Prudence and Intertemporal Utility

Piera Mazzoleni

Abstract

The intertemporal development of the consumption and saving plans has promoted a dynamic study also for the evolution of utility and the correspondent behavior towards risk.

The characterization of the risk aversion when facing multiple risks is particularly suitable to describe the behavior of sectors such as the banking and the insurance ones.

Recently, the analysis has been extended to an intertemporal framework and, referring to the well known recursive equation, a dynamic approach to risk aversion has been proposed.

The interrelation between time and risk deserves further evaluation in order to better state the dynamics of prudence. Indeed, it is no longer possible to confine prudence to management bounds: it has to become a proper description of the decision makers behavior.

Résumé

Dans cette recherche nous avons étudié la distribution temporel des comportements optimal avec prudence, soit dans le côté de la demand soit dans le côté de l'offre.

Keywords

Risk management, utility theory, dynamic programming.

Catholic University, Largo Gemelli 1, 20213 Milan (Italy); Tel: + 39-2-7234-2729, Fax: + 39-2-7234-2406-2210, E-mail: Mazzolen@Mi.Unicatt.It

1. Introduction

In a static framework it is worth considering the different certainty equivalent on the side of the offer and on the side of the demand. They may be read as trading prices and set equal to the minimum bound and to the maximum bound for trade, respectively.

For linear utility functions they cannot be distinguished and coincide with the expected value of the random amount. Alternatively the difference with respect to the expected value defines the risk premium which is related both with the variance, that is, with the riskiness of the amount under examination, and with the utility function, that is with the agent's preferences.

When the utility function is neither linear nor exponential equilibrium conditions may be set also in presence of inequalities between the selling and the purchasing prices.

A very simple introduction of time is related with the lag between the time at which the trade decision occurs and the time at which the random amount X is available (or due).

Then we may refer all the amounts at time T=0 and define an immediate certainty equivalent.

As it is well known in the economic analysis, marginal notions play a much more important rule than do their average or total analogues. Therefore we are led from the notion of the certainty equivalent to the notion of precautionary equivalent.

A typical field in which it is worth distinguishing demand and offer is the actuarial one. Consider for instance the standard non life insurance: a certain immediate premium is exchanged against a random future claim, deriving from the risk covered by the insurance policy.

Suppose it is possible to derive under the expected value.

Then if the marginal utility U' is convex, a Jensen type inequality is verified by the marginal utility and may be restored as equation by the insertion of an additional term.

At this point it is possible to treat the offer and the demand such as the one expressed by an insured to his insurer, separately, and analyze the properties of the prudential equivalent.

A compensating precautionary premium is defined as a function of the variance of the random amount (risk) and of the prudence measure $p_A = -U'''/U''$.

It is proved that such a new concept is non trivial and gives an actual improvement of the risk premium.

But what is even more interesting, Kimball (1990) used the two period consumption investment model by Rothschild-Stiglitz to provide an interpretation to prudence in terms of optimality first order conditions, which typically use the

marginal functions.

Indeed if we find an inequality rather than an equilibrium equation, some constraint will be active, cutting down the level of consumption: then, due to the classical complementarity conditions, a suitable compensation is needed.

A more significant analysis for prudence is carried out in the two period model and allows us to emphasize the roles of borrowers and lenders.

In a dynamic framework precautionary saving has been introduced under the assumption of an additive and separable utility function. In order to overcome this limitation a recursive utility has been defined with a dynamic programming approach. First uncertainty is resolved by calculating a certainty equivalent, which takes into account future information, and then the current value of the objective function is added, thus allowing the satisfaction of the optimality principle by Bellman.

But this way prudence is authomatically introduced, since the certainty equivalent is applied to the value function of the recursive equation of dynamic programming and it results to be referred to the optimizing process. The precautionary premium may be calculated only indirectly.

If we want to caracterize demand and offer we may formulate a two-person dynamic game, so that the saddle point gives the optimal demand and offer and in correspondence the bid-ask precautionary premium is indirectly found.

2. Bid-ask precautionary premium

When pricing a random amount it is worth taking into account the possible difference between the moment at which the price is paid and the one at which the asset produces a random money flow. But this is the typical distinction arising in the actuarial literature. Indeed, it is customary to separate the party ceding the risk and the counterpart assuming it.

As it is well known in the economic analysis, marginal notions play a much more important role than do their averages or total values. Therefore we are led from the notion of certainty equivalent to the notion of prudence equivalent.

Prudence is a kind of behavior which is always recommended in the bank and insurance management. Nevertheless it requires to be defined more precisely.

As well as plain risk may be sold and purchased, we face two different optimization problems: the one of the ceding party trying to maximize the benefit, the one of the assuming counterpart, accepting risk and in correspondence trying to minimize the loss. Even if the first order conditions look the same, it is not true that the amount required to keep optimality has the same either value or meaning.

Assume it is possible to derive under the expected operator. From the side of the insured we may take the Jensen's inequality:

$$E\left[U'\left(W_0 - X\right)\right] > EU'\left(W_0\right)$$

$W_a = W_0 + X$ being the initial wealth, allowed to be random, and X the random amount.

Then due to continuity properties we may ask whether it is possible to maintain the level of marginal utility when the individual offers the risk X.

If the addition of q_a is such that

$$E\left[U_a'\left(W_0 - X + q_a\right)\right] = EU_a'\left(W_0\right)$$

q_a is related with the expected value of the risky operation by $q_a = EX - \psi_a = x - \psi_a$:

ψ_a being called the equivalent precautionary premium.

The side of the insurer will require a reduction of the risk able to restate the equilibrium in the marginal utility.

Indeed, the minimization of the loss will require a reverse inequality

$$E\left[U'\left(W_0 + X\right)\right] < EU'\left(W_0\right)$$

and a corresponding compensation

$$E\left[U_b'\left(W_0 + X - q_b\right)\right] = EU_b'\left(W_0\right)$$

when purchasing the risk X. q_b is related with the expected value of the risk by the corresponding precautionary premium $q_b = EX - \psi_b = x - \psi_b$.

This means that in an optimization framework it is possible to restate equilibrium conditions on the side of the demand and on the one of the offer.

Since this equilibrium condition is stated in terms of marginal utility it may be read as a first order optimality condition for both the insurer and the insured.

This of course is trivial in the case of linear utility functions. Otherwise we have to take separate the decision variables.

In a discrete environment Castagnoli-Li Calzi (1994) suggested to read the expected utility as a bounded probability with respect to an ideal reference level.

If we take the first order difference instead of the derivative and transform it into a probability difference, we get

$$E\left[U_a\left(W_0 - X + q_a + \Delta\right)\right] - EU_a\left(W_0 - X + q_a\right) =$$
$$= \Pr ob\left(W_0 - X + q_a + \Delta \geq X_{U_a}\right) - \Pr ob\left(W_0 - X + q_a \geq X_{U_a}\right) =$$
$$= \Pr ob\left(X_{U_a} - \Delta \leq W_0 - X + q_a < X_{U_a}\right)$$

That is if we introduce a reference level X_{U_a} the probability before and after the risky operation belong to the interval $(X_{U_a} - \Delta, X_{U_a})$ after a suitable prudent adjustment q_a.

An analogous result holds for $W_0 + X - q_b$ with respect to X_{U_b}.

In order to characterize the conditions for trade, let us remind that $W_a = W_0 + X$: then the asking precautionary compensation depends on the expected value $x = EX$, $q_a = q_a(W_0 + x, x)$, $q_b = q_b(W_0, x) = q_b(W_0, 0) + x$.

From such expressions we get

$$\frac{dq_a}{dx} = \frac{\partial q_a}{\partial x} + \frac{\partial q_a}{\partial W_a}\frac{dW_a}{dx} = 1 + \frac{\partial q_a}{\partial W_a} = \frac{\partial q_b}{\partial x} - \frac{\partial \psi_a}{\partial W_a}$$

and the following relation states the sufficient conditions for trade if U exhibits decreasing prudence

$$\frac{dq_a}{dx} > \frac{\partial q_b}{\partial x} \Leftrightarrow -\frac{\partial \psi_a}{\partial W_a} > 0 \Leftrightarrow p'_A < 0.$$

Indeed it is immediate the following:

THEOREM - Assume $U_a = U_b = U$. If U displays constant non-null prudence, an equilibrium condition for the marginal utilities from the demand and offer sides holds, which guarantees that an insured is reaching his maximal benefit, the insurer his minimal loss. Alternatively, conditions arise to adjust the optimal decisions to the possible differential between the initial wealths from one side, $W_b - W_a$ and the adjustment prices $q_a + q_b$.

PROOF - Let W_0 and w_b the non random part of W_a and W_b and set $q_b(w_b, x) = w_b - W_0$. Then equality

$$U'(w_b) = EU'(w_b + X - q_b(w_b, x)) = EU'(W_0 + X) = U'(W_0 + q_a(W_0 + x, x))$$

entails $w_b - W_0 = q_b(w_b, 0) + x$. Therefore condition from changing optimality decisions with wealth is given by $x < x_0 = W_b - W_0 - q_b(w_b, 0)$. □

3. Prudence and intertemporal utility

The notion of certainty equivalent applies a compensation principle: if an individual takes on a risk, how much monetary compensation is necessary in order to maintain the same welfare.

Therefore we face either equilibrium conditions or willingness to trade.

Usually the same model represents the behavior of an agent acting as a seller and a buyer.

But if we are truly representing a trade, the equilibrium conditions arise from a game theoretic approach

$$\inf_{q_a} \sup_{q_b} EU\left(W_a - X + q_{a,}, W_b + X - q_b\right)$$

A saddle point under concavity assumption on the first argument and convexity assumption on the second argument require that the first order conditions are satisfied

$$EU_a'\left(W_a - X + q_a^*, W_b + X - q_b\right) = 0$$
$$EU_b'\left(W_a - X + q_{a,}, W_b + X - q_b^*\right) = 0$$

In a dynamic framework both time and uncertainty are taken into account.

A first example of dynamic analysis for the risk market is the following development of the Kimball (1990) illustrative model.

Consider the two period optimization problem to find the optimal consumption decision after ceding risk $\widetilde{X}$

$$\max \ g(C) + Eh\left(W_a - C - \widetilde{X}\right)$$

where g is the first period utility function, h the second period one, $W_a = W_0 + \widetilde{X}$ the initial wealth including risk $\widetilde{X} = x + X, EX = 0$, C the first period consumption. No explicit reference is done to the time lag.

First order conditions, also sufficient under the usual concavity assumption,

$$g'(C) = Eh'\left(W_a - C - \widetilde{X}\right)$$

have to be suitably modified after the sale of risk $\widetilde{X}$ with the addition of a compensation q_a in order to maintain the optimal level of consumption unchanged

$$h'(W_0 - C) = Eh'\left(W_a - C - \widetilde{X} + q_a\right)$$

q_a is known as the compensating precautionary asking price.

Suppose the initial wealth W_0 is augmented of risk $\widetilde{X}$ in the second period. Then the optimization problem to find the optimal consumption decision after purchasing risk $\widetilde{X}$ becomes

$$\max \ g(C) + Eh\left(W_0 - C + \widetilde{X}\right)$$

A compensating precautionary bid price q_b is required in order to restore optimal consumption

$$h'(W_0 - C) = Eh'\left(W_0 - C + \widetilde{X} - q_b\right)$$

If the problem is approached as a stochastic dynamic model, the attention is concentrated on the value function. Then the certainty equivalent is generalized for the objective rather than for the decision variable. This approach allows us

to manage the difference between the demand and offer sides, if we set a different certainty equivalent for the first and for the second argument.

Let us refer to Duffie-Epstein (1992). The utility process for a given financial flow $X \in D$, D being the space of investment flows, which we suppose to be l-valued square integrable processes, is defined by a semimartingale, V, that is, an adapted process that can be written in the form $H + K$, H being a finite variation process and K a local martingale.

For any time t the random variable V_t is treated as the utility for the continuation $\{(X_s, Y_s) : s \geq t\}$, where V_0 denotes the utility of the entire process $\{(X_s, Y_s) : s \geq 0\}$.

For any interval I of the real plane, let $P(I)$ denotes the space of probability measures on I whose mean exists.

By a certainty equivalent m we mean a function $m : P(I) \to R$ that associates to a probability measure p (representing the distribution of utility) its certainty equivalent $m(p)$.

In order to motivate the recursive utility, we will suppose that V is the utility-disutility process for a demand-offer investment process (X, Y).

In a discrete -time setting we get a representation as

$$m(\sim V_t | F_t) = G(X_t, Y_t, V_t)$$

for some $G = I \times R \to R$.

Thus the agent first computes the certainty equivalent $m(\sim V_t | F_t)$ of the conditional distribution $(\sim V_t | F_t)$ at time t, which is combined at a second step with the decision variables (X_t, Y_t) through an aggregator.

Assume differentiability with respect to Δt. We get

$$\frac{d}{ds} m_X(\sim V_{t+1} | F_t)_{s=0} = -f_a(X_t, Y_t^*, V_t) \text{ a.s.}$$

$$\frac{d}{ds} m_Y(\sim V_{t+1} | F_t)_{s=0} = -f_b(X_t^*, Y_t, V_t) \text{ a.s.}$$

Therefore it is no longer defined the certainty equivalent but its dynamic evolution law.

The pair (f, m) is called an *aggregator*. f determines the degree of intertemporal substitution of investment both as a sale and as a purchase.

Given f, risk attitudes of the seller and the buyer are fixed by the certainty equivalent m.

Optimality for a continuous-time recursive utility function is characterized by Bellman's equation, which is an extended version of the Hamilton-Jacobi-Bellman equation.

In a finite time horizon, a controlled state process $\{W(t)\}$ and a regular normalized aggregator f are given.

Let $\Gamma : I \times T \to 2^{X \times Y}$ define the admissible set of controls in the time interval $T = [0, T]$ for some finite T, in the sense that (X_t, Y_t) must be chosen from the set $\Gamma(w_a, w_b, t)$ at time t, when the current state $W(t)$ is $w_a, w_b \in R$.

For a given control $(X_t, Y_t) \in D$, the state process $(W(t))$ exists if it uniquely solves the equation

$$dW_a(t) = \rho_a(W_a(t), t, X_t, Y_t)dt + \sigma_a(W_a(t), t, X_t, Y_t)dB_t \quad (*)$$

$$dW_b(t) = \rho_b(W_b(t), t, X_t, Y_t)dt + \sigma_b(W_b(t), t, X_t, Y_t)dB_t$$

where $W(t)$ is given, ρ and σ are measurable.

We will restrict ourselves to the filtration $F = (F_t)$ of a standard Brownian motion B in R^n.

A process $(X, Y) = (X_t, Y_t)$ is an admissible control if there is an integrable state process $\{W(t)\}$ solving system $(*)$.

Let D^Γ denote the set of admissible control processes.

Let U denote the recursive utility function on D generated by f and consider the control problem

$$\inf_Y \sup_X U(X, Y) = U(X^*, Y^*)$$

Denote v the value function and $D^W v(w_a, w_b, t)$ the first order approximation. Then the Bellman equation

$$\inf_Y \sup_X D^W v(w_a, w_b, t) + f(X, Y, v(w_a, w_b, t)) = 0$$

gives a sufficient condition for a saddle point. The optimal policy (X^*, Y^*) is the argument of the saddle optimal value.

Therefore if we compare the two different situations with $B = 0$ and $B \neq 0$, then the difference of the optimal solutions represents the precautionary compensation.

Finally it is worth noticing that when the functional form of f keeps separate the influence of risk and time it is possible to recognize the insurance and saving attitude and it is especially the second which is related to prudence.

REFERENCES

E.Castagnoli-M.Li Calzi, "Expected uitlity without utility", Studi di Matematica Finanziaria ed Attuariale, n.13, Univ. Bocconi, Milano, dicembre 1994

D.Duffie.L.G.Epstein, "Stochastic differential utility", Econometrica, vol.60, n.2 (March 1992), pp.353-394

M.S.Kimball, "Precautionary saving in the small and in the large, Econometrica, vol.58, n.1 (January 1990), pp.53-73

P.Mazzoleni, "Prudence and deposit insurance", Techn.Rep, Catholic University, December 1994, presented at the XXII EFA Meeting, Milan, August 23-26, 1995

L.Peccati-G.Weinrich, "On certainty equivalent and time", presented at the VI FUR conference, Oslo, 1994