in three parts according to the states of nature, at maturity.

i) ***The firm is solvent***

If the firm is solvent (*i.e.*, $V \geq F$) the bondholder gets F, regardless of the wealth of the guarantor,

$$Bg_i = \int_0^\infty \int_0^\infty \int_F^\infty F\, L(\cdot) dV dW dQ.$$

[4] See Hull [1989] for more details.

ii) The firm is bankrupt and the guarantor has sufficient funds to honor its commitment

When the firm is bankrupt ($0 \leq V \leq F$) and the value of the guarantor exceeds the shortfall, $W \geq F - V$, the bondholder gets F

$$Bg_{ii} = \int_0^\infty \int_{F-V}^\infty \int_0^F F\, L(\cdot) dV dW dQ.$$

For integration purposes, it is convenient to rewrite the value of the insured debt in these two situations when the bondholder gets F, as the difference of two integrals.

$$Bg_i + Bg_{ii} = Bg_1 - Bg_2$$

where $Bg_1 = \int_0^\infty \int_0^\infty \int_0^\infty F\, L(\cdot) dV dW dQ,$ (6a)

$$Bg_2 = \int_0^\infty \int_0^{F-V} \int_0^V F\, L(\cdot) dV dW dQ. \tag{6b}$$

The first integral Bg_1 represents the face value of the loan regardless of the states of the firm and its guarantor. The second integral Bg_2 is the correction made when both firms go bankrupt.

iii) The firm is bankrupt but the guarantor does not have sufficient funds to honor its obligation

In this case, the value of the guarantor is lower than the shortfall ($0 < W < F - V$) and the bondholder receives the salvage values of the two companies V and W, instead of F.

$$Bg_3 = \int_0^\infty \int_0^{F-V} \int_0^F V\, L(\cdot) dV dW dQ. \tag{6c}$$

$$Bg_4 = \int_0^\infty \int_0^{F-V} \int_0^F W\, L(\cdot) dV dW dQ. \tag{6d}$$

The value of the guaranteed loan is then

$$Bg = Bg_1 - Bg_2 + Bg_3 + Bg_4. \tag{7}$$

This leads to the main proposition of this paper where three of the above four integrals are written in analytical forms amenable to numerical solutions.

Proposition I : *The value of a loan with private guarantee under Merton's [1973] stochastic unit bond price dynamics is given by*

$$Bg = Bg_1 - Bg_2 + Bg_3 + Bg_4 \tag{7}$$

where

$$Bg_1 = FQ, \tag{7.1}$$

$$Bg_2 = FQ\int_{-\infty}^{\ln(FQ/V)}\int_{-\infty}^{\ln[(FQ/W)-(V/W)\exp(x)]} n\begin{bmatrix}-.5\sigma^2_{V/Q} \\ -.5\sigma^2_{W/Q}\end{bmatrix}; \Sigma \Bigg] dydx, \tag{7.2}$$

with $\sigma^2_{V/Q} = \int_0^T\left(\sigma_V^2+\sigma_Q^2-2\sigma_{VQ}\right)ds$

$\sigma^2_{W/Q} = \int_0^T\left(\sigma_W^2+\sigma_Q^2-2\sigma_{WQ}\right)ds$

and,

$$\Sigma=\begin{bmatrix}\sigma^2_{V/Q} & \sigma_{V/Q.W/Q} \\ \sigma_{V/Q.W/Q} & \sigma^2_{W/Q}\end{bmatrix},$$

with $\sigma_{V/Q.W/Q} = \int_0^T\left(\sigma_{VW}-\sigma_{VQ}-\sigma_{WQ}+\sigma_Q^2\right)ds,$

$n(\mu\,;\Sigma)$ = Bivariate normal density function with vector mean μ and covariance matrix Σ.

$$Bg_3 = V\int_{-\infty}^{\ln(FQ/V)}\int_{-\infty}^{\ln[(FQ/W)-(V/W)\exp(x)]} n\begin{bmatrix}+.5\sigma^2_{V/Q} \\ -.5\sigma^2_{WQV}\end{bmatrix}; \Sigma \Bigg] dydx, \tag{7.3}$$

where $\sigma^2_{WQV} = \int_0^T\left(\sigma_W^2 - \sigma_Q^2 - 2\sigma_{VW} + 2\sigma_{VQ}\right)ds.$

And

$$Bg_4 = W\int_{-\infty}^{\ln(FQ/V)}\int_{-\infty}^{\ln[FQ/W)-(V/W)\exp(x)]} n\begin{bmatrix} +.5\sigma_{W/Q}^2 \\ -.5\sigma_{VQW}^2 \end{bmatrix}; \Sigma \Bigg] dydx, \tag{7.4}$$

where $\sigma_{VQW}^2 = \int_0^T \left(\sigma_V^2 - \sigma_Q^2 - 2\sigma_{VW} + 2\sigma_{WQ}\right)ds.$

For ease of notation, paramaters involving Q have not been indexed for time.

Proof : See Appendix.

Bg_1 represents the present value of the face amount F discounted at a stochastic (Gaussian) interest rate. Note that each of the other three integrals has the same integration limits and involves cumulative bivariate normal distributions with the same covariance matrix but different means or location parameters corresponding to risk adjustment in a risk-neutralized world.

Closed form solutions for the integrals can not be obtained because a dependent variable (*i.e.*, x) appears in the upper limit of integration. We use Drezner's [1978] and Simpson algorithms to compute Bg_2, Bg_3 and Bg_4. This is easier than trying to solve the PDEs.[5]

C. Value of the debt without guarantee

As first shown by Merton [1974], the value of a loan without a guarantee, $B(V,Q,\tau)$, satisfies the following equation :

$$\frac{1}{2}\sigma_V^2 V^2 B_{VV} + \frac{1}{2}\sigma_Q^2 Q^2 B_{QQ} + \sigma_{VQ} VQ B_{VQ} - B_\tau = 0. \tag{8}$$

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[5] In their survey of option-pricing applied to mortgages pricing, Hendershott and Van Order [1987] indicate that solving PDEs involving three state variables has not been attempted so far, due to the lack of computational technology (tractability and affordability). For an example of resolution of PDE containing two state variables, see for instance Titman and Torous [1989] among others.

The boundary condition in this case is

$$B(V, Q, 0) = Min\,(V, F), \tag{9}$$

or $\quad B = V - S = V - Call(V, F, Q, \tau).$

The value of the call option, Call (·), which is the equity, S, can be obtained from Merton [1973]

$$Call\,(V, F, Q, \tau) = S(V, Q, \tau) = [V\,erfc(-h_1) - FQ\,erfc(-h_2)]/2,$$

where $h_1 = [\ln(V/FQ) + .5\,I]/\sqrt{2I}$,

$h_2 = [\ln(V/FQ) - .5\,I]/\sqrt{2I}$,

$$I = \int_0^\tau \sigma_{V/Q}^2 ds = \int_0^\tau \left(\sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ}\right) ds = \int_0^\tau \left(\sigma_V^2 + \sigma_Q^2 - 2\rho_{VQ}\sigma_V\sigma_Q\right) ds$$

$erfc(z) = 1 - erf(z)$ with $erf(z) = \left(2/\sqrt{\pi}\right)\int_0^z \exp(-u^2)\,du$, and *erfc* and *erf* stand for error function complementary and error function, respectively. Hence,

$$B = V[1 - (erfc(-h_1))/2] + [FQ\,erfc(-h_2)]/2. \tag{10}$$

Note that equation (10) follows directly from Proposition I by setting W and σ_W equal to zero, *i.e.*, the value of the debt without guarantee is the value of the guaranted debt when there is no guarantor. To check our numerical algorithm, we use both Proposition I (*i.e.*, equation 7) and equation (10) above to derive the value of B.

D. Value of the guarantee

The value of the guarantee, G, is the difference between Bg and B as given by equations (7) and (10) respectively. Table 1 summarizes the end-of-period payoffs. The value of the guarantee is the risk-adjusted present value of the terminal payoff discounted at Merton's [1973] unit bond prices.

Table 1
Value of debts and guarantee at maturity

Case	Debt with guarantee (Bg)	Debt without guarantee (B)	Guarantee (G)
$0 < W < \infty$; $V \geq F$	F	F	0
$0 < V < F$; $V + W \geq F$	F	V	F-V
$0 < V < F$; $V + W < F$	V +W	V	W

The next section situates our model with stochastic interest rates in the loan guarantees literature.

II. CONTRIBUTION TO THE LOAN GUARANTEES LITERATURE

The existing financial guarantees literature which uses contingent claims analysis can be categorized by the nature of the guarantor : risky/private/vulnerable versus default-free/public/ government, and by the interest-rate regime : stochastic versus deterministic.

We present below three propositions on the value of loans guarantees corresponding to different scenarios of the interest rate regime and the nature of the guarantor. Proofs for the propositions follow from successive applications of Proposition I, under appropriate conditions, to obtain the value of the debt with guarantee *Bg* and the debt without guarantee *B*. The value of the guarantee, *G*, is simply the difference between these two values.

A. Government guarantee and constant interest rate

Following the seminal work of Merton [1977], most of the loan guarantees literature falls into this category. As matter of fact, there now exists a vast literature related to the valuation of deposit insurance.

The government is commonly viewed as a riskless institution ($\sigma_W = 0$) with unlimited assets ($W = \infty$). This condition combined with constant interest rate ($\sigma_Q = 0$) leads to the following proposition.

Proposition II : *The value of a government guarantee under constant interest rate regime,* $G_{f,r}$ *is given by* $G_{f,r} = Fe^{-rT}\left(1-N(h_2^*)\right) - V\left(1-N(h_1^*)\right)$.

The first term is the present value in a deterministic interest rate environment of the face value times the probability that the debt issuing firm is insolvent. The second term is interpreted as the expected present value of the borrowing firm given that its assets exceed the nominal value of the loan.

Proof : By letting W tend to infinity and σ_W and σ_Q tend to 0, it is immediate that Bg_2, Bg_3, and Bg_4 tend to 0, hence the value of government guaranteed debt under constant interest rate regime is given by

$$Bg = Bg_1 = FQ = Fe^{-rT},$$

where r is the (constant) risk-free rate.

In the same manner, by letting W, σ_W and σ_Q (σ_r) equal to 0 and Q equal to e^{-rT}, it follows from Proposition I that the uninsured debt, B, is

$$B = V - e^{-rT}\int_F^{\infty}(V-F)dV.$$

The integral portion represents the Black-Scholes formula for a European call option on the value of the firm V, with exercise price F and time to maturity T.

The value of the guarantee under a default-free and constant interest rate regime, $G_{f,r}$, is then

$$G_{f,,r} = Bg - B = Fe^{-rT}\left(1-N(h_2^*)\right) - V\left(1-N(h_1^*)\right)$$

where $N(\cdot)$ = cumulative normal function,

$$h_1^* = (\ln(V/F) + rT)/\sigma_V\sqrt{T} + 1/2\sigma_V\sqrt{T},$$

$$h_2^* = h_1^* - \sigma_V\sqrt{T}.$$ QED

B. Government guarantee and stochastic interest rate

The introduction of a stochastic interest rate (*i.e.*, $\sigma_Q \neq 0$) modifies the formulas in the following way.

Proposition III : *The value of a government guaranteed debt under Merton [1973] unit bond price process is given by* $G_{f,s} = FQ(1 - erfc(h_2)/2) - V[1 - erfc(-h_1)/2]$, *where h_1 and h_2 are as defined in Section I.C equation (10).*

The terms in the expression for $G_{f,s}$ are the "stochastic" counterparts of the two terms in Proposition II.

Proof : By letting W tend to infinity and σ_W to 0 we have, as in Proposition II, $Bg_2 = Bg_3 = Bg_4 = 0$. Hence, $Bg = FQ$, where as in Merton [1973], $Q = \exp\left(-rT - \alpha_r T^2/2 + \sigma_r^2 T^3/6\right)$ with α_r and σ_r being the drift and the volatility of the interest rate, respectively (see Footnote 3).

The value of the bond without guarantee B is

$$B = V - \int_0^{\infty}\int_F^{\infty}(V - F)dVdQ.$$ [6]

The value of the guarantee $G_{f,s}$ by a default-free guarantor under a stochastic interest rate regime is then given by

$$G_{f,s} = Bg - B = FQ(1 - erfc(-h_2)/2) - V(1 - erfc(-h_1)/2). \text{ QED}$$

C. Private guarantee and constant interest rate

When the guarantor is risky, the state variable W comes into play.

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[6] It could be shown that the double integral reduces to Merton's [1973] formula for a European call option on the value of the firm with exercise price F under a stochastic interest rate regime. Note the identity $N\left(\sqrt{2}h\right) = (1/2)erfc(-h)$.

Proposition IV : *The value of a private guarantee under constant interest rate, $G_{p,r}$, is given by*

$$G_{p,r} = G_{f,r} + \left(-Bg_2 + Bg_3 + Bg_4\right)$$

where

$$Bg_2 = Fe^{-rT} \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln\left[(Fe^{-rT}/W)-(V/W)e^{x}\right]} n\begin{bmatrix} -1/2\sigma_V^2 T \\ -1/2\sigma_W^2 T \end{bmatrix}; \Sigma \Bigg] dydx,$$

$$Bg_3 = V \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln\left[(Fe^{-rT}/W)-(V/W)e^{x}\right]} n\begin{bmatrix} +(1/2)\sigma_V^2 T \\ -(1/2)\left(\sigma_W^2 - 2\sigma_{VW}\right)T \end{bmatrix}; \Sigma \Bigg] dydx,$$

$$Bg_4 = W \int_{-\infty}^{\ln(Fe^{-rT}/V)} \int_{-\infty}^{\ln\left[(Fe^{-rT}/W)-(V/W)e^{x}\right]} n\begin{bmatrix} -(1/2)\left(\sigma_V^2 - 2\sigma_{VW}\right)T \\ +(1/2)\sigma_W^2 T \end{bmatrix}; \Sigma \Bigg] dydx,$$

and $$\Sigma = \begin{bmatrix} \sigma_V^2 & \sigma_{VW} \\ \sigma_{VW} & \sigma_W^2 \end{bmatrix} T.$$

Proof : The proof proceeds along the lines of Proposition II with the introduction of W and σ_W.[7]

Note that the difference between the value of the public and private guarantees, Propositions II and IV respectively, represents a market credit risk premium attached to an insurer with a full faith and credit rating. It is given by the expression $+Bg_2 - Bg_3 - Bg_4$.

Corrollary : The value of a government guarantee is equal to or greater than a private guarantee.

[7] This expression for the value of the guarantee can be reconciled with the expression obtained by Johnson and Stulz [1987] (equation 27, p. 279) with an appropriate change of the variables to standardized normal bivariates.

Proof : Since the expected proceeds to the bondholder when both the debt insuring firm and its guarantor go bankcrupt, $(Bg_3 + Bg_4)$, is less than the expected value of the face value of the discount debt (Bg_2) in the state of joint bankcruptcy of the two firms $F \geq (V+W)$, we have $Bg_2 \geq Bg_3 + Bg_4$. Therefore, from Proposition IV, $G_{f,r} \geq G_{p,r}$. QED

Table 2 presents a classification of the existing loan guarantee literature within our framework.

Table 2 : Loan Guarantees Literature

	Government guarantee	Private guarantee
Deterministic interest rate	*Proposition II* Merton [1977], Sosin [1980], Jones and Mason [1980], Chen et al. [1986], Selby et al. [1988] Deposit insurance : Merton [1977] and others	*Proposition IV* Johnson and Stulz [1987] Lai [1992]
Stochastic interest rate	*Proposition III* Deposit insurance : McCulloch [1985], Ronn and Verma [1986] Pennacchi [1987]	*Proposition I* This paper

We now turn to our numerical simulations to verify whether the incorporation of the stochasticity of interest rate affects significantly the valuation of loan guarantees by both private and public guarantors.

III. SIMULATION RESULTS

We use Drezner's [1978] algorithm for the computation of the bivariate normal integral for the portion where the integration limits (bounds) are constant.[8] For the variable bound part, we use a composite Simpson procedure (see Burden and Faires [1989]).

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[8] Hull (1989, Appendix 5A) and Stoll and Whaley (1993, Appendix 13.1) provide summaries of Drezner's [1978] algorithm. There exist alternative algorithms for approximating the bivariate normal integral, see for instance Divgi [1979].

For Merton's [1973] interest rates dynamics, following Chan, Karolyi, Longstaff, and Sanders [1992], we use $\alpha_r = 0.0055$, $\sigma_r = 0.02$ and $r = 0.067$ per annum.[9]

The present value of the borrowing firm (V) is set at \$1100 with a volatility (σ_V) at 0.3. The private guarantor value (W) is \$1500 with a risk (σ_V) of 0.3. Their cross correlations are put equal to 0.3. The principal of the loan (F) is \$1000 with a basic maturity (T) of 3 years.

To simulate the cases of default-free government guarantees, we set W arbitrarily large and σ_W arbitrarily small. Note again the presence of a stochastic interest regime from Merton [1973], which departs from the extant literature on government loan guarantees (*e.g.*, Merton [1977], Sosin [1980], Jones and Mason [1980]).

We conduct numerical simulations to derive the comparative statics for the value of the loan with private guarantee (Bg), the one with riskless guarantee (Bgf), the value of the uninsured loan (B) and the two corresponding guarantees (G) and (Gf).

Figure 1 shows the variations of Bgf, Bg, B, Gf, and G as a function of the size of the firms (V, W) and their respective risks (σ_V, σ_W).[10] As expected the value of both the insured and uninsured debt, Bg and B, increase with the value of the firm while the related guarantees, G and Gf decrease with V. Note that for present values of the borrowing firm V larger than \$800.00 the value of the two guarantees are essentially the same. For the range of economically plausible

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9 As specified by Merton [1973], the bond price volatility written as a deterministic function of time, is $\sigma_Q(T) = -\sigma_r T$. Hence the volatility of bond price at maturity is zero.

10 As a cross check, we compare the value B obtained from equation (7) (Proposition I) and the one computed directly from equation (10) employing approximation of the error function (see Abramowitz and Stegun [1972]). The results are identical to the order of 10^{-9}.

firm's asset risk, the borrowing firm risk σ_V has an increasing effect on the value of both the private and government guarantee.[11]

An increase in the claims paying capacity of the private guarantor W increases asymtotically the value of the private guarantee (see Cook and Spellman [1992]). On the other hand, an increase in the guarantor risk (σ_W) reduces its creditworthiness and consequently the value of the private guarantee. With large W or small σ_W, the value of the vulnerable guarantee is indistinguishable from the government default-free one.

Our results, obtained in a more general continuous-time framework with stochastic interest rates, are qualitatively the same as those of Lai [1992] who employs a discrete-time methodology.

Figure 2 shows the impact of the size of the guaranteed loan F, the loan maturity T, the short-rate r and the interest-rate volatility σ_r, on the debt values and guarantees. Obviously, a larger loan implies a larger guarantee. Both the insured and uninsured debts, Bg and B, decrease with the maturity T. However, their difference which is the guarantee increases and decreases with longer maturity.[12]

The other two plots in Figure 2 depict the effect of the interest rate level and its volatility on the loan guarantee variables. The effect of volatility of interest rates on loan guarantees has not been examined in the literature. An increase in the short-term rate r, for a given volatility σ_r = .02, reduces the present value of the price of the default-free unit discount bond Q and hence reduces the value of both the private and public guarantee. For Merton's [1973] interest rate dynamics, our simulation results indicate that both the value of the public and private guarantees

[11] This result contrasts with Lai [1992] in a discrete time framework where assets volatilities compound the time dimension. (See also Stulz and Johnson [1985] in a somewhat different context of a case of debt with a security provision).

[12] This result differs from the one of Sosin [1980] who finds that the guarantee is everywhere an increasing function of the term of the discount loan. Note however that Sosin allows for dividend payments.

increase monotonically with the volatility of interest rate. For the loan without guarantee, an increase in σ_r augments the total risk ($\sigma^2_{V/Q}$) which increases the value of the equity (the call on the borrowing firm asset), hence reduces the value of the debt. On the other hand, a larger total risk increases the put option value on the sum of the value of the firm and its guarantor, this augmentation is less than in the present value of the debt face value. The difference between Bg and B, *i.e.* G, increases with σ_r.

For our baseline parameters, the underestimation of the value of the loan guarantee associated with the use of a constant interest rate instead of a stochastic one with a volatility level of $\sigma_r =$ 0.02 (the average volatility documented by Chen et al. [1992]), is roughly 20%. At level of σ_r of 12% the government guarantee Gf is twice the guarantee computed with a deterministic rate ($\sigma_r = 0$). For the private guarantee an error of 100% occurs with an interest rate volatility of 14%. In economic parlance, for high interest-rates volatilities, the elasticity of the government guarantee to changes in interest rate volatilities is greater than the one of the private guarantee. Our result concurs qualitatively with empirical findings of McCulloch [1983], Pennacchi [1987] but not with those of Ronn and Verma [1987] who find under their models, that the incorporation of stochastic interest rate does not affect significantly the valuation of deposit insurance.[13]

Figure 3 shows the impact of the coefficients of correlation between the firm, the guarantor and the unit interest price Q on the loan guarantee. Given that Q varies inversely with the interest rate, our results show that the more the interest rates and firm value vary in the same direction (*i.e.*, positive correlation), the higher the guarantee. The impact of the correlation between interest rates and guarantor values is imperceptible . As in Lai [1992], the more the guarantor

[13] Note that there exists differences, albeit imperceptible in our figures, between the government and private guarantee.

resembles the insured firm, (*i.e.*, $\rho_{VW} > 0$) the less valuable the private guarantee is in the credit enhancement.

We summarize below the numerical comparative statics results by the signs of the partial derivatives for *Bgf*, *Bg*, *B*, *Gf* and *G*.

$$Bgf = Bgf\ (\overset{0}{V},\ \overset{0}{W},\ \overset{0}{\sigma_V},\ \overset{0}{\sigma_W},\ \overset{-}{r},\ \overset{+}{\sigma_r},\ \overset{0}{\rho_{VQ}},\ \overset{0}{\rho_{WQ}},\ \overset{0}{\rho_{VW}},\ \overset{0}{F},\ \overset{-}{T}),$$

$$Bg = Bg\ (\overset{+}{V},\ \overset{+/0}{W},\ \overset{0/-}{\sigma_V},\ \overset{-}{\sigma_W},\ \overset{-}{r},\ \overset{+}{\sigma_r},\ \overset{0}{\rho_{VQ}},\ \overset{0}{\rho_{WQ}},\ \overset{0}{\rho_{VW}},\ \overset{+}{F},\ \overset{-}{T}),$$

$$B = B\ (\overset{+}{V},\ \overset{0}{W},\ \overset{-}{\sigma_V},\ \overset{0}{\sigma_W},\ \overset{-}{r},\ \overset{-}{\sigma_r},\ \overset{+}{\rho_{VQ}},\ \overset{-}{\rho_{WQ}},\ \overset{-}{\rho_{VW}},\ \overset{-}{F},\ \overset{-}{T}),$$

$$Gf = Gf\ (\overset{-}{V}\ \overset{0}{W},\ \overset{+}{\sigma_V},\ \overset{0}{\sigma_W},\ \overset{-}{r},\ \overset{+}{\sigma_r},\ \overset{0}{\rho_{VQ}},\ \overset{0}{\rho_{WQ}},\ \overset{0}{\rho_{VW}},\ \overset{+}{F},\ \overset{+/-}{T}),$$

$$G = G\ (\overset{-}{V}\ \overset{+/0}{W},\ \overset{+}{\sigma_V},\ \overset{-}{\sigma_W},\ \overset{-}{r},\ \overset{+}{\sigma_r},\ \overset{+}{\rho_{VQ}},\ \overset{-}{\rho_{WQ}},\ \overset{-}{\rho_{VW}},\ \overset{+}{F},\ \overset{+/-}{T}).$$

IV. CONCLUSIONS

Using the continuous time option-pricing methodology, we develop a "simplified" model of private financial guarantees that takes account of both stochastic interest rates and potential default by the guarantor. The extant literature generally assumes a nonstochastic interest rate and a riskless default-free guarantor as buttressed by the works of Merton [1977], Jones and Mason [1980], Chen, Chen, and Sears [1986], Selby, Franks, and Karki [1988] in the context of government loan guarantees. With the exception of Johnson and Stulz's [1987] work on pricing options with default risk and Lai's [1992] analysis of private loan guarantees, both under nonstochastic interest rates, evaluating loan guarantees by vulnerable guarantors under stochastic interest rates has not been attempted in the literature. To demonstrate the impact of stochastic interest rates on the valuation of loan guarantees, we derive a model under Merton's [1973] lognormal bond price process which is used to compute both the values of insured and uninsured debts. We obtain a model which necessitates numerical integration of the bivariate normal density function to circumvent the need to solve partial differential equations with three state variables.

Using numerical simulations, we obtain comparative statics for the debts (uninsured and insured) and the value of the guarantees.

The impact of incorporating the term structure of interest rates is gauged. We find that consistent with the conjecture of Jones and Mason [1980], guarantee values computed under nonstochastic interest rates are low estimates of the "correct" values. In other words, their conjecture is substantiated, and loan guarantees are increasing monotonically with the interest rate volatilities. The value of both the private and public guarantees are found to be an ambiguous function of term of the loan. Our other results are consistent with those of Lai [1992] and the literature on government loan guarantees.

To account for more realistic interest rate process (*e.g.*, Cox-Ingersoll-Ross [1985b]), we have to solve a system of PDEs with more than two state variables which is not tractable at present. This task remains a challenging agenda for further research.

APPENDIX

Derivation of the integrals in the guaranteed loan formula

Let n_{xyz} $(\mu;\Sigma)$ represent the trivariate normal density function with mean vector μ and covariance matrix Σ, *i.e.*, $n(\cdot) = n_{xyz}(\mu;\Sigma) =$

$$(2\pi)^{-3/2}\left|\Sigma^{-1/2}\right|\exp\left\{-1/2\left[(\mathbf{x}-\mu)'\,\Sigma^{-1}(\mathbf{x}-\mu)\right]\right\},$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mu = \begin{bmatrix} \mu_x \\ \mu_y \\ \mu_z \end{bmatrix}$

It can be shown that

$$e^x e^y n_{xyz}(\mu;\Sigma) = e^{\mu_x+\mu_y+(1/2)(\sigma_x^2+2\sigma_{xy}+\sigma_y^2)}\, n\begin{bmatrix} \mu_x+\sigma_x^2+\sigma_{xy} & \\ \mu_y+\sigma_{xy}+\sigma_y^2 & ;\Sigma \\ \mu_z+\sigma_{xy}+\sigma_{yz} & \end{bmatrix}.$$

This property, which we call Property (*) is needed in subsequent derivations. See Rubinstein (1976, Appendix) for a proof. We want to compute the integrals for Bg. Letting E denote the expectation operator, subscript 0 stand for today and T for maturity, we have

$$Bg = (Q_0/E\{Q_T\}) \int_0^\infty\int_0^\infty\int_0^\infty FQ_T L(\cdot)dWdVdQ$$

$$- (Q_0/E\{Q_T\}) \int_0^\infty\int_0^{FQ_T}\int_0^{FQ_T-V_T} FQ_T L(\cdot)dWdVdQ$$

$$+ (Q_0/E\{Q_T\}) \int_0^\infty\int_0^{FQ_T}\int_0^{FQ_T-V_T} V_T L(\cdot)dWdVdQ$$

$$+ (Q_0/E\{Q_T\}) \int_0^\infty\int_0^{FQ_T}\int_0^{FQ_T-V_T} W_T L(\cdot)dWdVdQ$$

Let's make the following changes of the random variables, *i.e.*, $L1(\cdot) = L(\cdot)$ with new variables

$$R_W = W_T/W_0;\ R_V = V_T/V_0;\ R_Q = Q_T/Q_0$$

so :

$$Bg = (Q_0/E\{Q_T\}) \int_0^\infty \int_0^\infty \int_0^\infty FQ_0\, R_Q L1(\cdot)\, dR_W\, dR_V\, dR_Q$$

$$-(Q_0/E\{Q_T\}) \int_0^\infty \int_0^{FQ_0 R_Q/V_0} \int_0^{FQ_0 R_Q/W_0 - V_0 R_V/W_0} FQ_0\, R_Q L1(\cdot)\, dR_W\, dR_V\, dR_Q$$

$$+(Q_0/E\{Q_T\}) \int_0^\infty \int_0^{FQ_0 R_Q/V_0} \int_0^{FQ_0 R_Q/W_0 - V_0 R_V/W_0} V_0\, R_V L1(\cdot)\, dR_W\, dR_V\, dR_Q$$

$$+(Q_0/E\{Q_T\}) \int_0^\infty \int_0^{FQ_0 R_Q/V_0} \int_0^{FQ_0 R_Q/W_0 - V_0 R_V/W_0} W_0\, R_W L1(\cdot)\, dR_W\, dR_V\, dR_Q.$$

Let's make another change of variables:

$$r_{W/V} = \ln(R_W/R_V); \qquad r_{V/Q} = \ln(R_V/R_Q)$$

$$r_{W/Q} = \ln(R_W/R_Q); \qquad r_Q = \ln(R_Q)$$

Substituting, with $n(\cdot) = N'(\cdot)$ = Normal density and $E\{R_Q\} = e^{+\alpha_Q T}$, we obtain

$$Bg = e^{-\alpha_Q T} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty FQ_0 e^{r_Q} N'(\cdot)\, dr_{W/Q}\, dr_{V/Q}\, dr_Q$$

$$-e^{-\alpha_Q T} \int_{-\infty}^\infty \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})]} FQ_0\, e^{r_Q} N'(\cdot)\, dr_{W/Q}\, dr_{V/Q}\, dr_Q$$

$$+e^{-\alpha_Q T} \int_{-\infty}^\infty \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})]} V_0\, e^{r_{V/Q}} e^{r_Q}\, N'(\cdot)\, dr_{W/Q}\, dr_{V/Q}\, dr_Q$$

$$+e^{-\alpha_Q T} \int_{-\infty}^\infty \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})]} W_0\, e^{r_{W/Q}}\, e^{r_Q} N'(\cdot)\, dr_{W/Q}\, dr_{V/Q}\, dr_Q$$

Now,

$$N'(\cdot)dr_{W/Q}dr_{V/Q}dr_Q = n\begin{bmatrix} \mu_{rW/Q} & \sigma^2_{W/Q} & \sigma^2_{W/Q.V/Q} & \sigma_{W/Q.Q} \\ \mu_{rV/Q} & ;\ \sigma_{W/Q.V/Q} & \sigma^2_{V/Q} & \sigma_{V/Q.Q} \\ \mu_{rQ} & \sigma_{W/Q.Q} & \sigma_{V/Q.Q} & \sigma^2_Q \end{bmatrix} dr_{W/Q}dr_{V/Q}dr_Q$$

where

$$\mu = \begin{bmatrix} \mu_{rW/Q} \\ \mu_{rV/Q} \\ \mu_{rQ} \end{bmatrix} = \begin{bmatrix} -(1/2)(\sigma^2_W - \sigma^2_Q)T \\ -(1/2)(\sigma^2_V - \sigma^2_Q)T \\ \alpha_Q - (1/2)(\sigma^2_Q)T \end{bmatrix}$$

because in a risk-neutral world $\alpha_W = \alpha_V = \alpha_Q$, and

$$\Sigma_{(3)} = \begin{bmatrix} \sigma^2_{W/Q} & \sigma_{W/Q.V/Q} & \sigma_{W/Q.Q} \\ \sigma_{W/Q.V/Q} & \sigma^2_{V/Q} & \sigma_{V/Q.Q} \\ \sigma_{W/Q.Q} & \sigma_{V/Q.Q} & \sigma^2_Q \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2_W + \sigma^2_Q - 2\sigma_{QW} & \sigma_{WV} - \sigma_{WQ} - \sigma_{VQ} + \sigma^2_Q & \sigma_{WQ} - \sigma^2_Q \\ \sigma_{WV} - \sigma_{WQ} - \sigma_{VQ} + \sigma^2_Q & \sigma^2_V + \sigma^2_Q - 2\sigma_{VQ} & \sigma_{WQ} - \sigma^2_Q \\ \sigma_{WQ} - \sigma^2_Q & \sigma_{VQ} - \sigma^2_Q & \sigma^2_Q \end{bmatrix} \times T$$

Write $Bg = Bg_1 - Bg_2 + Bg_3 + Bg_4$

$$Bg_1 = e^{-\alpha_Q T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} FQ_0\, e^{r_Q}\, n(\mu_{(3)};\ \Sigma_{(3)})\,dr_{W/Q}dr_{V/Q}dr_Q$$

Using property (*) after simplification yields:

$$Bg_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n\begin{bmatrix} -(1/2)(\sigma^2_W + \sigma^2_Q - 2\sigma_{WQ})T & ;\ \Sigma_{(2)} \\ -(1/2)(\sigma^2_V + \sigma^2_Q - 2\sigma_{VQ})T & \end{bmatrix}$$

$\Sigma = \Sigma_{(2)}$ since the third column and row of $\Sigma_{(3)}$ are eliminated with r_Q as "numeraire". From matrix partioning, we eliminate the outer integral (which equals to 1) involving r_Q

$$\Sigma_{(2)} = \begin{bmatrix} \sigma_W^2 + \sigma_Q^2 - 2\sigma_{WQ} & \sigma_{WV} - \sigma_{WQ} - \sigma_{VQ} + \sigma_Q^2 \\ \sigma_{WV} - \sigma_{WQ} - \sigma_{VQ} + \sigma_Q^2 & \sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ} \end{bmatrix} T$$

so that $Bg_1 = FQ_0$.

$$Bg_2 = FQ_0 \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln\left[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})\right]} n \begin{bmatrix} -(1/2)(\sigma_W^2 + \sigma_Q^2 - 2\sigma_{WQ})T \\ -(1/2)(\sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ})T \end{bmatrix} ; \Sigma_{(2)} \Bigg] dr_{W/Q} dr_{V/Q} dr_Q$$

$$Bg_3 = e^{-\alpha_Q T} \int_{-\infty}^{\infty} \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln\left[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})\right]} V_0\, e^{r_{V/Q}} e^{r_Q}$$

$$n \begin{bmatrix} -(1/2)\left(\sigma_W^2 - \sigma_Q^2\right)T \\ -(1/2)\left(\sigma_V^2 - \sigma_Q^2\right)T ; \Sigma_{(3)} \\ \left(\alpha_Q - 1/2\sigma_Q^2\right)T \end{bmatrix} dr_{W/Q} dr_{V/Q} dr_Q.$$

Likewise applying Property (*) and simplifying, we obtain,

$$Bg_3 = V_0 \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln\left[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})\right]} n \begin{bmatrix} -(1/2)(\sigma_W^2 - \sigma_Q^2 - 2\sigma_{WV} + 2\sigma_{VQ})T \\ +(1/2)(\sigma_V^2 + \sigma_Q^2 - 2\sigma_{VQ})T \end{bmatrix} ; \Sigma_{(2)} \Bigg] dr_{W/Q} dr_{V/Q} dr_Q$$

and from

$$Bg_4 = e^{-\alpha_Q T} \int_{-\infty}^{\infty} \int_{-\infty}^{\ln(FQ_0/V_0)} \int_{-\infty}^{\ln\left[FQ_0/W_0 - (V_0/W_0)\exp(r_{V/Q})\right]} W_0 e^{r_{W/Q}} e^{r_Q}$$

$$n\begin{bmatrix} -(1/2)\left(\sigma_W^2-\sigma_Q^2\right)T & \\ -(1/2)\left(\sigma_V^2-\sigma_Q^2\right)T\,; & \Sigma_{(3)} \\ \left(\alpha_Q-(1/2)\sigma_Q^2\right)T & \end{bmatrix} dr_{W/Q}\,dr_{V/Q}\,dr_Q$$

In a similar fashion as for Bg_3, we obtain finally

$$Bg_4 = W_0\int_{-\infty}^{\ln(FQ_0/V_0)}\int_{-\infty}^{\ln\left[FQ_0/W_0-(V_0/W_0)\exp(r_{V/Q})\right]}$$

$$n\begin{bmatrix} +(1/2)(\sigma_W^2-\sigma_Q^2+2\sigma_{WQ})T & \\ -(1/2)(\sigma_V^2+\sigma_Q^2-2\sigma_{WV}+2\sigma_{WQ})T & ;\ \Sigma_{(2)} \end{bmatrix} dr_{W/Q}\,dr_{V/Q}.$$

Set $\Sigma = \Sigma_{(2)}$, $x = r_{V/Q}$ and $y = r_{W/Q}$, $T = \int_0^T ds$ and reorder the variables, we have the results in the text.

Note that for ease of notation we have suppressed time in the variances and covariances terms involving Q.

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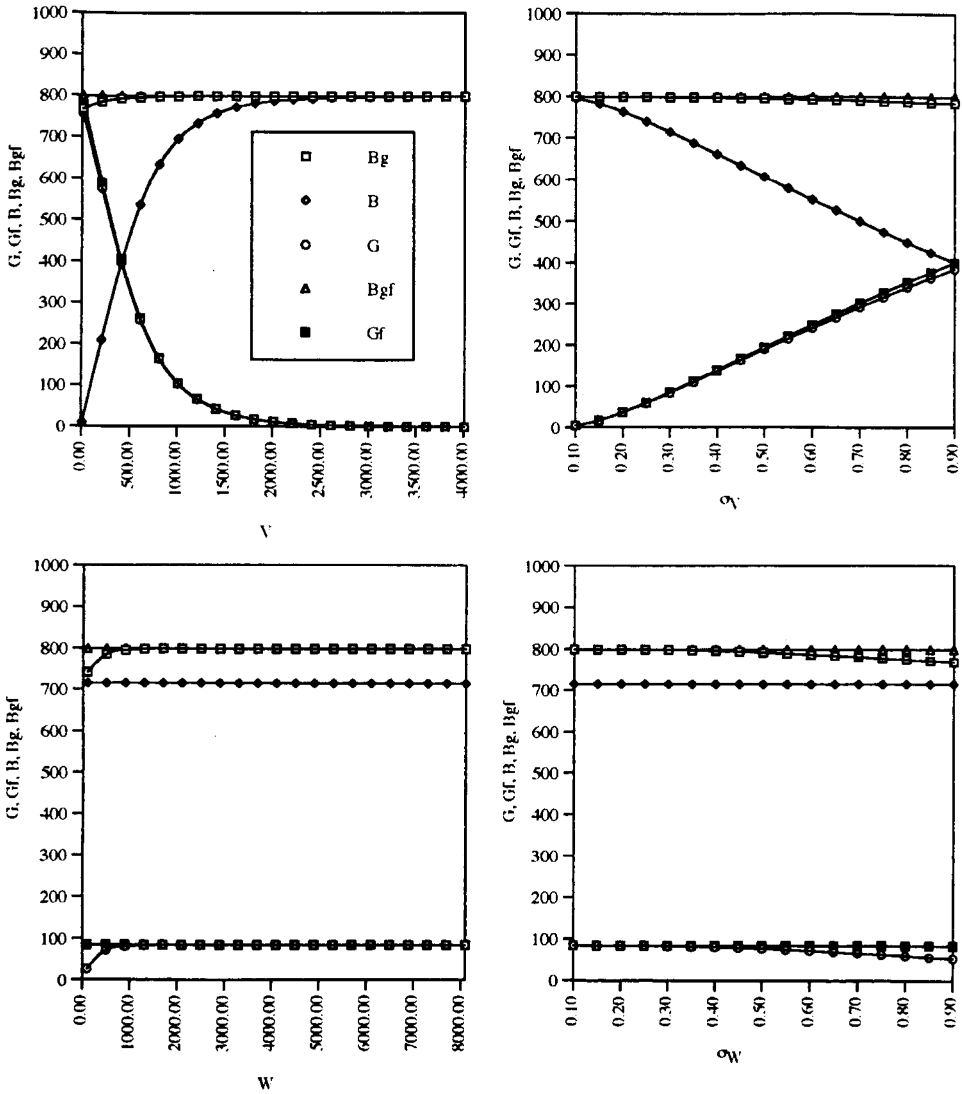

Figure 1: Impact of the size of the firm (V), the claims paying capacity of the guarantor (W) and their respective risks (σ_V) and (σ_W) on the value of the loan with guarantee (Bg), the one without guarantee (B) and the guarantee (G). The subscript (f) denotes the cases corresponding to the default-free guarantee.
Note: We conduct the experiments with V = 1100, $\sigma_V = 0.3$, W = 1500, $\sigma_W = 0.3$, r = 0.067, $\sigma_r = 0.02$, $\rho_{WV} = 0.3$, $\rho_{VQ} = 0.3$, $\rho_{WQ} = 0.3$, T = 3, F = 1000, while varying the studied parameters in the comparative statics ceteris paribus.

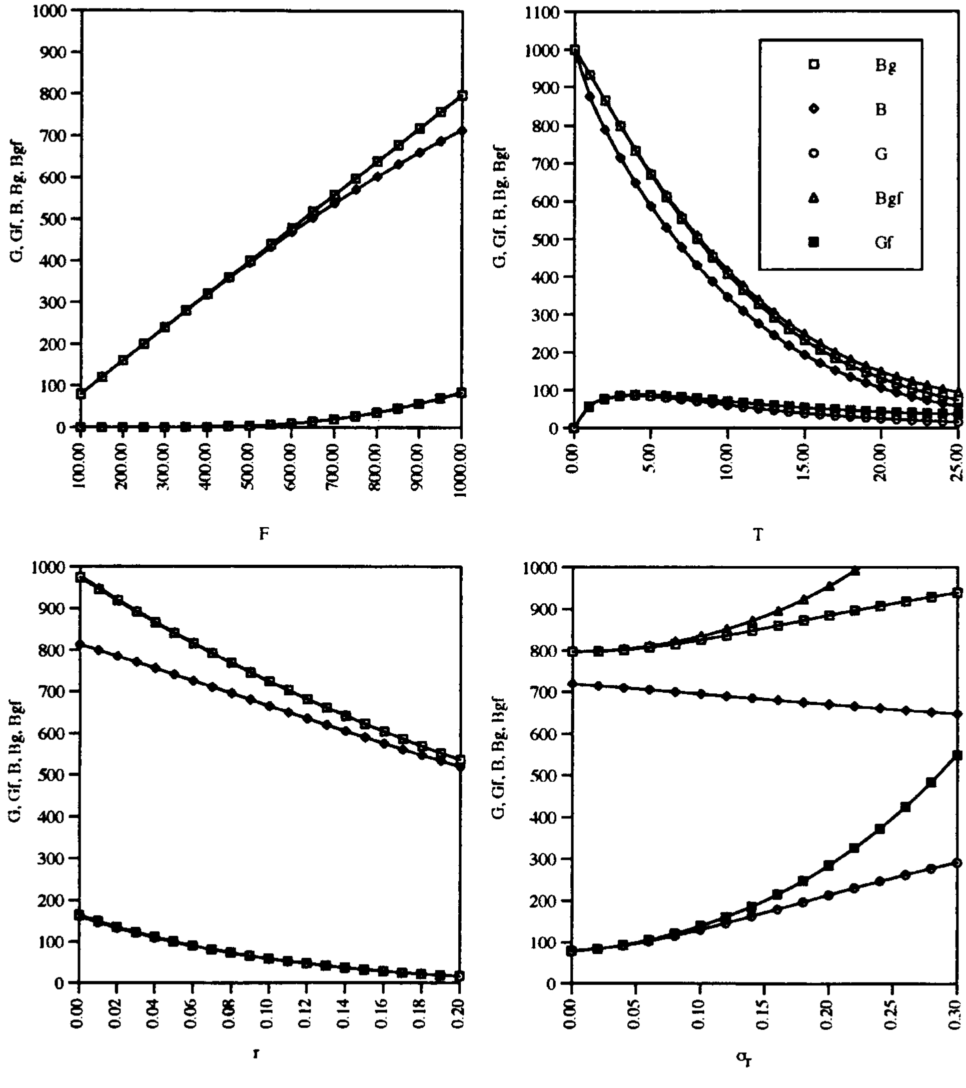

Figure 2: Impact of the size of the loan (F), the time to maturity (T), the short term interest rate (r), the volatility of rates (σ_r) on the value of the loan with guarantee (Bg), the one without guarantee (B) and the guarantee (G). The subscript (f) denotes the cases corresponding to the default-free guarantee.

Note: We conduct the experiments with $V = 1100$, $\sigma_V = 0.3$, $W = 1500$, $\sigma_W = 0.3$, $r = 0.067$, $\sigma_r = 0.02$, $\rho_{WV} = 0.3$, $\rho_{VQ} = 0.3$, $\rho_{WQ} = 0.3$, $T = 3$, $F = 1000$, while varying the studied parameters in the comparative statics ceteris paribus.

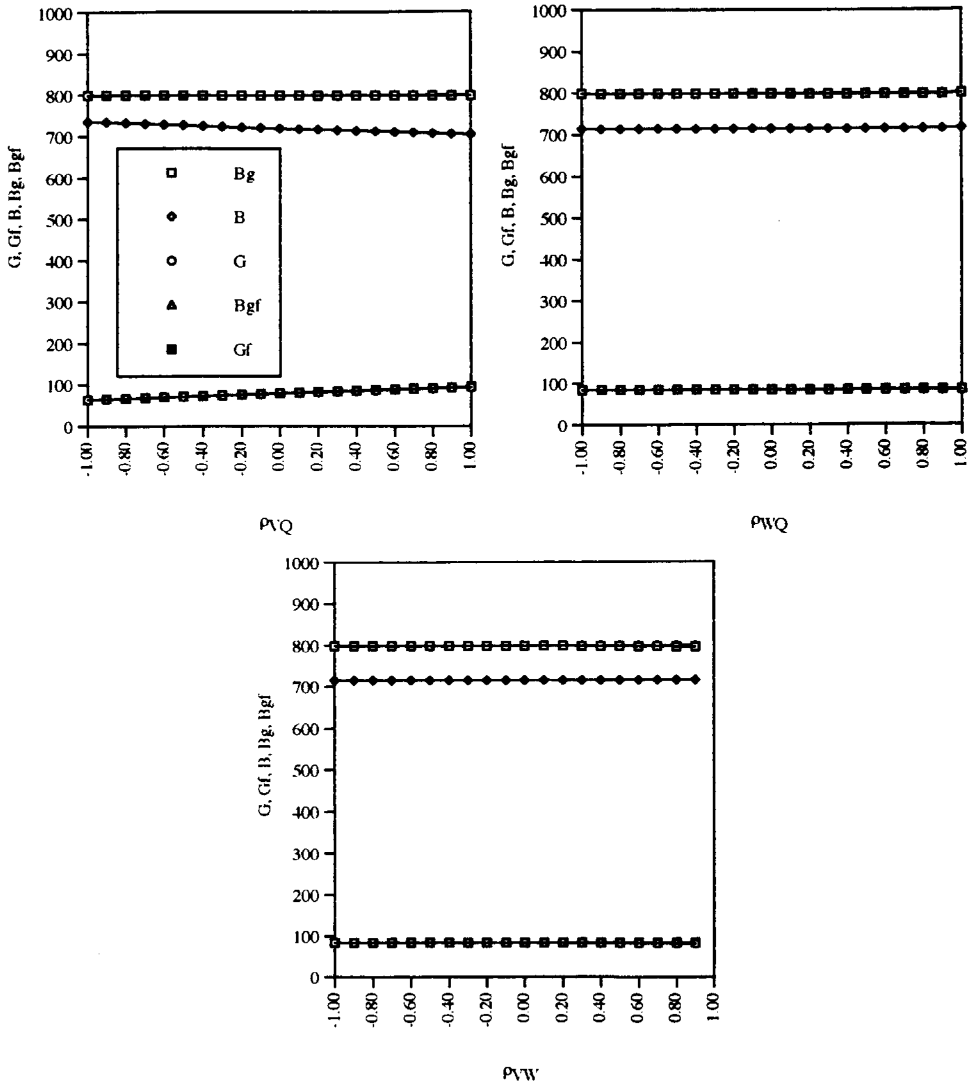

Figure 3: Impact of the coefficients of correlation between the firm (V), the guarantor (W) and the price of a default-free unit discount bond on the value of the loan with guarantee (Bg), the one without guarantee (B) and the guarantee (G). The subscript (f) denotes the cases corresponding to the default-free guarantee.

Note: We conduct the experiments with $V = 1100$, $\sigma_V = 0.3$, $W = 1500$, $\sigma_W = 0.3$, $r = 0.067$, $\sigma_r = 0.02$, $\rho_{WV} = 0.3$, $\rho_{VQ} = 0.3$, $\rho_{WQ} = 0.3$, $T = 3$, $F = 1000$, while varying the studied parameters in the comparative statics ceteris paribus.