Baseline for Exchange Rate - Risks of an International Reinsurer

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Abstract

In the insurance or reinsurance business it is a natural question to ask which asset structures in terms of currency exposure imply the lowest risk to overall shareholder-value.

The paper starts with the analogy in the classical banking business and then considers the stochastic asset-liability, and currency structures of an international (re-)insurer. A first approach to the general solution is presented by using a simple Monte Carlo model. A more sophisticated approach is then presented within the framework of stochastic control theory. An instructive example gives a rule of thumb for simple cases. Some ideas for generalizations close the article.

Résumé

Pour une société d´assurance ou de réassurance il est naturel de ce poser la question: quelle structure des actifs en regard du risque de taux de change doit une companie d´assurance ou de réassurance avoir afin de minimizer le risque global pour l´astionnaire final.

Cet article commence par une comparaison avec le secteur bancaire, puis considère l´aspect stochastic de l´actif et du passif d´une société internationale de réassurance. Une première solution est présentée utilisant la méthode des simulations de Monte Carlo. Puis une approche plus sophistiquée reposant sur la théorie du control optimal est présentée. Un exemple ainsi que des idées d´extension concluent l´article.

Keywords

Asset liability matching, currency matching, Value at Risk, stochastic control.

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1. Introduction

More and more institutional investors are beginning to measure their asset-risks. A very popular method to do this is the Value at Risk approach (VaR) which is described in Risk Metrics- Technical Document of J.P. Morgan [1]. This approach is widely used in banking business, where risky positions on the asset-side are considered together with liabilities. If in this case the liability exposure in a given currency is matched by assets of the same order in the same currency, VaR assumes that there is no currency-risk in this currency. A similar argument is used in the case of interest rate risks, when the duration of the (deterministic) liabilities equals the duration of the assets. In the present paper we describe the general problem. We then study how to combine an exchange rate risk exposure due to stochastic liabilities with the VaR concept on the asset side. The latter case is important if the VaR method is applied to the assets of a (re-)insurance company.

In this case the following question arises: which is the zero-line for currency - asset - risk-measurement ? As the structure of the liabilities in the (re-)insurance case is more stochastic than in the banking case, things become more complicated: a priori there is no simple rule for currency matching because the reserves have to be bigger than in the classical banking business in order to reduce the probability of ruin. Should the reserves be invested in the individual currencies according to the expected claims in each country or should higher quantiles be the basis for this splitting ? To decide this question more sophisticated methods have to be applied **to solve the problem of which relative amounts of the total value of the reserves have to be invested in which currency in order to minimize the volatility of the shareholder-value, given the stochastic structure of assets, liabilities and exchange-rates.**

This relative distribution of total assets to the corresponding currencies can serve as the basis for risk-measurement on the currency side. We call this problem (more precise definition following) **the currency-baseline-problem of an international (re-)insurer.** The currency-baseline-problem is therefore imbedded in the more general problem of asset-liability-management (ALM) of such companies.

In the 5-th AFIR colloquium a paper of Ars-Janssen [2] was published which treated general ALM - problems on a very aggregated level for insurance companies. In this paper, assets were modeled by one single diffusion process and liabilities by one single mixed process, which is a product of a diffusion process and a compound Poisson process.

We use this approach to model both assets, liabilities and exchange-rates, but more processes have to be used. As in each currency assets and liabilities have to be modeled separately. In the simplest case of 2 currencies, 5 stochastic processes have to be modeled: 2 asset-processes, 2 liability-processes and 1 exchange rate process. As this complicates the situation, at first we do not try to solve the problem analytically, but use a numerical method: Monte Carlo simulation. The general solution and corresponding stochastic control algorithms are then developed in the second part of the paper. The first method is demonstrated in an instructive, simple example which illuminates the differences to the classical banking approach and suggests a rule of thumb for the currency-baseline which has still to be tested with more examples.

In the baseline problem the riskiest factor of international invested assets that influence both the asset and the liability side is considered: the exposure to exchange rates. In a future paper we will try to model the more general case of ALM with similar methods in order to define also a baseline for the interest rate risks.

2. Currency - Duration - Matching in Classical Banking Business

The problem of reducing risks by immunization in classical banking business has a long tradition. The first contribution to this problem was made by Macaulay 1938 [3], where the concept of duration was defined. This definition got forgotten but in 1952 the british actuary F.M. Redington used similar methods in life insurance problems [4]. An elementary introduction into the principles of managing the interest rate risks is given in [5]. In order to be able to formulate the general problem of the application of VaR in the context of banking business with a given structure of liabilities we introduce the classical first order quantity "duration" which explains the overwhelming part of the interest rate risk [7].

The classical Duration of Macaulay for a straight bond is defined as follows:

$$D = \frac{\sum_{t=1}^{n} \frac{tc_t}{(1+y)^t}}{P}$$

where y is the yield to maturity, c_t an individual payment (coupon or final payment) at time t and P the price of the bond. Duration is a linear quantity in the sense that it can be extended to a whole portfolio as a weighted sum of the durations of the individual bonds of the portfolio, if the yields are the same for all bonds in the portfolio. The last assumption corresponds to the simple scenario, where interest rates movements in one currency are modeled by a small parallelshift of flat interest rate curves.

It is easy to derive the following relationship, which is the basis for classical first order immunization procedures:

$$\frac{dP}{P} = -D\frac{dy}{1+y}$$

As deterministic liabilities can be considered as a bondportfolio whose payments have negative signs, the last equation leads to a simple immunization principle for classical banking business:

1. Currency hedging:

   Invest the sum of the present values of all liabilities in a currency in this currency

2. Interest rate hedging:

   Choose the investments in such a way, that the duration of the assets in each currency is the same as the duration of the liabilities in this currency.

Principle (2) can be extended by introducing the second order moments (convexity or dispersion) in the case of interest rate hedging. As the scenario of parallelshifts of flat interest rates is too simple in many cases, modern theories however use stochastic interest rate models with various assumptions for the stochastic processes.

Principle (1) seems to be quite natural, as no theory is used for it. However we will see in our example, that it has to be modified in the case of stochastic liabilities.

3. Abstract Modeling of the (Re-)Insurance Business and of the Assets

We use a similar framework as described in the generalized Janssen's model [2].

On a filtered complete probability space $(\Omega,F,(F_t)_{0<t<\infty},P)$ satisfying the usual conditions, stochastic processes for three types of values are considered:

$\{A_t^{(k)}, t > 0\}$ asset processes in currency k (k = 1,...,m)

$\{L_t^{(k)}, t > 0\}$ liability processes in currency k (k = 1,..,m)

$\{E_t^{(k)}, t > 0\}$ processes for the exchange rate of the amount of the accounting currency 1 for one unit of foreign currency k (k = 2,...,m)

It is very important to be aware of the asymmetry in the definitions above: not all currencies are equally important, because the shareholder value will be expressed in the accounting currency. This asymmetry will produce some very important consequences in the solution of the currency baseline problem, as will be shown in the example at the end.

As in the generalized Janssen's model we assume that the asset process is an m-dimensional Brownian motion defined by a trend vector and a variance-covariance matrix Q, describing the values of the assets in the local currencies. For the exchange rates we assume that they are described by m-1 one dimensional Brownian motions describing the exchange rate dynamics of the local currencies with respect to the accounting currency 1.

The liability processes are modeled by m combined processes each of which is a product of a Brownian motion and a compound Poisson process. In contrast to the generalized Janssen's model however, we do not require an explicitly given correlation between the assets and the liabilities in a local currency, because we have an implicit correlation between these two quantities in the accounting currency, as both assets and liabilities in a local currency are multiplied by the same exchange rate.

As described in [2] the asset process (and the exchange rate processes) can be written (omitting the index k) in the following way:

$$A_t = A_0 \exp[(\mu_A - \sigma_A^2 / 2)t + \sigma_A W_t^{(A)}] \quad (3.1)$$

The liability process is given by:

$$L_t = L_0 \exp\ [(\mu_L - \sigma_L^2 / 2)t + \sigma_L W_t^{(L)}\ ]S_t \quad (3.2)$$

where W_t is a standard Brownian motion and S_t is a compound Poisson process which can be interpreted as catastrophes on the liability side:

$$S_t = \prod_{1 \le i \le N_t} (1 + Y_i)$$

with $N = (N_t, t > 0)$ being a Poisson process of parameter λ and Y_i being i.i.d. random variables.

Given the asset, liability and exchange rate processes in the above formulas, it is possible to calculate numerical solutions for the currency baseline problem either by Monte Carlo simulations or by stochastic control algorithms. Both procedures shall be described in the following.

4. The Monte Carlo Approach

4.1 Simulation of random numbers of a prescribed distribution

The basis for a Monte Carlo simulation is the availability of good random number generators in most programming languages that produce real random numbers in the unit interval. These random numbers can easily be transformed into arbitrarily distributed random numbers by some principles [6]. One such principle to get one dimensionally distributed random numbers which are distributed according to a cumulative distribution function F, is to transform the uniformly [0,1]-distributed random numbers u by the inverse $F^{-1}(u)$ of the F. It is shown in [6] that $F^{-1}(u)$ is distributed according to F.

Algorithms for the simulation of random numbers which are distributed according to a compound Poisson distribution or according to a multivariate normal distribution are also given in [6]. All the elements for a simulation of the processes given by the formulas (3.1) and (3.2) of chapter 3 are therefore available.

4.2 The description of the model

The formulas (3.1) and (3.2) define the probability distributions of assets and liabilities after a time horizon of t periods and enable us to simulate the corresponding random variables with Monte Carlo techniques.

Assets and liabilities are thought to be in m currencies. Let k=1 be the accounting currency and $\{A_t^{(k)}, k = 1,2,\ldots,m\}$ be the set of asset processes defined by (3.1) in these currencies and $\{L_t^{(k)}, k = 1,2,\ldots,m\}$ be the corresponding set of claims to be paid after t time periods.

The exchange rates $E_t^{(k)}$ = currency 1/currency k are also modeled as diffusion processes (as the assets): omitting the indices we have:

$$E_t = E_0 \exp\left[(\mu_E - \sigma_E^2 / 2)t + \sigma_E W_t^{(E)}\right] \qquad (4.1)$$

We limit ourselves to a one period model and consider therefore the set of settled claims and not the set of claims occurred during that period.

We think $\{p_k, k = 1,\ldots,m\}$ as a given vector of percentages of the reserves which are invested at time t = 0 in currencies k = 1,2,...,m.

Let V_0 be the total initial reserves in the accounting currency at the beginning. We have:

$$V_0 = A_0^{(1)} + A_0^{(2)} E_0^{(2)} + \ldots + A_0^{(m)} E_0^{(m)} \qquad (4.2)$$

Without considering interest payments and premiums during the period which influence the volatility of the reserves in the case of our currency study only in a second order way, the reserves at time t=1 can be written as:

$$V_1 = A_1^{(1)} - L_1^{(1)} + (A_1^{(2)} - L_1^{(2)})E_1^{(2)} + \ldots + (A_1^{(m)} - L_1^{(m)})E_1^{(m)} \qquad (4.3)$$

We define now the standard deviation of the final reserves V_1. as a function of

$$p_1, p_2, \ldots, p_m$$

$$\mathrm{Std}(V_1) = F(p_1, \ldots, p_m)$$

$$= \mathrm{Std}\left\{ \sum_{k=1}^{m} \left[p_k V_0 \exp\left(\mu_A^{(k)} - (\sigma_A^{(k)})^2 / 2 + \sigma_A^{(k)} W_1^{(k,A)}\right) - E_0^{(k)} L_1^{(k)} \right] E_1^{(k)} \Big/ E_0^{(k)} \right\} \qquad (4.4)$$

The value of F for a given vector $\{p_k, k = 1, \ldots, m\}$ can be calculated by a sufficiently big number of Monte Carlo simulations. The last step of the model consists in calculating the minimum

$$P = \{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_m\}$$

of F by applying a standard non-linear optimisation procedure.

We say that P defines the currency baseline.

The currency baseline defines the distribution of the initial reserves in the currencies in such a way that the standard deviation of the final reserves in the accounting currency is a minimum.

It is therefore the stochastic analogy for the insurance environment to the deterministic currency matching in the classical banking business. **The input for the exchange rate exposure to the VaR-model in the (re-)insurance environment should therefore consist of the deviations to the currency baseline.** This guarantees that the currency hedging effects produced by the international (re-)insurance business are integrated in the VaR-model as in the deterministic case of the classical banking business in a consistent way.

5. The General Approach Within the Framework of Stochastic Control Theory

General Asset/Liability Management. In generalization of the above, we now consider the problem of effectively managing a multi-currency portfolio of assets (e.g., cash, bonds, stocks, futures and options) and liabilities (e.g., life and non-life (re-)insurance

contracts) over some extended period $[0,H]$ of time. We call H the associated ALM horizon. For $2 \le i \le m$, let $E_i(t)$ denote the exchange rate at time t of foreign currency i ($i=1$ is the index of the domestic or accounting currency). Furthermore, for $1 \le j \le n_i^A$ and $1 \le j \le n_i^L$, respectively, let $A_{ij}(t)$ denote the market value at time t of asset j available in currency i and, similarly, let $L_{ij}(t)$ denote the cost at time t associated with liability j in currency i. If now

$$v_i^A(t) = \left(v_{i1}^A(t),..,v_{ij}^A(t),..,v_{in_i^A}^A(t)\right) \text{ and } v_i^L(t) = \left(v_{i1}^L(t),..,v_{ij}^L(t),..,v_{in_i^L}^L(t)\right)$$

are the currency i asset and liability portfolio positions at time t, respectively, and

$$v^A(t) = \left(v_2^A(t),..,v_i^A(t),..,v_m^A(t)\right) \text{ and } v^L(t) = \left(v_2^L(t),..,v_i^L(t),..,v_m^L(t)\right)$$

the corresponding total foreign currency asset and liability positions, then the value at time t, denominated in the domestic currency, of the entire multi-currency asset/liability portfolio is

$$V_v^v(t) = \left[v_1^A(t)^T A_1(t) - v_1^L(t)^T L_1(t)\right] + \sum_{i=2}^{m}\left[v_i^A(t)^T A_i(t) - v_i^L(t)^T L_i(t)\right]E_i(t) \qquad (5.1)$$

[where the initial value is $V_v^v(0) = v$, $v(t) = \left(v_1^A(t), v^A(t), v_1^L(t), v^L(t)\right)$ is the total asset/liability portfolio at time t and

$$A_i(t) = \left(A_{i1}(t),..,A_{ij}(t),..,A_{in_i^A}(t)\right) \text{ and } L_i(t) = \left(L_{i1}(t),..,L_{ij}(t),..,L_{in_i^L}(t)\right)$$

holds for the components of the two vectors $A(t) = \left(A_1(t), A_2(t),..,A_m(t)\right)$ of assets and $L(t) = \left(L_1(t), L_2(t),..,L_m(t)\right)$ of liabilities, respectively]. At this stage of the ALM modeling process, we introduce a general Markov jump diffusion state variable

$$x(t) = \left(X(t), V_v^v(t),...\right) \qquad (5.2)$$

in which the component $X(t) = (A(t), L(t), E(t),...)$ represents the multi-currency asset/liability market [and $E(t) = \left(E_2(t),..,E_m(t)\right)$]. For a detailed justification and characterization of this modeling assumption within the microeconomic theory of

asset/liability markets, see references [8.8] to [8.18]. References [8.19] to [8.25] provide the necessary mathematical background information.

Jump Diffusion Market Variable. The uncontrolled RCLL state dynamics of the asset/liability market $X(t)$ in our stochastic control model for optimal multi-currency asset/liability management are then determined by the differential equation

$$dX(t) = B(t, X(t))dt + C(t, X(t))dW(t) + dJ(t)$$
$$J(t) = \int_0^t \int_{R^M} D(s, X(s-), y)N(dsdy) = \sum_{t_n \le t} D(t_n, X(t_n-), y_n) \qquad (5.3)$$

where the coefficients $B(t, X) \in R^M$ and $C(t, X) \in R^{M \times N}$ of the diffusion part satisfy the usual conditions that guarantee a corresponding unique strong solution with bounded absolute moments. The additional (Poisson) jump process $J(t)$ is characterized by the bounded and measurable parameter $D(t, X, y) \in R^M$ which is continuous in time t and state X and a Poisson random measure $N(dtdy)$ with intensity

$$E[N(dtdy)] = \lambda dt \Pi(dy)$$

on the Borel σ-algebra $B([0, \infty) \times R^M)$ [where the associated probability measure $\Pi(dy)$ on the Borel sets $B(R^M)$ has compact support $\Gamma \subseteq R^M$]. It therefore has the continuous jump rate

$$\lambda(t, X) = \lambda \int_{\{y: D(t, X, y) \neq 0\}} \Pi(dy)$$

and the corresponding continuous (in time t and state X) jump distribution

$$\overline{\Pi}(t, X, Q) = \int_Q \overline{\Pi}(t, X, dy) = \int_{\{y: D(t, X, y) \in Q \wedge D(t, X, y) \neq 0\}} \Pi(dy).$$

Under these assumptions, the above stochastic evolution equation for the market state variable has a unique strong solution $X(t)$ (with at most finitely many jumps in the time interval $[0, H]$ representing the relevant ALM horizon) for each initial condition $X(0) = X$. Furthermore, the Ito formula

$$f(t,X(t)) = f(0,X(0)) + \begin{bmatrix} \int_0^t [Af](s,X(s))ds + \\ \int_0^t [\nabla_X f^T C](s,X(s))dW(s) \end{bmatrix} + J_f(t) \qquad (5.4)$$

with the associated integro-differential operator

$$[Af](t,X) = \begin{matrix} [\frac{\partial f}{\partial t} + B^T \nabla_X f + \frac{1}{2} tr(CC^T \nabla_X^2 f)](t,X) + \\ \lambda(t,X) \int_{\Gamma} [f(t,X+y) - f(t,X)] \overline{\Pi}(t,X,dy) \end{matrix} \qquad (5.5)$$

and the martingale

$$J_f(t) = \begin{matrix} \sum_{s \le t} [f(s,X(s)) - f(s,X(s-))] - \\ \int_0^t \lambda(s,X(s)) \left[\int_{\Gamma} [f(s,X(s)+y) - f(s,X(s))] \overline{\Pi}(s,X(s),dy) \right] ds \end{matrix} \qquad (5.6)$$

holds for $C^{1,2}$ functionals $f(t,X)$ of the jump diffusion market state variable $X(t)$.

5.1 Digression: General Asset Management for Banks

Before proceeding with the analysis of our general asset/liability management model for a large, internationally operating (re-)insurance company, we shall briefly examine the special case where the state dynamics of the asset market

$$X(t) = (A(t), E(t), \ldots) \qquad (5.7)$$

have continuous sample paths, i.e., the general asset or investment management problem of a bank. Its solutions are called risk arbitrage strategies (for more details, see references [8.26] and [8.27]; in the sequel, we shall briefly summarize some of the results presented in [8.27] using the same notation). Intuitively, risk arbitrage strategies are trading or portfolio management strategies in the securities and derivatives markets that guarantee (with probability one) a limited risk exposure over the entire investment horizon $[0,H]$ and at the same time achieve a maximum (with guaranteed floor) rate of portfolio value appreciation over each individual trading period. They ensure an efficient allocation of investment risk in these international financial markets and can further be characterized as the solutions of the constrained expected utility maximization problem

$$\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))] \qquad (5.8)$$
$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v$$

[where $U^c(t,c(t))$ is the utility of intertemporal fund consumption and $U^V(V_v^{c\theta}(H))$ the utility of final wealth] with drawdown control

$$D(t)V_v^{c\theta}(t) > \alpha M_v^{c\theta}(t),\ 0 \le t \le H$$

[$M_v^{c\theta}(t) = \max_{0\le s\le t} D(s)V_v^{c\theta}(s)$], limited risk arbitrage objectives

$$|v(t)^T\Delta(t)| \le \delta(t),\ 0 \le t \le H \quad \text{(instantaneous investment risk)}$$

$$|v(t)^T\Gamma(t)| \le \gamma(t),\ 0 \le t \le H \quad \text{(future portfolio risk dynamics)}$$

$$v(t)^T\Theta(t) \ge \vartheta(t),\ 0 \le t \le H \quad \text{(portfolio time decay dynamics)}$$

$$v(t)^T\Lambda(t) \ge \lambda(t),\ 0 \le t \le H \quad \text{(portfolio value appreciation dynamics)}$$

[$\theta(t) = I_X(t)v(t)$] and additional inequality and equality constraints

$$g(t,X(t),D(t),\varsigma(t),v(t)) \le 0,\ 0 \le t \le H$$

$$h(t,X(t),D(t),\varsigma(t),v(t)) = 0,\ 0 \le t \le H$$

(e.g., market frictions, etc.) in a diffusion type securities and derivatives market

$$dX(t) = I_X(t)[M(t)dt + \Sigma(t)dW(t)]$$
$$dD(t) = -D(t)r(t)dt \qquad d\varsigma(t) = -\varsigma(t)[r(t)dt + A(t)^T dW(t)] \qquad (5.9)$$
$$M(t) = \begin{bmatrix} M_1(t) \\ \vdots \\ M_L(t) \end{bmatrix} \quad \Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \cdots & \Sigma_{1N}(t) \\ \vdots & & \vdots \\ \Sigma_{L1}(t) & \cdots & \Sigma_{LN}(t) \end{bmatrix} \quad I_X(t) = \begin{bmatrix} X_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_L(t) \end{bmatrix}$$

with associated [expressed in terms of an underlying Markov risk exposure assessment and control model $(t,S(t))$] instantaneous investment risk, future derivatives risk dynamics, options time decay dynamics and asset value appreciation dynamics

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) \\ \vdots \\ \Delta_L(t) \end{bmatrix} \quad \Gamma(t) = \begin{bmatrix} \Gamma_1(t) \\ \vdots \\ \Gamma_L(t) \end{bmatrix} \quad \Theta(t) = \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_L(t) \end{bmatrix} \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ \vdots \\ \Lambda_L(t) \end{bmatrix} \qquad (5.10)$$

[where $\Delta_i(t) = \nabla_S X_i(t,S(t))$ is the delta (N-vector), $\Gamma_i(t) = \nabla_S^2 X_i(t,S(t))$ the gamma ($N \times N$-matrix), etc. of traded asset $X_i(t,S(t))$ in the market, $1 \le i \le L$, and the market prices of risk associated with the exogenous sources $W(t)$ of market uncertainty are $A(t) = \Sigma(t)^T K(t)^{-1}[M(t) - r(t)1_L]$ with the asset price covariance matrix $K(t) = \Sigma(t)\Sigma(t)^T$]. If this financial economy is dynamically complete, then (in a Markovian framework) the value function

$$V_v^{c_{y_v},\theta_{y_v}}(t) = V_{\omega_{y_v} v}^{c_{y_v},\theta_{y_v}}(t) = V(t,X(t),Z(t))$$

of the investor's optimal or limited risk arbitrage (LRA) portfolio satisfies the linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B \alpha_{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (5.11)$$

with boundary conditions $V(0,X,Z) = v$ and $V(H,X,Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ \vdots \\ X_N M_N^{\omega_{y_v}} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \quad B = \begin{bmatrix} X_1\Sigma_{11} & \cdots & X_1\Sigma_{1N} \\ \vdots & & \vdots \\ X_N\Sigma_{N1} & \cdots & X_N\Sigma_{NN} \\ -Z\alpha_1^{\omega_{y_v}} & \cdots & -Z\alpha_N^{\omega_{y_v}} \end{bmatrix} \quad \nabla = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ \vdots \\ \frac{\partial}{\partial X_N} \\ \frac{\partial}{\partial Z} \end{bmatrix} = \begin{bmatrix} \nabla_X \\ \nabla_Z \end{bmatrix}$$

$$I^c(t,c) = [\frac{\partial U^c(t,c)}{\partial c}]^{-1} \quad I^V(V) = [\frac{dU^V(V)}{dV}]^{-1}$$

[and $\alpha_\omega(t) = \Sigma(t)^T K(t)^{-1}[M_\omega(t) - r_\omega(t)1_N]$, $M_\omega(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t^\alpha)1_N$ and $r_\omega(t) = r(t) + \delta(\omega(t)|K_t^\alpha)$ holds]. The optimal trading strategy (risk arbitrage strategy) itself is

$$\theta_{y_v} = [B\Sigma^{-1}]^T \nabla V = I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N] Z \nabla_Z V. \qquad (5.12)$$

In the incomplete case, we have the quasi-linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2}\mathrm{tr}(BB^T \nabla^2 V) - \nabla V^T B \tilde{\alpha}_{\tilde{a}_{y_v}}^{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t,Z) = 0 \qquad (5.13)$$

with the boundary conditions $V(0,X,Y,Z) = v$ and $V(H,X,Y,Z) = I^{V}(Z)$ for the portfolio value function

$$V_{v}^{c_{y_v}\theta_{y_v}}(t) = V_{\omega_{y_v} v}^{c_{y_v}\theta_{y_v}}(t) = V(t, X(t), Y(t), Z(t))$$

where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ \vdots \\ X_L M_L^{\omega_{y_v}} \\ Y_1 \bar{a}_1^{z_v} \\ \vdots \\ Y_{N-L} \bar{a}_{N-L}^{z_v} \\ -Z r_{\omega_{y_v}} \end{bmatrix} \quad B = \begin{bmatrix} X_1 \Sigma_{11} & \cdots & X_1 \Sigma_{1N} \\ \vdots & & \vdots \\ X_L \Sigma_{L1} & \cdots & X_L \Sigma_{LN} \\ Y_1 b_{11} & \cdots & Y_1 b_{1N} \\ \vdots & & \vdots \\ Y_{N-L} b_{N-L1} & \cdots & Y_{N-L} b_{N-LN} \\ -Z \bar{\alpha}_1^{\omega_{y_v} \bar{a}_{z_v}} & \cdots & -Z \bar{\alpha}_N^{\omega_{y_v} \bar{a}_{z_v}} \end{bmatrix} \quad \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \\ \nabla_Z \end{bmatrix}$$

[and $\alpha_{\omega}(t) = \Sigma(t)^{T} K(t)^{-1} [M_{\omega}(t) - r_{\omega}(t) 1_L]$, $M_{\omega}(t) = M(t) + \omega(t) + \delta(\omega(t) | K_t^{\alpha}) 1_L$ and $r_{\omega}(t) = r(t) + \delta(\omega(t) | K_t^{\alpha})$ holds, and moreover

$$v_{\bar{a}_{z_v}} = \frac{B_Y{}^{T} \nabla_Y V}{Z \nabla_Z V} = b^{T} \frac{I_Y \nabla_Y V}{Z \nabla_Z V} \qquad (5.14)$$

for the completion premium $v_{\bar{a}_{z_v}} \in K(\Sigma)$, $\bar{\alpha}_{\bar{a}_{z_v}}^{\omega_{y_v}} = \alpha_{\omega_{y_v}} + v_{\bar{a}_{z_v}}$, associated with the market prices of risk $\alpha_{\omega_{y_v}} \in K^{\perp}(\Sigma)$]. The optimal LRA asset allocation (risk arbitrage strategy) is similar to above

$$\theta_{y_v} = [B \Sigma^{T} K^{-1}]^{T} \nabla V = I_X \nabla_X V - K^{-1} [M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L] Z \nabla_Z V. \qquad (5.15)$$

All the notation used above is briefly described in the glossary of terms and symbols below. For more details, especially on the two important concepts of "securities market variation" $[K_t^{\alpha}, \delta(\omega(t) | K_t^{\alpha}), \omega_{y_v}(t)]$ and "securities market completion" $[K(\Sigma), K^{\perp}(\Sigma), \bar{a}_{z_v}(t)]$ that go well beyond the scope of this brief introduction, see reference [8.27]. During the construction process that led to these optimal solutions of the above stochastic control problem for strictly limited risk investments in (highly

geared) derivative financial products, several assumptions about an investor's utility functions $U^c(t,c)$ and $U^V(V)$ had to be made, especially

$$R^c(t,c) = -\frac{c\dfrac{\partial^2 U^c(t,c)}{\partial c^2}}{\dfrac{\partial U^c(t,c)}{\partial c}} \le 1 \text{ and } R^V(V) = -\frac{V\dfrac{d^2 U^V(V)}{dV^2}}{\dfrac{dU^V(V)}{dV}} \le 1 \qquad (5.16)$$

for the associated coefficients of relative risk aversion. In a general dynamic programming framework, all these restrictions (beyond the standard differentiability and boundedness assumptions) on the investor's overall risk management objectives can be removed and furthermore efficient alternative numerical solution methods obtained (for more details, see references [8.22] and [8.27]): We consider a discrete-time Markov chain approximation

$$x_h(t) \qquad (5.17)$$

(derived with the explicit finite difference method in a viscosity solution setting) with one step transition probabilities

$$\pi_t^u(x, x \pm \delta e_l) = \frac{h}{2\delta^2}\left[c_{ll}(t,x,u) - \sum_{\substack{k=0\\k\neq l}}^{L}\left|c_{kl}(t,x,u)\right| + 2\delta a_l^{\pm}(t,x,u)\right]$$

$$\pi_t^u(x, x + \delta e_k \pm \delta e_l) = \frac{h}{2\delta^2}c_{kl}^{\pm}(t,x,u) \quad \pi_t^u(x, x - \delta e_k \pm \delta e_l) = \frac{h}{2\delta^2}c_{kl}^{\pm}(t,x,u) \qquad (5.18)$$

$$\pi_t^u(x,x) = 1 - \frac{h}{\delta^2}\sum_{l=0}^{L}\left[c_{ll}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0\\k\neq l}}^{L}\left|c_{kl}(t,x,u)\right| + \delta\left|a_l(t,x,u)\right|\right]$$

[where $c(t,x,u) = b(t,x,u)b(t,x,u)^T$ is the covariance matrix of the approximated diffusion process $x(t) \in R^{L+1}$, $e_0,..,e_L$ is the standard basis in R^{L+1} and $\pi_t^u(x,y) = 0$ for all other grid points y on the associated lattice structure $X_h \subseteq R^{L+1}$] of the controlled continuous-time state dynamics

$$dx(t) = a(t,x(t),u(t))dt + b(t,x(t),u(t))dW(t)$$

$$a(t) = \begin{bmatrix} X_1(t)M_1(t) \\ \vdots \\ X_L(t)M_L(t) \\ \theta(t)^T[M(t) - r(t)1_L] \\ +V(t)r(t) - c(t) \end{bmatrix} \quad b(t) = \begin{bmatrix} X_1(t)\Sigma_{11}(t) & \cdots & X_1(t)\Sigma_{1N}(t) \\ \vdots & & \vdots \\ X_L(t)\Sigma_{L1}(t) & \cdots & X_L(t)\Sigma_{LN}(t) \\ & \theta(t)^T\Sigma(t) & \end{bmatrix} \quad (5.19)$$

[where $x(t) = (X(t), V(t))$, $u(t) = (c(t), \theta(t))$ are the controls and the coefficients $a(t,x,u) \in R^{L+1}$ and $b(t,x,u) \in R^{(L+1)\times N}$ satisfy the usual conditions that guarantee a unique strong solution of the associated evolution equation with bounded absolute moments]. The discrete-time Hamilton-Jacobi-Bellman (HJB) dynamic programming equation is then

$$V_h(t,x) = \max_{u \in \bar{A}_{tx}} \left[\sum_{y \in X_h} \pi_t^{u(t)}(x,y) V_h(t+h,y) + hL(t,x,u(t)) \right] \quad (5.20)$$

with boundary condition $V_h(H,x) = \psi(x)$ [where

$$L(t,x(t),u(t)) = U^c(t,c(t)) \text{ and } \psi(x(H)) = U^V(V(H))$$

are the investor's overall risk management objectives and $\bar{A}_{tx}$ is the set of all feasible controls $u(s)$ on the time interval $[t,H]$ when the time t state is x] and an associated optimal Markov control policy $\bar{u}_h(t,x)$ maximizes the expression

$$\sum_{y \in X_h} \pi_t^{u(t)}(x,y) V_h(t+h,y) + hL(t,x,u(t)) \quad (5.21)$$

in $\bar{A}_{tx}$ [backwards in time from $H-h$ to 0]. Furthermore, we have the uniform convergence

$$\lim_{\substack{(\tau,\xi)\to(t,x) \\ h\downarrow 0}} V_h(\tau,\xi) = V(t,x) \quad (5.22)$$

[where

$$J(t,x,u) = E_{tx}[\int_t^H L(s,x(s),u(s))ds + \psi(x(H))]$$

$$V(t,x) = \sup_{u \in \bar{A}_{tx}} J(t,x,u)$$

is the corresponding continuous-time expected utility maximization criterion] of the discrete-time Markov chain control problem to the continuous-time diffusion process control problem. All the notation used above is briefly described in the glossary of terms and symbols below. For more details, see reference [8.27]. It turns out that the above methods can also very easily be applied to the case where the liabilities $L(t)$ are related to life (re-)insurance contracts. This is the topic of our forthcoming paper [8.28] with Mark H. A. Davis. Returning to the general asset/liability management problem of a large (re-)insurance company, i.e., to the case of a market state variable

$$X(t) = (A(t), L(t), E(t), \ldots) \qquad (5.23)$$

with RCLL sample paths, we shall now continue our stochastic analysis using the notation and limited risk arbitrage (LRA) techniques of reference [8.27]. For further details, see [8.25], [8.27] and our forthcoming paper [8.29] with M. H. A. Davis.

5.2 Impulse Control Approach for (Re-)insurance Companies

Singular Controls. Similar to our LRA asset management approach for banks outlined above, we now consider the controlled state variable

$$x(t) = \left(X(t), V_v^v(t), \ldots\right) \qquad (5.24)$$

which would at least have the following additional components:

$\Delta_v^v(t)$	(instantaneous investment risk)
$\Gamma_v^v(t)$	(future portfolio risk dynamics)
$\Theta_v^v(t)$	(portfolio time decay dynamics)
$\Lambda_v^v(t)$	(portfolio value appreciation dynamics)

(on the liability side exposure constraints in the form of jump size restrictions or solvency capital restrictions are possible). Denoting the controls with $u(t) = (v(t), \ldots)$ and with $\overline{G} = (0, \delta, \gamma, \vartheta, \lambda, \ldots) \subseteq R^n$ the region in state space (ALM tolerance band) defined by corresponding inequality constraints $V_v^v(t) \geq 0$, $|\Delta_v^v(t)| \leq \delta$, ..., we want to

make sure that with optimal stochastic control $x(t) \in \overline{G}$ holds, $0 \le t \le H$: Started at an admissible point $x \in G$ the state variable $x(t)$ evolves in time until it comes close to the boundary ∂G of the ALM tolerance band. At each boundary point $y \in \partial G$, a set $R(y)$ of admissible reflection directions is assumed to be given [e.g., the interior normals $n(y) \perp \partial G$ on the hyperplanes $(0, \delta, \gamma, \vartheta, \lambda, \ldots)$ at all points $y \in \partial G$ where they exist] and the state evolution is then reflected back into G in one of these admissible directions. We also allow (relaxed) intertemporal control of the state variable while it meets the ALM constraints, i.e., resides in $\overline{G}$, and therefore consider the general singular (reflected jump diffusion) control model

$$x(t) = x + \begin{bmatrix} \int_0 \int_U a(s, x(s), u) m_s(du) ds + \\ \int_0 \int_U b(s, x(s), u) m_s(du) dW(s) + \\ \int_0 \int_{R^n} \int_U q(s, x(s-), u, y) m_{s-}(du) N(dsdy) \end{bmatrix} + F(t) + z(t)$$

$$x + q(t, x, u, y) \in \overline{G}, \; x \in \overline{G} \tag{5.25}$$

$$F(t) = \sum_{i=1}^{l} r_i f_i(t) \quad r_i \in R(y_i), \; y_i \in \partial G \quad f_i(t) \ge 0, \; df_i(t) \ge 0$$

$$z(t) = \int_0 r(s) d|z|(s), \; r(s) \in R(x(s)) \; [\mu_z(ds) \text{ a.e.}] \quad |z|(t) = \int_0 1_{x(s) \in \partial G} d|z|(s)$$

which is based on a Lipschitz continuous solution mapping in the Skorokhod problem for (G, R) and under our above assumptions [and the usual compact control space $U \subseteq R^p$] has a unique strong solution $x(t) \in \overline{G}$, $0 \le t \le H$, for every $x \in G$. Note that any (conventional) progressively measurable control process $u(t) \in U$, $0 \le t \le H$, has a relaxed control representation $m_t^{u(\cdot)}(du)$ [by an adapted random measure on the Borel sets $B(U)$] such that

$$a(t, x(t), u(t)) = \int_U a(t, x(t), u) m_t^{u(\cdot)}(du)$$

$$b(t, x(t), u(t)) = \int_U b(t, x(t), u) m_t^{u(\cdot)}(du)$$

$$q(t, x(t-), u(t-), y) = \int_U q(t, x(t-), u, y) m_{t-}^{u(\cdot)}(du)$$

holds. The control-theoretical value function associated with general impulsive asset/liability management is then

$$J(t,x,m,f) = E_{tx}\begin{bmatrix} \int_t^H \int_U L(s,x(s),u)m_s(du)ds + \\ \int_t^H M(s,x(s))^T df(s) + \\ \int_t^H N(s,x(s))^T dz(s) - \\ \psi(x(H)) \end{bmatrix} \qquad V(t,x) = \inf_{AS} J(t,x,m,f) \qquad (5.26)$$

where $\psi(x(H)) = U^V(V(H))$ is the utility of final wealth [the utility $U^c(t,c(t))$ of intertemporal fund consumption would be included in the term $L(t,x(t),u(t))$] and the continuous reflection part of the bounded and continuous total risk exposure control costs (L,M,N) satisfies $N(t,y)^T r \geq 0$, $r \in R(y)$, on ∂G. The infimum is taken over all admissible (relaxed/singular) control systems. and the corresponding (formal) HJB dynamic programming equation is of the form

$$\min\left\{ \inf_{m_{tx}(du)} \begin{bmatrix} [A^{m_{tx}(du)}V](t,x) + \\ \int_U L(t,x,u)m_{tx}(du) \end{bmatrix}, \min_{1\leq i\leq l}\left[\frac{\partial V}{\partial t}(t,x) + \nabla_x V(t,x)^T r_i + M_i(t,x) \right] \right\} = 0$$

$$\frac{\partial V}{\partial t}(t,x(t)) + [\nabla_x V(t,x(t)) + N(t,x(t))]^T r(t) = 0,\ x(t) \in \partial G \qquad (5.27)$$

$$V(H,x) = -\psi(x)$$

where the parabolic integro-differential operator $[A^{m_{tx}(du)}V](t,x)$ (in relaxed control notation) is defined with the controlled jump diffusion parameters, i.e. ,

$$[A^{m_{tx}(du)}V](t,x) = \begin{bmatrix} \frac{\partial V}{\partial t} + \\ \int_U \begin{bmatrix} a(t,x,u)^T \nabla_x V + \frac{1}{2}\mathrm{tr}\left(b(t,x,u)b(t,x,u)^T \nabla_x^2 V\right) + \\ \lambda(t,x,u)\int_{\Gamma}[V(t,x+y) - V(t,x)]\overline{\Pi}(t,x,u,dy) \end{bmatrix} m_{tx}(du) \end{bmatrix}(t,x).$$

<u>Markov Chain Approximation</u>. With a discrete-time Markov chain approximation

$$x_h(t) = x + \begin{bmatrix} \int_0^t a(s, x_h(s), u_h(s))ds + \\ \int_0^t b(s, x_h(s), u_h(s))dW_h(s) \end{bmatrix} + J_h(t) + F_h(t) + z_h(t) + \widetilde{F}_h(t) + \widetilde{z}_h(t) + \varepsilon_4^h(t)$$

$$J_h(t) = \sum_{t_n^h \le t} q_h(t_n^h, x_h(t_n^h -), u_h(t_n^h -), y_n^h) \tag{5.28}$$

$$E[\sup_{0 \le t \le H} \|\widetilde{F}_h(t)\|^2] \xrightarrow{h \downarrow 0} 0 \quad E[\sup_{0 \le t \le H} \|\widetilde{z}_h(t)\|^2] \xrightarrow{h \downarrow 0} 0$$

$$E[\sup_{0 \le t \le H} \|\varepsilon_4^h(t)\|] \xrightarrow{h \downarrow 0} 0$$

that is locally consistent with the controlled continuous-time reflected jump diffusion state dynamics

$$x(t) = \left(X(t), V_v^v(t), \Delta_v^v(t), \Gamma_v^v(t), \Theta_v^v(t), \Lambda_v^v(t), \ldots\right) \tag{5.29}$$

an approximation $V_h(t,x)$ of the value function of the above impulse control approach to optimal asset/liability management satisfies the discrete-time HJB dynamic programming equation

$$V_h(t,x) = \min \left\{ \begin{array}{c} \min_{u \in U} \left[\begin{array}{c} (1 - \lambda \Delta t_{txu}^h - \delta_{txu}^h) \sum_{y \in X_h} \pi_{txu}^h (y|D) V_h(t + \Delta t_{txu}^h, y) + \\ (\lambda \Delta t_{txu}^h + \delta_{txu}^h) \int_\Gamma V_h(t + \Delta t_{txu}^h, x + q_h(t,x,u,y)) \Pi(dy) + \\ \Delta t_{txu}^h L(t,x,u) \end{array} \right], \\ \min_{1 \le i \le I} \left[\sum_{y \in X_h} \pi_{tx}^h (y|r_i^h) V_h(t,y) + hM_i(t,x) \right] \end{array} \right\}$$

$$V_h(t,x) = \sum_{y \in X_h} \pi_{tx}^h (y|R) V_h(t,y) + N(t,x)^T \Delta z_h(t,x), \; x \in \partial G_h \tag{5.30}$$

$$V_h(H,x) = -\psi(x).$$

By solving this equation backwards in time from $t_{K_h - 1}^h$ $[t_{K_h}^h = H]$ to 0, we can determine an optimal Markov control policy $\bar{u}_h(t,x)$ [which we denote by $\overline{m}_{tx}^h(du)$ in relaxed control notation], the corresponding (optimal) intertemporal singular control impulses $\Delta \bar{f}_h(t,x)$ and the necessary reflection impulses $\Delta \bar{z}_h(t,x)$ at the boundary ∂G_h (where $\overline{G}_h$ is a corresponding discretization of the given ALM tolerance band $\overline{G}$), i.e., an optimal discrete-time impulsive risk exposure control strategy $(\bar{u}_h = \overline{m}_h, \bar{f}_h, \bar{z}_h)$ and

the associated state evolution $\bar{x}_h(t) \in \bar{G}_h$, $0 \le t \le H$. A weak convergence argument [with an embedded time scale adjustment to ensure tightness of the singular control and boundary reflection parts of the approximating discrete-time state dynamics $x_h(t)$] then also establishes an optimal solution of the initially given continuous-time impulsive asset/liability management problem, i.e., a continuous-time optimal impulsive risk exposure control strategy $(\bar{m}, \bar{f}, \bar{z})$ and the associated state evolution $\bar{x}(t) \in \bar{G}$, $0 \le t \le H$. All the notation used above is briefly described in the glossary of terms and symbols below. For more details, see references [8.27] and [8.29] (forthcoming). With the above, we have developed a consistent mathematical framework within which various generalizations of the currency-baseline problem of a (re-)insurance company can be evaluated and implemented (the simple Monte Carlo simulation presented earlier would be replaced by lattice approaches, Markov chain approximations, along the above lines). Apart from establishing a continuous-time solution of the complex asset/liability management problem of a large, internationally operating (re-)insurance company, the value of the above control theory framework especially lies in the fact that it explicitly defines corresponding stable, discrete-time numerical approximations with good convergence properties (see references [8.22] and [8.25]).

5.3 Glossary of Terms and Symbols

The main terms and symbols used above are listed here together with a brief description. For further details, see reference [8.27].

Securities and Derivatives Market Model

$r(t)$	risk-free interest rate
$M(t)$	asset return rate vector
$\Sigma(t)$	asset volatility matrix
$K(t)$	asset covariance matrix
$W(t)$	Wiener process
$X(t)$	asset price process
$I_X(t)$	matrix with diagonal $X(t)$
$A(t)$	market prices of risk

$D(t)$	risk-neutral discount factor
$\varsigma(t)$	Arrow-Debreu price system
$\Delta(t)$, $\Gamma(t)$, $\Theta(t)$, $\Lambda(t)$	derivatives risk parameter
$c(t)$	fund withdrawal rate
$v(t)$, $\theta(t)$	trading strategy
K_t^α	risk/arbitrage constraint set
$V_v^{c\theta}(t)$, $V_v^{c_{y_v}\theta_{y_v}}(t)$	portfolio value process
$(c,\theta) \in A(v)$	admissibility
$U^c(t,c)$, $U^V(V)$	utility function
$I^c(t,c)$, $I^V(V)$	inverse of marginal utility
$R^c(t,c)$, $R^V(V)$	relative risk aversion
$Z(t)$	marginal utility of wealth

Securities Market Completion

$Y(t)$	additional state variables
$b(t)$	orthonormal volatility matrix
$S(t)$	market state vector
$K(\Sigma)$	volatility matrix kernel
$K^\perp(\Sigma)$	volatility matrix range
$\hat{a}_{z_v}(t)$	market variation parameter
$v_{\hat{a}_{z_v}}(t)$	completion premium
$\hat{\alpha}_{\hat{a}_{z_v}}^{\omega_{y_v}}(t)$	market prices of risk

Securities Market Variation

$\omega(t)$, $\omega_{y_v}(t)$	market variation parameter
$\delta(\omega(t)\vert K_t^\alpha)$	support function of K_t^α
$r_\omega(t)$, $r_{\omega_{y_v}}(t)$	risk-free interest rate
$M_\omega(t)$, $M_{\omega_{y_v}}(t)$	asset return rate vector
$\alpha_\omega(t)$, $\alpha_{\omega_{y_v}}(t)$	market prices of risk
$V_{\omega_{y_v} v}^{c_{y_v}\theta_{y_v}}(t)$	portfolio value process

Dynamic Programming

$x(t)$, $x_h(t)$	state dynamics
$\pi_t^u(x,y)$	one step transition probability
X_h	associated lattice structure
$L(t,x,u)$, $\psi(x)$	optimization criterion

$V(t,x)$, $V_h(t,x)$	value function
$u(t)$, $\bar{u}_h(t,x)$	admissible control
$u \in \overline{A}_{tx}$	feasibility

Impulse Control Approach

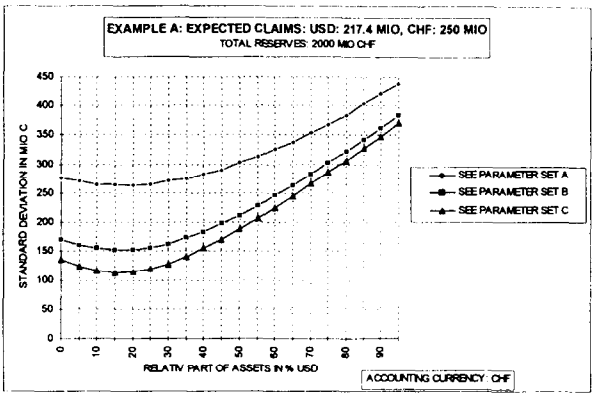
$W(t)$, $W_h(t)$	Wiener process
$J(t)$, $J_h(t)$	Poisson jump process
$q(t,x,y)$, $q_h(t,x,y)$	associated parameter
$N(dtdy)$	Poisson random measure with parameter λ
$\Pi(dy)$	associated probability measure with support Γ
$\overline{G} = (0,\delta,\gamma,\vartheta,\lambda,\ldots)$	ALM tolerance band
$R(y)$, $y \in \partial G$	admissible directions
$u(t)$, $u_h(t)$, $\bar{u}_h(t,x)$	ordinary control process
$m_t(du)$, $m_t^h(du)$, $\overline{m}_t(du)$, $\overline{m}_{tx}^h(du)$	relaxed ordinary control
$F(t)$, $F_h(t)$, $\overline{F}(t)$, $\overline{F}_h(t)$	impulse (singular) control
$z(t)$, $z_h(t)$, $\bar{z}(t)$, $\bar{z}_h(t)$	boundary reflection
Δt_{txu}^h	interpolation interval
δ_{txu}^h	jump diffusion parameter
$\pi_{tx}^h(y\|D)$, $\pi_{tx}^h(y\|r_i^h)$, $\pi_{tx}^h(y\|R)$	one step transition probability
$x(t)$, $x_h(t)$, $\bar{x}(t)$, $\bar{x}_h(t)$	state dynamics

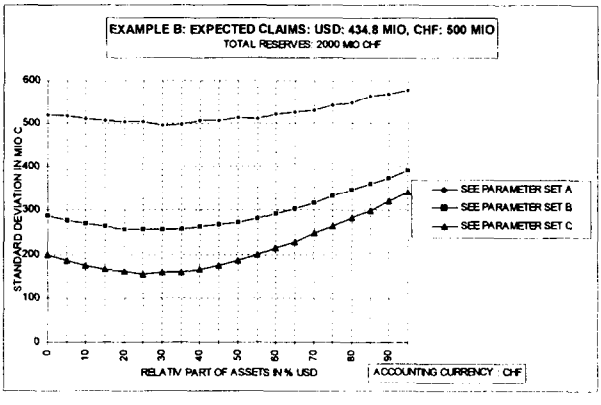
6. Two Instructive Examples

In this section we provide two numerical examples (1 period, 2 currencies), solved with the Monte Carlo method. We consider two examples of standard deviations of net reserves before the new premium arrives after one period where CHF (Swiss franc) is the accounting currency.

In both examples we have independent assets and liabilities only in two currencies: in CHF and in USD.

In the examples claims reserves are assumed to be identical and to have an amount of CHF 2000 mio.





Parameter sets for the examples

Assets		μ_A =0.05	σ_A =0.05
Currencies	E_0 =1.15	μ_E =0	σ_E =0.02

Parameter set for liabilities example A:

	L_0 (USD)	L_0 (CHF)	μ_L	σ_L	λ
Curve A	217.4	250	0.02	0.5	0.5

Curve B	217.4	250	0.02	0.25	0.25
Curve C	217.4	250	0.02	0.125	0.125

Parameter set for liabilities example B:

	L_0 (USD)	L_0 (CHF)	μ_L	σ_L	λ
Curve A	434.8	500	0.02	0.5	0.5
Curve B	434.8	500	0.02	0.25	0.25
Curve C	434.8	500	0.02	0.125	0.125

In example B the liabilities are expected to be twice as risky as in example A whereas the assets are thought to perform in the same manner in both examples. Curves corresponding to three similar parameter sets are used in each example. These parameter sets correspond to different levels of the volatilities of the liability-processes. The curves were obtained by 100'000 Monte Carlo simulations. It is important to see that the minimum of the curves, expressed as optimum relative percentage of the assets in USD is nearly the same for all three curves in each example. This means that the currency baseline is not directly very sensitive to the volatility-parameters but indirectly as the ratio of expected claims to reserves changes. As the ratio of total reserves divided by expected claims reflects the chosen probability of ruin, the currency baseline is a function of the probability of ruin. These two examples lead us to suppose the following rule of thumb for the currency baseline in a one period model:

Rule of thumb

> ***A first approximation for the currency-baseline in a one-period-model is obtained by reserving in each currency the expected value of the claims to be paid during the next period and to shift the whole rest of the reserves into the accounting currency.***

This means, as seen in the above examples, that even if there is identical business in two currencies, a big shift to the accounting currency will occur for the baseline because by

ruin probability arguments the total reserves have to be much bigger than the expectation of the total claims: In example A 12.5% of the total reserves should be in the USD-baseline according to our rule of thumb, whereas 50% of the business is made in this currency.

7. Some Possible Generalisations

In this study we were interested in the currency baseline for the exchange rate risk in the VaR model. Our argumentation was based on a one-period model. It is an interesting question whether modelling a (re-) insurance of longtail business with multiperiod jump diffusion models would confirm or reject our rule of thumb (stochastic control approach).

It is also interesting to see the deviations of the exact baseline values from the rule of thumb values for other parameter combinations.

Similar ways are possible to calculate the interest rate risk baseline for the modeling of the interest rate risks with the VaR model. In this case however, it is necessary to model longtail business with multiperiod jump diffusion models.

8 References

Currency-Baseline Problem.

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Future Work.

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