Pricing PCS-Options with the Use of Esscher-Transforms

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Abstract

This paper introduces some formulas for pricing PCS Catastrophe Insurance Options before the date of their expiration. A model for the underlying process of the options is specified, the option prices are then derived by the so-called "risk-neutral Esscher transform" of the law of this process.

Résumé

Ce document présente quelques formules pour fixer le montant des tarifs pour la couverture des options d'assurance catastrophe PCS avant la date d'expiration. On spécifie un modèle servant au processus sous-jacent des options. On détermine ensuite les tarifs des options par la transformation de la distribution de ce processus qu'on appelle "transformation neutre du risque d'Esscher".

Keywords

PCS-Options, Esscher transform.

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1. Introduction

In September 1995 the Chicago Board of Trade (CBOT*) introduced a new class of options - the PCS Catastrophe Insurance Options. Unlike "normal" options the underlying of the option is not an asset, but a catastrophe losses index provided by the Property Claim Services (PCS), a division of the American Insurance Group Inc. For the exact construction of this index cf. SCHRADIN 1996.

Since the underlying index is determined by singular events causing jumps in the path of the process, Geometrical Brownian Motion as used in the classical option pricing formula of BLACK / SCHOLES 1973 is not the appropriate choice for the underlying process. So in the second chapter a more suitable model for this process is described.

In chapter three the necessary details for PCS-options are given.

Chapter four gives a short survey of the risk-neutral Esscher transform following closely the guidelines of GERBER / SHIU 1994 a and b. Applying this method, explicit formulas for PCS Call and Put Options are developed, chapter five gives pricing formulas for the development period, chapter six the corresponding results for the loss period.

The paper concludes with a critical discussion about the adequacy of the used method.

2. A Model for the Underlying Process

For the determination of the underlying PCS index we have to distinguish between the loss period $[0, u)$ and the development period $[u, v]$ of the particular index. The loss period of the index is the period in which the catastrophe events occur, that determine the index. To simplify matters we assume that the loss caused by any catastrophe event is instantaneously calculated into the index at least in an approximate amount. Due to more elaborated estimates of the loss amounts in the

development period directly following the loss period subsequent corrections of the index value are made.

Since the amounts of catastrophe losses are difficult to handle, the original amounts are divided by 100 million USD and - a point we shall neglect in our model - rounded to the first decimal point. For more details on the determination of the index cf. CBOT 1995 or SCHRADIN 1996.

So let $L(t)$ denote the index value at time t for the divided PCS index with loss period $[0, u)$ and development period $[u, v]$. We assume that for all $t \in [0, v]$ $L(t)$ is of the form

$$L(t) = L(0) \exp\big(X(t)\big) , \qquad (2.1)$$

$X(t)$ being a stochastic process with independent and stationary increments.

For the loss period $[0, u)$ we define $X(t)$ to be a compound Poisson process with frequency parameter λ and jump amounts exponentially distributed [1] with parameter a, i.e. it holds

$$X(t) = \sum_{n=1}^{N(t)} Y_n \qquad , t \in [0, u) \qquad (2.2)$$

with $N(t)$ being a homogeneous Poisson process with parameter λ, Y_n being independent and identically distributed random variables following an exponential distribution with parameter a, and $\{Y_n\}$ and $\{N(t)\}$ being independent families of random variables.

For the development period $[u, v]$ we take $X(t)$ as a Brownian Motion with drift μ and volatility σ, i.e. the increments $X(t) - X(u)$ are normally distributed with mean $\mu(t - u)$ and variance $\sigma^2 (t - u)$.

3. Options on the PCS Index

There are two basic options on the PCS index which will be referred to as the PCS Call Option and the PCS Put Option. All option strategies used for hedging catastrophe risks can then be derived as appropriate combinations of these two options, for some examples cf. SCHRADIN / MÖLLER 1996 or SCHRADIN 1996. Both options can only be exercised at the end of the development period, hence they are of European type.

The exercise price X as well as the option premium are measured in index points, each point equals 200 USD when settled at maturity. For more flexibility PCS Options are available as small caps (exercise prices X from 0 to 200 points) and large caps (exercise prices X from 200 to 500 points). For PCS Call Options these caps of 200 resp. 500 points furthermore work as an upper bound for the option value. Hence the value $C(v)$ at time v of a PCS Call Option with exercise price X can be presented as

$$C(v) = \min \{\max \{0; L(v) - X\}; K - X\} \tag{3.1}$$

with $K = 200$ and $0 \leq X \leq 200$ for a small cap and $K = 500$ and $200 < X \leq 500$ for a large cap. The value $P(v)$ at maturity of a PCS Put Option with exercise price X can be presented as

$$\tag{3.2}$$
$$P(v) = \max \{X - L(v); 0\}$$

with $0 \leq X \leq 200$ for a small cap and $200 < X \leq 500$ for a large cap.

4. Finding the Martingale Measure by Esscher Transforms

In this chapter we will give a short survey of the ingeniously simple method of deriving the equivalent martingale measure of a stochastic process by the Esscher transform as it was first described in GERBER / SHIU 1994a.

Let us assume that the random variable $X(t)$ as in (2.1) has a probability density function (pdf.) $f(x, t)$, then the moment generating function $M(z, t)$ of $X(t)$ is defined by

$$M(z,t) \; := \; E[\exp(zX(t))] \;\; = \;\; \int_{-\infty}^{\infty} \exp(zx) \, f(x,t) \, dx \; , \qquad (4.1)$$

and the Esscher transform with parameter h of f is defined by

$$f(x,t;h) \; := \; \frac{\exp(hx) \, f(x,t)}{M(h,t)} \; . \qquad (4.2)$$

For the corresponding moment generating function of $f(x, t; h)$

$$
\begin{aligned}
M(z,t;h) \; &= \; \int_{-\infty}^{\infty} \exp(zx) f(x,t;h) dx \\[2mm]
&= \; \frac{M(z+h,t)}{M(h,t)} \; .
\end{aligned}
\qquad (4.3)
$$

holds. Since (if $M(z, t)$ is continuous in t) $M(z, t) = [M(z, 1)]^t$ holds we can derive by formula (4.3) that

$$M(z,t;h) \; = \; [M(z,1;h)]^t \; . \qquad (4.4)$$

Now the idea of Gerber and Shiu is to choose the parameter $h = h^*$ in the proper way to make the process $\{\exp(-\delta t) \, L(t)\}$ discounted by the riskless rate of interest δ be a martingale. Denoting E^* the expectation with respect to the so transformed measure in particular we must have

$$L(0) = \exp(-\delta t) \, E^* \, [L(t)] \qquad (4.5)$$

and the parameter h^* can be determined by

$$\exp(\delta t) = M(1, t; h^*) \qquad (4.6)$$

or, using (4.4), by

$$\exp(\delta) = M(1, 1; h^*) \qquad (4.7)$$

Equation (4.7), as shown in GERBER / SHIU 1994b, has a unique solution for h^* which then defines the unique risk-neutral Esscher measure[2].

5. Pricing PCS Options in the Development Period

Now let us introduce explicit formulas for the values $C(t)$ of the Call Option and $P(t)$ of the Put Option at time t in the development period ($u \leq t \leq v$).
Under the assumption that for $u \leq t \leq v$ $\{X(t)\}$ is a Brownian Motion with drift μ and volatility σ we get for the risk-neutral Esscher parameter

$$h^* = \frac{\delta - \mu}{\sigma^2} - \frac{1}{2} \quad , \qquad (5.1)$$

for details cf. SCHRADIN / MÖLLER 1996, appendix B, or GERBER / SHIU 1994a, chapter 3.1.

Hence the Esscher transformed process with parameter h^* is again a Brownian Motion but with drift $\delta - \frac{1}{2}\sigma^2$ and volatility σ.

If $\mathcal{F}_t$ denotes the σ-Algebra generated by the process up to time t then it is a well-known fact that we get the value $C(t)$ at time t of a Call Option as

$$C(t) = \exp(-\delta\ (v-t))\ E^*\ (C(v)\,|\,\mathcal{F}_t) \quad . \tag{5.2}$$

With $\kappa(t) := \ln\ (K\,/\,L(t))$, $\xi(t) := \ln\ (X\,/\,L(t))$ and Φ denoting the distribution function of the standard normal distribution we get for this expression

$$
\begin{aligned}
C(t) \;=\; L(t)\ &\left[\Phi\left(\frac{\kappa(t)-(\delta+\frac{1}{2}\sigma^2)\ (v-t)}{\sigma\sqrt{v-t}} \right) - \Phi\left(\frac{\xi(t)-(\delta+\frac{1}{2}\sigma^2)\ (v-t)}{\sigma\sqrt{v-t}} \right) \right] \\[2mm]
&+ \exp\left(-\delta(v-t)\right)\ K\ \Phi\left(\frac{-\kappa(t)+(\delta-\frac{1}{2}\sigma^2)\ (v-t)}{\sigma\sqrt{v-t}} \right) \\[2mm]
&- \exp\left(-\delta(v-t)\right)\ X\ \Phi\left(\frac{-\xi(t)+(\delta-\frac{1}{2}\sigma^2)\ (v-t)}{\sigma\sqrt{v-t}} \right) \quad ,
\end{aligned}
\tag{5.3}
$$

for a proof cf. SCHRADIN / MÖLLER 1996.

By analogy we get for $P(t)$, the price of a Put Option at time t

$$P(t) = \exp(-\delta(v-t))\ E^*\ (P(v)\,|\,\mathcal{F}_t) \tag{5.4}$$

for which we get the formula

$$P(t) = \exp(-\delta(v-t)) \, X \, \Phi \left[\frac{\xi(t)-(\delta+\frac{1}{2}\sigma^2)\,(v-t)}{\sigma\sqrt{v-t}} \right]$$

$$- L(t) \, \Phi \left[\frac{\xi(t)-(\delta-\frac{1}{2}\sigma^2)\,(v-t)}{\sigma\sqrt{v-t}} \right] \quad . \tag{5.5}$$

6. Pricing PCS Options in the Loss Period

Since there is a structural discontinuity in the index process at time u, for $0 \leq t <$ u we have to use the formula

$$C(t) = \exp(-\delta \, (v-t)) \, E^*(C(v)|\mathscr{F}_t)$$
$$= \exp(-\delta \, (v-t)) \, E^*(E^*(C(v)|\mathscr{F}_u)|\mathscr{F}_t) \tag{6.1}$$

to derive the value of a Call Option at time t. While the conditional expectation E^* $(C(v) \mid \mathscr{F}_u)$ is given as a special case of (5.3) it remains to calculate the outer conditional expectation. For this purpose we have to calculate the pdf. f $(x, u-t;$ $h^*)$ of the risk-neutral process.

With the assumption (2.2) in SCHRADIN / MÖLLER 1996, appendix C, it is shown that the risk neutral parameter h^* holds to be

$$h^* = (a-\frac{1}{2}) - \sqrt{\frac{1}{4} + \frac{\lambda a}{\delta}} \quad . \tag{6.2}$$

From this we get

$$f(x, u-t; h^*) = \sum_{k=0}^{\infty} \left[\frac{(\lambda a(u-t))^k}{\Gamma(k)k!} \exp\left[-\frac{\lambda a(u-t)}{a-h^*} \right] x^{k-1} \exp\left(-(a-h^*)x\right) \right] \quad (6.3)$$

Hence we have

$$
\begin{aligned}
C(t) \ = \ & \exp\left(-\delta(u-t)\right) L(t) \int_{-\infty}^{\infty} \exp(x)\, \Phi\left[\frac{-x+\kappa(t)-(\delta+\frac{1}{2}\sigma^2)\,(v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx \\[2ex]
& - \exp(-\delta(u-t))L(t) \int_{-\infty}^{\infty} \exp(x)\, \Phi\left[\frac{-x+\xi(t)-(\delta+\frac{1}{2}\sigma^2)(v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx \\[2ex]
& + \exp\left(-\delta(v-t)\right) K \int_{-\infty}^{\infty} \Phi\left[\frac{x-\kappa(t)-(\delta-\frac{1}{2}\sigma^2)\,(v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx \\[2ex]
& - \exp\left(-\delta(v-t)\right) X \int_{-\infty}^{\infty} \Phi\left[\frac{x-\xi(t)-(\delta-\frac{1}{2}\sigma^2)\,(v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx
\end{aligned}
\qquad (6.4)
$$

with $f(x, u-t, h^*)$ as in (6.3).

With analogous considerations we finally get for the value of a put option

$$P(t) =$$

$$\exp\left(-\delta(v-t)\right) X \int_{-\infty}^{\infty} \Phi \left[\frac{-x+\xi(t)-(\delta+\frac{1}{2}\sigma^2)\ (v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx$$

$$- \exp\left(-\delta(u-t)\right) L(t) \int_{-\infty}^{\infty} \exp(x)\ \Phi \left[\frac{-x+\kappa(t)-(\delta+\frac{1}{2}\sigma^2)\ (v-u)}{\sigma\sqrt{v-u}} \right] f(x,u-t;h^*)dx \quad .$$

$$\tag{6.5}$$

7. Final Considerations

One may remark that one of the basic assumptions for the validity of option pricing models is that the underlying of the option is continuously traded in an arbitrage free market, which for PCS options certainly is not true. Hence the formulas derived in this paper may not have any empirical content.

But we argue that the crucial question for using such option pricing formulas as a reference point is not whether the underlying *is* actually traded but rather whether it *could* be traded.

This, however, is also true for the PCS options, if we imagine traded certificates with a value corresponding to the actual value of the PCS catastrophe loss index.

Furthermore, the importance of such option pricing formulas as the famous one of Black and Scholes is not primarily due to the validity of its assumptions. Actually, also for "classical" stock options, the assumption does not fully hold. The relevance of the pricing formulas is rather due to the fact that people believe that the formulas give approximately the fair prices and no one is willing to spend much more or to ask by far less than the formulas indicate. Hence option pricing formulas work

more or less as a point of reference, in real markets, however, deviations are possible.

End notes

1) The Exponential distribution is obviously not an optimal choice to represent catastrophe losses since its skewness $\gamma = 2$ is by far too small for catastrophe events. But distributions that have a higher risk do unfortunatelly not have the moment generating function which is necessary for our calculation (e.g. log normal or pareto distribution) or do not allow an analytical solution of the equivalent martingale measure by the Esscher transform (e.g. Gamma distribution).

2) This does not mean uniqueness of the equivalent martingale measure, however, as Gerber and Shiu mention. For a study on equivalent martingale measures to a compound Poisson process, for instance, cf. DELBAEN / HAEZENDONCK 1989.

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