

An Empirical Comparison of Valuation Models for Interest Rate Derivatives

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Summary

Interest rate derivatives are much more difficult to value than stock options. This paper discusses the basic approaches to price interest rate derivatives and presents the first comprehensive study of different models which can be used to manage the risk of interest rate derivatives. One and two factor models of the Heath / Jarrow / Morton type and inversion models of the Hull / White type are considered.

Résumé

Les dérivatifs sur taux d'intérêt sont bien plus difficiles à évaluer que les options sur actions. Cet article discute l'approche de base d'évaluation des dérivatifs sur taux d'intérêt et présente une revue étendue de différents modèles pouvant être utilisés pour gérer le risque lié à ces dérivatifs. Les modèles à un et deux facteurs du type Heath / Jarrow / Morton et des modèles à inversion du type Hull / White sont considérés.

Keywords

Interest rate derivatives, valuation, empirical study.

1 Introduction

The valuation and management of interest rate derivatives is one of the most burning issues in investment banking today. Valuation models of interest rate derivatives are applied in three main fields. The first field is comprised of the short-term trading of exchange traded or OTC-interest rate options. Here fine tuned models are required which take into account the peculiarities of a specific market segment to supply realistic values for option sensitivities and implied volatilities. For short-term options the use of Black's model (1976) is often an appropriate solution. The second field refers to pricing issues related to more complex and possibly longer dated options such as call features, prepayment options, caps, and collars. In principle these options should be priced regarding all information concerning the term and volatility structure of interest rates. However, approaches used in the banking industry often rest on simpler models, which are based on the stochastics of one bond price or one interest rate, only. The third field where valuation models for interest rate derivatives are applied is that of systems to manage the total interest rate position of a financial intermediary. Whether a scenario technique, a simulation approach or the value at risk concept is used to measure and control the exposure to interest rate risk, the value of interest rate options at future points in time must be known.

In this study we empirically compare models which could be used for application in the aforementioned second and third fields. These models must possess the following two features:

(1) Usage of all information available in the market.

In order to come up with realistic option values it is necessary to use the information about the current term structure of interest rates and, equally important, the current term structure of volatilities.

(2) Consistent modelling of the stochastic dynamics of the term structure of interest rates.

The key problem in valuing interest rate derivatives is that of obtaining an arbitrage-free, feasible, and realistic description of how the term structure of interest rates – a theoretically infinite dimensional process – evolves over time.

In the literature four approaches for the valuation of interest rate options have been suggested. The first models the endogenous term structure of interest rates in a no-arbitrage (Vasicek (1977), Brennan and Schwartz (1979, 1982), Langetieg (1980)) or in an equilibrium framework (Cox, Ingersoll, and Ross (1985), Longstaff and Schwartz (1992)) as a function of a few exogenous factors. The second group of models follow closely the approach of Black and Scholes (1973) and use the price of the underlying bond as an exogenous variable (Ball and Torous (1983), Schaefer and Schwartz (1987), Bühler (1990)). Both approaches are deficient in a number of ways as discussed intensively in the literature. Above all, they do not satisfy the two conditions stated above and are therefore not considered in our study.

When one considers those models which fully exploit the information available from the current term structure of interest rates, there are essentially two additional approaches. The third approach, pioneered by Ho and Lee (1986) and Heath, Jarrow, and Morton (1990, 1992) begins with the evolution of the entire zero-coupon price curve. The fourth approach, conceived by Black, Derman, and Toy (1990), Hull and White (1990, 1993), Black and Karasinski (1991), Jamshidian (1991), and Uhrig and Walter (1993, 1996), specifies the spot rate process and determines the parameters in such a way that the model is consistent with the current term structure. Whereas the models of the Heath, Jarrow, and Morton (HJM) type are consistent with the current term structure by construction, the fourth approach type models can be regarded as generalizations of the first approach. In order to be able to incorporate all information of the current term structure of interest rates into these models, it is necessary to inverse the endogenous zero bond prices. Following the terminology of HJM these models will be denoted as inversion models.

The majority of recent academic literature which considers the problem of valuing interest rate options is almost exclusively theoretically oriented. This is especially the case for the extension of the martingale approach pioneered by Harrison and Kreps (1979), Harrison and Pliska (1981) to interest rate derivatives (e.g. Artzner and Delbaen (1989)). Very few papers study the empirical performance of a single model. Dietrich-Campbell and Schwartz (1986) value interest rate options on US government bonds and treasury bills using the two-factor model of Brennan and Schwartz (1982). Bühler and Schulze (1993a, 1995), Flesaker (1993), and Amin and Morton (1994) present empirical studies of the HJM model and its variant, the Ho and Lee model. The first two studies analyze the value of callable bonds of German public issuers in the German bond market. The third and fourth papers present results for Eurodollar futures options.

We are not aware of any empirical study which has to date compared different valuation models for interest rate options. This paper is the first assessment of the empirical quality of one- and two-factor models of the HJM type and of one- and two-factor inversion models. It therefore contributes to our understanding of the performance of these models for a number of important reasons:

- (1) As the same data set of German interest rate warrants is used, the differences between the quoted prices and the theoretical prices of the different valuation models can be directly compared.
- (2) The time to maturity of the interest rate warrants covers periods of up to 2.9 years. Therefore not only short-lived options are contained in the sample.
- (3) The empirical quality of a valuation model and the quality of the estimation procedure for the input data of the model are unavoidably assessed together. Therefore as far as possible, the same data and the same statistical methodology are used throughout. For all models

the estimation of the term structures of interest rates is based on the identical set of German government bonds. Furthermore, diffusion and drift coefficients of the different models are estimated by means of time series of the same length.

- (4) The different valuation models are not only assessed by their ability to predict observed option prices (empirical quality). In addition the following three criteria are taken into account: Difficulty of the estimation of the input data, problems in fitting the endogenous term and volatility structures of interest rates, and numerical problems in solving the valuation model.

The paper is organized as follows. In Section 2, we present some results of our data snooping. The main purpose of this section is to provide an insight into the selection and modelling of the factors in the inversion models. A short survey on the valuation models used in the empirical part of our paper is given in Section 3, and in Section 4, the basic characteristics of the data is described. In Section 5, we outline the estimation procedures used for the input data, and we present the results on the empirical quality of the different models. In Section 6, we apply the four criteria described under (4) above to assess the different models. This assessment results in the recommendation of one model.

2 Data Exploration

Three main problems have to be solved if interest rate options are valued. First, the current term structure of interest rates has to be estimated. Second, the transition behaviour of future term structures must be modelled. Third, the interest rate option under consideration is valued, typically by means of a numerical procedure. Each of these steps involves a number sub-problems which are themselves closely interrelated as well as being related

to subproblems of other steps. In this section we present results of a data exploration which will justify some of the decisions which we made in constructing the valuation models of Section 3 and in planning the empirical design of the study.

2.1 Estimation of Term Structures of Interest Rates

For reasons which will become clear in Section 4, the term structures of interest rates will be estimated for the homogeneous and most liquid market segment of German government bonds. As the German government does not issue zero bonds, and as its coupon bonds are not stripped into zero bonds as in the US-STRIPS program, the term structure of interest rates has to be estimated from traded coupon bonds.

An analysis of the traded government bonds' maturity structure and of the quoted prices reveals the following facts:

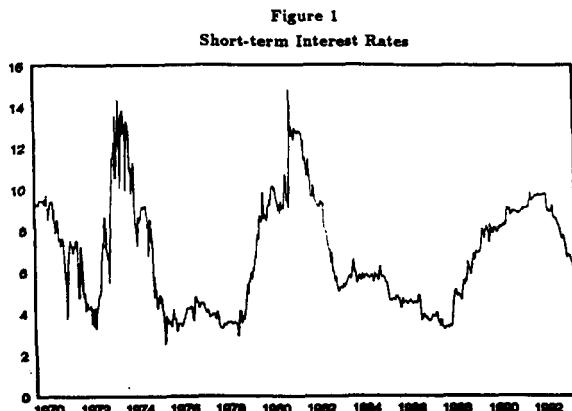
- (1) The number of coupon dates exceeds the number of traded bonds. Therefore there are more unknown zero bond prices (zero bond yields) than estimation equations. This means that, in principle, the observed bond prices can be explained without error by the estimated term structure of interest rates. However, this assertion has to be qualified by the following two observations.
 - (2) A few coupon bonds mature on the same day. As the observed prices of these bonds are not perfectly in line with each other, the bonds with identical maturity dates result in an unavoidable, but small observation error.
 - (3) Noise in bond prices with maturity dates which are close to each other may result in negative forward rates. If the nonnegativity of forward rates is imposed as a constraint, the estimation error increases, but remains nevertheless small.

- (4) The estimation procedure for the term structure of interest rates affects the valuation of interest rate options in two ways. First, the underlying of the option is valued using the term structure of interest rates. Second, in models of the HJM type the volatility of the forward rates will be estimated from a time series of the term structures of interest rates. In valuation models of the inversion type, the estimated zero bond prices directly affect the time-dependent parameter, in our study being the market price of risk of an interest rate factor. These two applications have different consequences for the valuation of interest rate options.

The first application leads to the recommendation of the implementation of an estimation procedure which minimizes the deviations between observed and theoretical prices of the underlyings since these deviations transfer directly to differences between the observed and theoretical option values which can not be attributed to the valuation model. An estimation procedure which results only in the occurrence of unavoidable errors of type (2) and (3) is appropriate for this application. We will denote this procedure as the "minimum error method (MEM)".

The MEM fully transfers noise in the data of coupon bonds to the term structure of interest rates. This noise in the term structure of interest rates results in irregular time-dependent market prices of risk in inversion models and leads to unreasonably high volatility estimates of forward rates. In both cases the empirical quality of the valuation models turned out to be very low. The second application therefore suggests a smoothing out of the irregular behaviour of the estimates obtained from the MEM. This is achieved in a balanced way by a cubic spline method. By smoothing the term structure of interest rates for the research period from 1990 to 1993, the average absolute deviation of prices increased from DM 0.07 to DM 0.15 per DM 100 face value.

Figure 1
Short-term Interest Rates

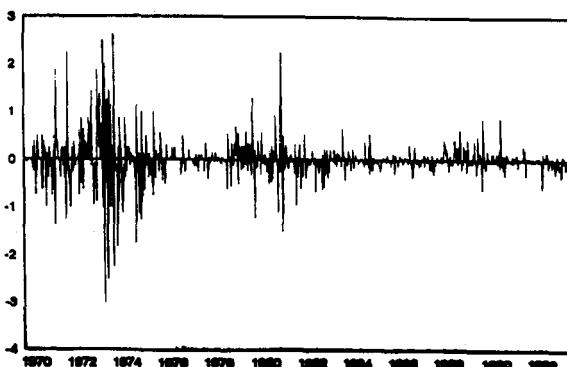


2.2 Short-Term Interest Rates

Bid and offer rates in the German money market are available for one day, one month, two, three, six, twelve, and twenty four months. As the daily rate fluctuates strongly, and as the level and the changes of this rate are only loosely related to other short-, medium- and long-term rates, the daily rate cannot reasonably be used to explain the evolution of the whole term structure of interest rates. These criticisms do not hold to the same extent for the second shortest rate, the monthly rate. This rate is therefore selected as "the" short rate. The level and the weekly changes of the monthly rate are shown in Figures 1 and 2.

The evolution of the monthly rate shows a mean reversion property around the long-term mean of about 6.8%. A comparison of the two figure shows that the largest interest rate movements take place in the periods of high interest rates around 1973/74 and 1980/81. Therefore modeling the short rate r as a mean reverting process and its conditional volatility proportional to a

Figure 2
Weekly Changes of Short-term Interest Rates
Figure 2
Weekly Changes of Short-term Interest Rates



positive power of r appears to be reasonable for a one-factor model. However, as the changes of the monthly rate exhibit typical volatility clusterings also a model with stochastic volatility of the short rate (i.e. a two-factor model) could be an appropriate description of the data.

2.3 Long-term Interest Rate, Spread, and Correlations

Multi-factor models provide a more realistic description of the transition behaviour of term structures of interest rates. In fact, Bühler and Schulze (1993b) find that 95% of the variation of interest rates in the German government bond market can be explained by two factors only. Like in the US bond market one factor can be identified with the level of interest rates, the second one is closely related to the spread between a long-term rate and the one-month rate. The correlations between different zero bond yields, the

Table 1
Correlation between Interest Rates (1980 – 1993)

Time to Maturity	1 M	1 Y	2 Y	3 Y	4 Y	5 Y	6 Y	7 Y	8 Y	9 Y	10 Y
1 M	1.00										
1 Y	0.97	1.00									
2 Y	0.94	0.99	1.00								
3 Y	0.90	0.97	0.99	1.00							
4 Y	0.87	0.95	0.98	0.99	1.00						
5 Y	0.86	0.94	0.97	0.99	0.99	1.00					
6 Y	0.84	0.93	0.98	0.98	0.99	0.99	1.00				
7 Y	0.81	0.90	0.94	0.96	0.98	0.99	0.99	1.00			
8 Y	0.80	0.89	0.93	0.96	0.98	0.98	0.99	0.99	1.00		
9 Y	0.79	0.88	0.92	0.95	0.97	0.97	0.98	0.99	0.99	1.00	
10 Y	0.79	0.88	0.91	0.94	0.95	0.96	0.96	0.98	0.98	0.98	1.00
Spread	-0.92	-0.82	-0.75	-0.68	-0.63	-0.61	-0.58	-0.52	-0.51	-0.50	-0.48

one-month rate, and the spread are shown in Table 1. A natural conclusion of these results is a valuation model which uses both a long rate and the spread between a long rate and a short rate as explaining factors.

As a proxy for the long rate we use the yield to maturity of a nine-year zero-bond. The weekly changes of the nine-year yield are represented in Figure 3. These changes show no volatility clustering. Furthermore, neither the mean nor a coefficient which characterizes mean reversion are significantly different from zero on the 1%-level. Therefore it is reasonable to model the long rate as a martingale.

The evolution of the difference of the long-term and the short-term interest rate is presented in Figure 4. Obviously the spread takes on both positive and negative values. As, in addition, the mean reversion parameter is significant on the 5%-level, the spread will be modelled as an Ornstein-Uhlenbeck process.

2.4 Forward rates

Models of the HJM type are based on the dynamics of instantaneous forward rates. Figure 5 shows the weekly changes of smoothed instantaneous

Figure 3
Weekly Changes of the the Nine-Year Yield (1980 -1993)

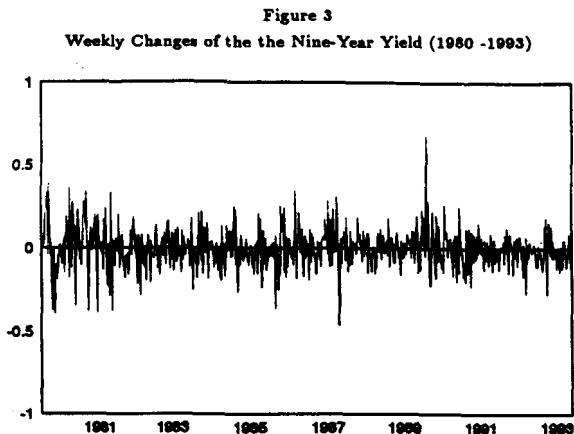


Figure 4
Evolution of the Spread (1980-1993)

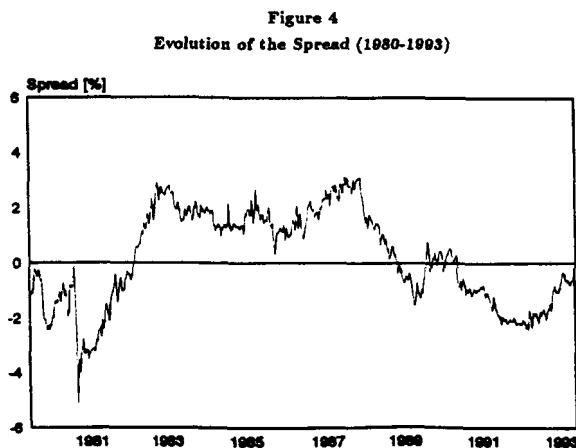
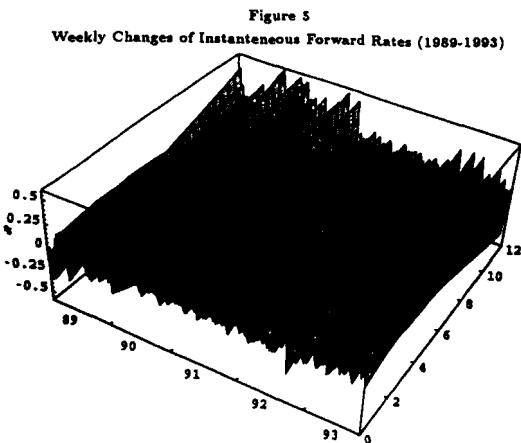


Figure 5
Weekly Changes of Instantaneous Forward Rates (1989-1993)



forward rate functions from 1989 to 1993 for maturities of up to 12 years. These differences are partly constant, and partly dependent on the time to maturity. This "rough" observation will be used in the parametric formulation of the diffusion coefficients of forward rates. A factor analysis of the instantaneous forward rates shows similar results as in the spot market. Two factors explain almost 96% of the variation in forward rates. One factor is highly correlated (0.99) with the 8- and 10-year forward rate, whereas the second one is related to the 2-year forward rate.

3 A Short Survey of the Empirically Tested Models for Valuing Interest Rate Derivatives

In this section, we shortly summarize the structure of the HJM models and the inversion models which were compared in our empirical investigation. For details the reader is referred to the literature.

3.1 The HJM Type Models

HJM start with a fixed number of unspecified factors which are driving the dynamics of the instantaneous forward rates

$$df(t, T) = \alpha(t, T, \cdot)dt + \sum_{i=1}^N \sigma_i(t, T, f)dz_i(t). \quad (1)$$

$f(t, T)$ denotes the instantaneous forward interest rate at date t for borrowing or lending at date T . $z_1(t) \dots z_n(t)$ are independent one-dimensional Brownian motions and $\alpha(t, T, \cdot)$ and $\sigma_i(t, T, f)$ are the drift¹ and the volatility coefficients of the forward rate of maturity T . Under a number of regularity conditions and a standard no-arbitrage condition HJM show, that the drift of the forward rates under the risk neutral measure is uniquely determined by the volatility functions $\sigma_i(t, T, f)$

$$\alpha(t, T, \cdot) = \sum_{i=1}^N \sigma_i(t, T, f) \int_t^T \sigma_i(t, s, f)ds. \quad (2)$$

Exogenous to the model of HJM are the current forward rate curve $f(0, T)$, which is needed as initial value for the stochastic differential equation (1),

¹In general, the drift depends on the forward rates of all maturities between t and T .

the number N of factors, and the structure of the instantaneous forward rates, defined by the diffusion coefficients $\sigma_i(t, T, f)$.

In this paper, we focus on 5 different one-factor models² and 2 different two-factor models. The common feature of these models is that they are all parsimoniously parametrized. However the dependence of the diffusion coefficients on the length of the time interval $T - t$ and on the forward rate f differs in the models. The parametric specification of the volatility functions is shown in Table 2.

Table 2
HJM Models under Consideration

One-Factor Models		
(1)	$\sigma(t, T, f) \equiv \sigma$	Absolute I
(2)	$\sigma(t, T, f) = \sigma\sqrt{f}$	Square Root
(3)	$\sigma(t, T, f) = \sigma f$	Proportional I
(4)	$\sigma(t, T, f) = \sigma_0 + \sigma_1(T - t)$	Linear Absolute
(5)	$\sigma(t, T, f) = (\sigma_0 + \sigma_1(T - t))f$	Linear Proportional
Two-Factor Models		
(6)	$\sigma_1(t, T, f) = \sigma_1(t, T)$ $\sigma_2(t, T, f) = \sigma_2(t, T)$	Absolute II
(7)	$\sigma_1(t, T, f) = \sigma_1(t, T)f$ $\sigma_2(t, T, f) = \sigma_2(t, T)f$	Proportional II

3.2 One-Factor Inversion Model

Essentially, all one-factor models for valuing interest rate derivatives use the instantaneous short-term interest rate as the exogenous factor. We assume

²The one-factor models were also considered by Amin and Morton (1994).

that the dynamics of the short-term interest rate $r(t)$ exhibit mean-reversion and that the diffusion coefficient may depend on the level of the short rate. In a continuous-time setting the process is characterized by the following stochastic differential equation

$$dr = \kappa(\gamma - r)dt + \sigma r^\epsilon dz(t), \quad (3)$$

where $z(t)$ is a Brownian motion and t represents the calendar time. κ , γ , σ , and ϵ are positive constants.

If the market price of risk λ is given, this (classical) interest rate model is fully specified and the current term structure of interest rates results as a solution to a well-known parabolic partial differential equation subject to an initial or terminal condition. The basic idea behind the procedure of Hull and White (1990) is to allow for time-dependent parameters in the risk neutralized process $\tilde{r}(t)$ corresponding to (3), and to determine these parameters in such a way that the endogenous term structure of interest rates coincides with the observed one. Excluding special cases, e.g. the Vasicek model, this calibration has to be carried out numerically. Hull and White (1993) suggest a numerical method which is based on a trinomial tree. We use the inverted implicit finite difference method introduced by Uhrig and Walter (1993). Generally, one is free to choose one of the parameters as a time-dependent function. However, by a number of economic and technical reasons we are convinced that selecting the market price of risk as a time-dependent function is the best choice.³

The model can be fitted additionally to an exogenously given volatility structure, if besides the market price of risk one of the parameters in (3) is assumed to depend on time.⁴ This procedure, however, has an important drawback. The calibration guarantees that the current volatility structure

³For a discussion of the appropriate parameter to fit the model, cf. Uhrig and Walter (1993).

⁴Cf. for example Hull and White (1990, 1993) and Black and Karasinski (1991).

is adequate. However, an analysis of the endogenous volatility structure for future points in time showed, that the time-dependent parameters of the stochastic differential equation (3) result in instable and partially unrealistic future volatilities.⁵ Therefore, the calibration to an exogenously given volatility structure by a second time-dependent parameter is not a feasible procedure.

As the volatilities of interest rates with longer maturities have a pronounced effect on the values of interest rate derivatives on long-term instruments, it is compulsory to integrate their current values into the model. Uhrig and Walter (1996) suggest to choose the (constant) parameter κ , which has a strong influence on the volatility structure, in such a way that the volatility of a long-term rate coincides with the corresponding observed volatility. Obviously, in this procedure only the volatilities of the short rate and a long rate coincide with the observed volatilities. Volatilities of intermediate rates are interpolated endogenously by the model. The advantage of this compromise is that the model results in stable future volatilities.

In order to implement this one-factor inversion model the current discount function, the parameters describing the dynamics of the short-term rate, and the volatility of a long-term rate, all have to be estimated.

3.3 Inversion Model with Long-Term Rate and Spread

Walter (1995) proposes a two-factor inversion model which uses a long-term rate l and the spread s between the long-term rate l and the short-term rate r as stochastic factors. Like Schaefer and Schwartz (1984) he exploits the empirical results presented in Section 2 that the correlation between the two factors is small. Within this model the dynamics of the two state variables

⁵Hull and White (1993) report similar results.

is described by the following stochastic differential equations

$$\begin{aligned} dl &= \sigma_l \sqrt{l} dz_l(t), \\ ds &= \kappa_s(\gamma_s - s)dt + \sigma_s dz_s(t), \end{aligned} \quad (4)$$

where σ_l , κ_s , γ_s , and σ_s are positive constants and $z_l(t)$ and $z_s(t)$ represent independent one-dimensional Brownian motions.

As discussed in Section 2, the long-term rate shows a drift close to zero. The factor l is therefore modelled by a martingale. In order to exclude negative long-term rates a square root representation of the diffusion coefficient is chosen. The spread process is assumed to follow an Ornstein-Uhlenbeck process in line with the observation that the spread process can take on both positive and negative values. In addition this choice ensures that the endogenous term structure of interest rates can be adapted to every observed one. This is not generally true for two-factor models, which use nonnegative state variables.

The assumption of uncorrelated factors provides an important simplification of the valuation problem: In the general case of interrelated factors, the values of zero coupon bonds result from the solution of a partial differential equation of two state variables and one variable of time subject to an initial or terminal condition, which can be solved numerically only. A separation of variables however reduces the problem to two unrelated partial differential equation with only one state variable and one variable of time, each. Due to the special structure of the stochastic processes (4) both partial differential equations can be solved explicitly.

The model is calibrated to the current term structure of interest rates by means of a time-dependent market price of spread risk. Due to the separability of variables and the choice of an Ornstein-Uhlenbeck process for the spread, this problem can be solved analytically. The market price of long-term interest rate risk is used to overcome a problem which is typical for two-factor models where both factors are interest rates, and which is partly

overlooked in the literature: The state variable l is labeled “long rate” but it does not represent a long-term interest rate in its original meaning.⁶ By choosing the mean-reverting parameter κ , and the long-term market price of risk appropriately, the model can be specified in such a way that the long rate l actually has the economical meaning of a long rate in the model.

In order to implement the inversion model with long-term rate and spread, the current term structure of interest rates and the volatility parameters σ_l and σ_s have to be estimated.⁷

3.4 Inversion Model with Stochastic Interest Rate Volatility

In Section 2, some empirical evidence is presented that the short rate exhibits volatility clusters which can be modelled approximately by stochastic volatility. The inversion model with stochastic interest rate volatility used in this study is a generalized version of the model of Longstaff and Schwartz (1992) proposed by Uhrig (1995). Longstaff and Schwartz base their model on assumptions about the stochastic evolution of two abstract independent factors. The dynamics of the two unspecified state variables are driven by the stochastic differential equations

$$\begin{aligned} dx &= \alpha_x(\gamma_x - x)dt + \sigma_x\sqrt{x}dz_1(t) \\ dy &= \alpha_y(\gamma_y - y)dt + \sigma_y\sqrt{y}dz_2(t), \end{aligned} \tag{5}$$

where α_x , γ_x , σ_x , α_y , γ_y , and σ_y are (positive) parameters. While both factors affect the mean of the instantaneous rate of return of the production process, only the second factor has an impact on its instantaneous variance, therefore risk due to the first factor is not priced in this economy.

⁶See also Duffie and Kan (1993), p. 7.

⁷A specification of the parameter γ , within the drift of the spread process is not necessary. Cf. Walter (1995).

Following the general equilibrium framework of Cox, Ingersoll, and Ross (1985) Longstaff and Schwartz derive a fundamental partial differential equation for the value of any derivative security in zero net supply.

Furthermore, the processes of the short-term rate r and its instantaneous variance V are determined endogenously as part of the equilibrium:

$$\begin{aligned} r &= x + y \\ V &= \sigma_x^2 x + \sigma_y^2 y \end{aligned} \quad (6)$$

Using this system of linear equations the authors are able to express the valuation equation in terms of the observable state variables r and V .

In order to achieve consistency with the current term structure, the model is generalized by allowing for a time-dependent risk parameter. Due to the separability of the partial differential equation in the state variables x and y , the adaption of the endogenous to the exogenous term structure of interest rates can be reduced to the adaption problem within the one-factor Cox, Ingersoll, and Ross model. This problem is solved by applying the inverted implicit finite difference method proposed by Uhrig and Walter (1993).

In order to implement the extended Longstaff and Schwartz model both the current term structure of interest rates and the constant parameters of the stochastic differential equation (5) for the state variables x and y have to be estimated.

4 The Data

Three types of interest rate options are traded in Germany: Options on the 10-year BUND-Future and on the 5-year BOBL-Future, interest rate warrants, and OTC-interest rate options of all types. As the market of options on futures is not particularly liquid and as data on OTC-options are not available to us, the empirical study is based on the valuation of interest rate warrants.

Table 3
Statistics of the One-Month Rate (1970 - 1993)

Variable	# Observ.	Mean	Volatility	Minimum	Maximum
One-month rate r_t	1253	6.8287	2.602	2.500	14.7500
One-month rate change $r_{t+1} - r_t$	1252	-0.0031	0.362	-3.0000	2.6250

Three types of data are necessary to implement the models outlined in Section 3:

1. Time series of the short-term and long-term interest rate.
2. Prices and terms of straight bonds issued by the German government (BUND- and BOBL-bonds).
3. Prices and terms of interest rate warrants.

The data is available from the German Financial Data Base Mannheim/Karlsruhe.⁸

The short-term rate is represented by the one-month interbank money rate. The time series of this rate and its weekly changes is shown in Section 2. Some statistics of this rate and its weekly changes are summarized in Table 3. The long-term rate is given by the yield to maturity of a 9-year zero-bond. The time series behaviour of this interest rate has also been presented in Section 2. Summary statistics of this rate and its weekly changes are given in Table 4.⁹ The prices, coupons, and maturities of BUND-

⁸This data base was established under the research program "Empirical Capital Market Research", supported by the German National Science Foundation.

⁹Cf. Walter (1995).

Table 4
Statistics of the 9-Year Interest Rate (1980 - 1993)

Variable	# Observ.	Mean	Volatility	Minimum	Maximum
long rate l_t	726	7.467	1.067	5.243	10.185
long rate change $l_{t+1} - l_t$	7252	-0.002	0.119	-0.464	0.679

and BOBL-bonds are used to estimate the term structure of interest rates and the long-term rate. These bonds define a homogeneous market segment with respect to bankruptcy risk, liquidity, and taxes. In addition, a subsample of these bonds represents the underlyings of the interest rate warrants considered in the study. In the period from 1990 to 1993 the initial maturity of these bonds varied between 5 and 10 years apart from one extreme case of a bond with an initial maturity of 15 years. Typically BUNDs are issued with an initial maturity of 10 years and BOBLs with an initial maturity of 5 years. The first group are termed long-term, the second medium-term bonds. The coupons varied between 5% and 10.75%. Summary statistics of the deviations between the theoretical and observed bond prices are presented in Section 5.2.

German interest rate warrants began trading at the end of 1989. In this study we used all call and put options listed on the "Amtlicher Handel" and the "Geregelter Markt" of the Frankfurt Stock Exchange within the sample period January 1990 through November 1993. During this period, 19 different calls and 14 different puts traded on 13 different German government bonds. 10 of the 13 underlying bonds were BUND-bonds with 10-year maturity, the remaining 3 were BOBL-bonds with an initial maturity of 5 years. Table 5 displays details of the underlying bonds and the options written on each bond. During the sample period, the time-to-maturity of

Table 5
Bond and Interest Rate Warrant Data

Bond		Bond price		Exercise price		calls	Obs.	Puts	Obs.	
		min	max	min	max					
BUND	7%	89/99	86.67	96.50	100	100	1	42	1	42
BUND	7.25%	90/00	87.85	96.93	99	100	2	105	1	49
BUND	7.75%	90/00	91.01	99.11	100	100	1	84	1	84
BUND	8.75%	90/00	98.05	103.15	100	100	1	74	1	74
BUND	9%	90/00	100.27	106.35	100	100	1	78	0	0
BOBL	8.75%	90/95	98.40	101.03	99	101	1	49	1	49
BUND	8.25%	91/01	99.57	105.23	100	100	1	49	1	49
BOBL	7.5%	92/97	102.73	108.19	102	103	2	88	2	88
BOBL	8%	92/97	98.28	107.08	98	99	1	58	1	58
BUND	8%	92/02	98.85	113.77	101	102	2	144	2	144
BUND	7.125%	92/02	101.70	108.23	103	104	2	68	2	68
BUND	8%	92/02	103.34	114.13	100	102	3	189	0	0
BUND	6.5%	93/03	102.93	104.51	103	103	1	9	1	9
Total:						19	1037	14	714	

the bonds ranged from 6.9 to 9.1 years for the BUNDs and from 3.4 to 3.8 years for the second segment. The average time-to-maturity for the options was 0.85 years, the maximum being 2.91 years. With the exception of three European interest rate warrants, the options under consideration were American-type options. Weekly observations were used and the total number of option prices collected amounts to 1751.

5 Implementation of the Models and Valuation Results

In order to value the interest rate warrants within the models described in Section 3 we followed the procedure described below:

5.1 Estimation of the Current Term Structure

For each of the 204 valuation days we estimated the current term structure of interest rates from prices of German government bonds. Each estimate used all currently traded German government bonds with a time to maturity of between 0.5 and 10 years. For each estimation 100 bonds on average were used. To uphold the yield curve at short maturities we included German money market rates with a time to maturity of one day, one, three, and six months.

As discussed in Section 2.1 a compromise between the accuracy in explaining observed bond prices and the smoothness of the term structure of interest rates has to be made. We primarily placed particular emphasis on accuracy. A quadratic programming approach with linear constraints was used as a "minimum error method". The estimation result of this approach was a discrete term structure of interest rates which explained most of the observed coupon bond prices exactly, and resulted in very low estimation errors reported in Section 2.1. The irregularity of the discrete term structure of interest rates resulted in time-dependent market prices of risk which were extreme variable. The estimation of the diffusion coefficients of the instantaneous forward rates using the discrete term structure of interest rates also proved to be infeasible.

Therefore in order to obtain a smooth yield curve the corresponding term structure of interest rates was approximated in a second step through the use of cubic splines.¹⁰ Table 6 shows the deviations between the theoretical bond prices and the market prices of the bonds for the sample period of January 1990 through November 1993.

¹⁰For details to this estimation procedure, cf. Uhrig and Walter (1994).

Table 6
Summary Statistics of the Estimation Quality of Term
Structures of Interest Rates

	1990 - 1993	1990	1991	1992	1993
valuation days	204	52	52	52	48
total number of observations	20308	5392	5310	5052	4554
# bonds per valuation day	99.55	103.69	102.12	97.15	94.88
mean absolute deviation	0.1477	0.1330	0.1556	0.1854	0.1138
standard deviation	0.1658	0.1409	0.1862	0.1900	0.1252
best valuation day	20/08/93	22/06/90	08/02/91	14/02/92	20/08/93
mean absolute deviation	0.0848	0.1025	0.1025	0.1370	0.0848
worst valuation day	30/04/92	21/12/90	27/12/91	30/04/92	08/01/93
mean absolute deviation	0.2446	0.1832	0.2173	0.2446	0.1939
number of bonds with an absolute deviation					
< 0.05	28.18%	30.29%	27.98%	21.97%	32.81%
< 0.10	51.09%	53.17%	51.05%	41.90%	58.88%
< 0.20	77.08%	79.54%	76.78%	67.34%	85.31%
< 0.50	95.56%	97.03%	94.44%	92.70%	98.30%
< 1.00	99.52%	99.83%	99.23%	99.41%	99.62%

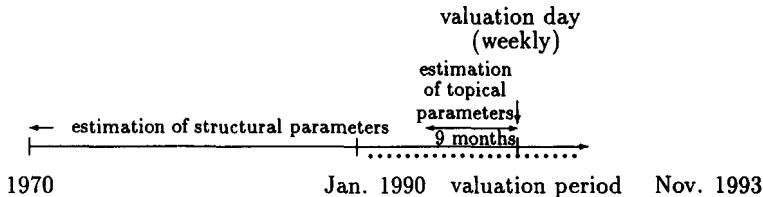
The deviations are in DM per DM 100 nominal value.

5.2 Determination of the Parameters of the Model

In order to determine the parameters of the models we used an historical estimation procedure. We distinguish between two types of parameters, structural and topical parameters. The structural parameters – those parameters that are relevant for the principle structure of the model – were estimated using a long estimation period of at least 20 years. The topical parameters have to reflect the current market information. They were therefore estimated using observations of the previous nine months only. Figure 6 illustrates this procedure.

As the valuation models require the estimation of widely differing parameters, we give a brief description of the estimation procedure for each of the models:

Figure 6



Parameter Estimation for the HJM Models

For the HJM models the volatility parameters are interpreted as topical parameters and were therefore estimated from a historical time series of observations from the previous nine months. The parameters of the one-factor models were obtained directly from the forward rate changes. For the estimation of the parameters for the two-factor models a factor analysis was performed at the beginning in order to obtain the realizations of the two factors as well as the corresponding factor loadings. Depending on the specification of the model, the dynamics of the forward rates was discretized and the volatility parameters were estimated directly from the volatilities of the two factors and the corresponding factor loadings.¹¹

Parameter Estimation for the One-Factor Inversion Model

Within the one-factor inversion model, the long-term mean γ and the elasticity parameter ϵ are interpreted as structural parameters. In order to

¹¹For details, see Weber (1995).

estimate these parameters of the continuous-time process we discretized the model by applying the Euler scheme. We then determined the parameter estimates that provided the best fit to a time series of German money market rates from January 1970 to the current valuation day. Contrary to these structural parameters, the volatility of the short rate σ and the volatility of the long rate which is required for the estimation of the parameter κ , were estimated from the weekly changes of the spot rate and the weekly changes of 9-year rate using observations of the previous 9 months only. All estimates are obtained by means of the maximum likelihood method.¹²

Parameter Estimation for the Inversion Model with Long-Term Rate and Spread

In order to apply the two-factor inversion model with the long-term rate and the spread only the volatilities of these two factors are required. Analogous to the estimation procedure of the volatility parameters within the one-factor inversion model, the parameters were estimated from the weekly changes of the 9-year rate and the weekly changes of the spread based on the observations of the 9 months preceding the valuation day.¹³

Parameter Estimation for the Inversion Model with Stochastic Interest Rate Volatility

Within the inversion model with stochastic interest rate volatility the estimation procedure is slightly more complex as the volatility of the short-term rate is not observable directly. Following Longstaff and Schwartz (1993) we use a two-step approach: In the first step we estimated the volatility of the short-term rate using a GARCH-procedure. The second step consists

¹²For details, see Walter (1995).

¹³For details, see Walter (1995).

of the estimation of the parameters describing the movement of the short-term interest rate and its volatility. In order to estimate these parameters we equated the first two moments of the long-run stationary unconditional distribution of r and V with their historical counterparts. In addition to these four equations, two further conditions were obtained by choosing the volatility parameters σ_x^2 and σ_y^2 as the minimum and the maximum of the ratio $\frac{V(t)}{r(t)}$ respectively. Through the use of these six conditions the six parameters of the model could be calculated simply by solving a nonlinear system of six equations.

The relationship between the short-term interest rate and its volatility according to the GARCH-model was interpreted as a structural link. We therefore estimated the GARCH-model using a time series of German money market rates from January 1970 to the beginning of the valuation period in 1990. With this relationship the volatility of the short-term rate can be estimated given its evolution. In order to estimate the moments of the short-term interest rate we used observations from January 1970 to the current valuation day. In contrast the estimation of the moments of the volatility and the maximum and minimum of the ratio between volatility and interest rate were based on observations of the 9 months preceding the valuation day only. The motivation to choose this asymmetrical treatment is the observation that the short-term rate has a mean-reverting feature with cycles lasting for some years, whereas the volatility oscillates with fairly short cycles. Although this procedure is not justified from a theoretical point of view, we believe that estimating the moments of the volatility using short periods allows us to reflect current market expectation more realistically.¹⁴

5.3 Valuation of Interest Rates Warrants

The last step in the empirical study consists of the valuation of German interest rate warrants and of an analysis of the deviations. For the HJM

¹⁴For details, see Uhrig (1995).

Table 7
Deviation between model and market values – all options

Model	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
Absolute I	0.31	-0.15	0.24	0.49
Square Root	0.31	-0.17	0.24	0.48
Proportional I	0.31	-0.17	0.25	0.48
Linear Absolute	0.29	-0.14	0.22	0.46
Linear Proportional	0.30	-0.17	0.24	0.46
Absolute II	0.36	-0.27	0.31	0.54
Proportional II	0.37	-0.28	0.32	0.54
Short-Term Rate	0.35	0.13	0.37	0.47
Long-Term Rate and Spread	0.30	-0.17	0.21	0.44
Short-Term Rate and Volatility	0.30	-0.09	0.23	0.41

models we discretized the dynamics of the forward rates under the risk-neutral measure by building a binomial-type model with seven time steps. In order to compute the theoretical option values for the inversion models we used a finite difference method. Whereas for the one-factor model a full implicit scheme was applied, the option values for the two-factor inversion models were computed using the alternating direction implicit method.¹⁵ For the American options the value of the option if not exercised is replaced by its intrinsic value if the latter is larger than the first.

A first impression on the performance of the ten models is given by Tables 7 to 9. These tables present some simple summary statistics for the deviations between theoretical values and market prices of the warrants. Table 7 is based on the whole sample of calls and puts. Table 8 refers to calls, Table 9 to puts. Within these tables columns 2 to 4 report the average absolute

¹⁵Cf. McKee and Mitchell (1970) and Press, Flannery, Teukolsky, and Vetterling (1988). In contrast to the valuation of zero coupon bonds within the two-factor models, the valuation of options on coupon bonds makes it necessary to solve a partial differential equation in two state variables, because the terminal condition cannot be separated.

Table 8
Deviation between model and market values – calls

Model	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
Absolute I	0.30	-0.14	0.21	0.44
Square Root	0.30	-0.16	0.22	0.44
Proportional I	0.30	-0.16	0.22	0.45
Linear Absolute	0.30	-0.13	0.21	0.44
Linear Proportional	0.31	-0.16	0.22	0.44
Absolute II	0.37	-0.27	0.26	0.54
Proportional II	0.39	-0.29	0.27	0.56
Short-Term Rate	0.36	0.22	0.31	0.43
Long-Term Rate and Spread	0.30	-0.16	0.17	0.41
Short-Term Rate and Volatility	0.28	-0.03	0.17	0.33

Table 9
Deviation between model and market values – puts

Model	Mean abs. dev. [DM]	Mean dev. [DM]	Mean rel. abs. Dev	St. dev. [DM]
Absolute I	0.33	-0.17	0.28	0.54
Square Root	0.32	-0.18	0.28	0.53
Proportional I	0.32	-0.19	0.28	0.53
Linear Absolute	0.29	-0.16	0.25	0.49
Linear Proportional	0.29	-0.19	0.27	0.49
Absolute II	0.34	-0.26	0.40	0.53
Proportional II	0.34	-0.27	0.41	0.52
Short-Term Rate	0.35	0	0.48	0.50
Long-Term Rate and Spread	0.30	-0.18	0.26	0.47
Short-Term Rate and Volatility	0.34	-0.19	0.31	0.51

pricing errors, the average pricing errors (defined as model value minus market value), and the average relative absolute pricing errors.¹⁶ Column 5 shows the standard deviation of the absolute pricing error.

The mean option price of the sample as a whole was DM 3.13. The average absolute pricing errors ranged from DM 0.29 for the best model, namely the linear absolute one-factor HJM model, to DM 0.37 for the worst case, namely that of the proportional two-factor HJM model. The 3rd column indicates that with the exception of the one-factor inversion model, all models underpriced the options on average. The one-factor inversion model resulted in an average pricing error of DM 0.13. The underpricing of the other models ranged from the best value of DM -0.09 to the worst value of DM -0.28. The average absolute relative pricing errors varied between 21% and 37%.

A comparison of the results in Tables 8 and 9 shows that the absolute relative pricing error was consistently lower for the call values than for the put values. A reason for this might be that the calls' average price of DM 3.51 is higher than the puts' average price of DM 2.56.

The one-factor HJM models showed fairly similar patterns of mispricing. The observation also holds for the two-factor HJM models.

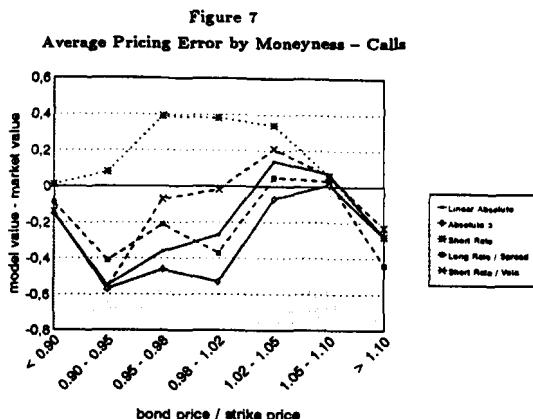
It is surprising that the two-factor models performed consistently worse than the one-factor models.¹⁷

For the inversion models the relation between the one- and two-factor models is as expected. The two-factor models outperform consistently the one-factor model. One reason for the minor quality of the one-factor model

¹⁶To calculate the relative pricing errors we removed all observations where the market price of the option was less than DM 0.10. The total number of observations eliminated was 228, of which 81 were calls and 147 were puts.

¹⁷One explanation for this observation could be a higher sensitivity of the two-factor diffusion coefficient to in-the-sample variations, which leads to worse out-of-the-sample valuation results.

Figure 7
Average Pricing Error by Moneyness – Calls



could be the high sensitivity of the option values to the implicitly estimated mean-reversion parameter κ .

Figures 7 to 10 show the pricing errors for calls and for puts by the moneyness of the options. As one would expect in general increasing moneyness leads to decreasing relative absolute pricing errors. In many cases even the absolute pricing error decreased when moneyness increased. All but the one-factor inversion model underpriced out-of-the-money and at-the-money calls and all but the two-factor HJM model overpriced in-the-money calls. However, the deep-in-the-money calls were underpriced by all the models. With the exception of the deep-in-the money puts, the puts were also underpriced in general.

The average pricing errors and the average absolute pricing errors by time to maturity are visually displayed in Figures 11 and 12. All the models with the exception of the one-factor inversion model overpriced options with only a few months to expiration, whereas options with a longer time to maturity

Figure 8
Average Pricing Error by Moneyness – Puts

Figure 8
Average Pricing Error by Moneyness – Puts

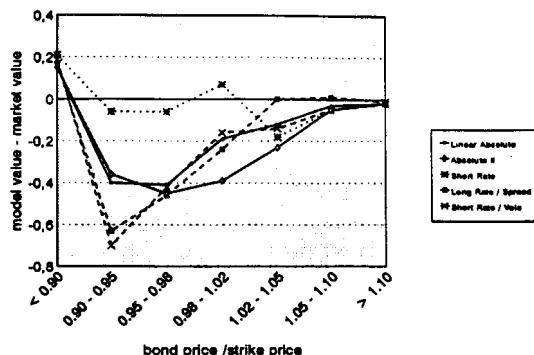


Figure 9
Average Absolute Pricing Error by Moneyness – Calls

Figure 9
Average Absolute Pricing Error by Moneyness – Calls

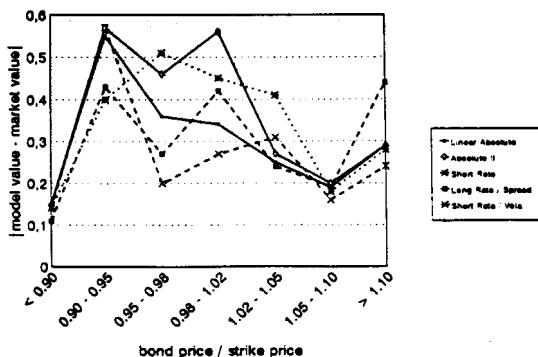


Figure 10
Average Absolute Pricing Error by Moneyness – Puts

Figure 10
Average Absolute Pricing Error by Moneyness – Puts

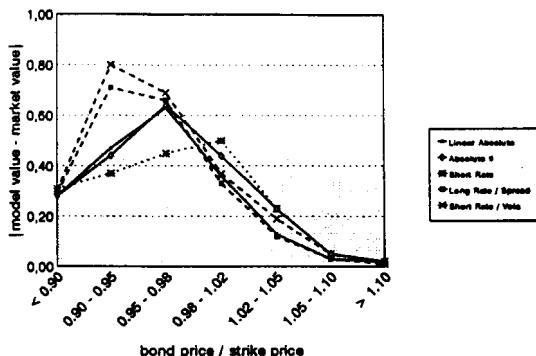


Figure 11
Average Pricing Error by Time to Maturity

Figure 11
Average Pricing Error by Time to Maturity

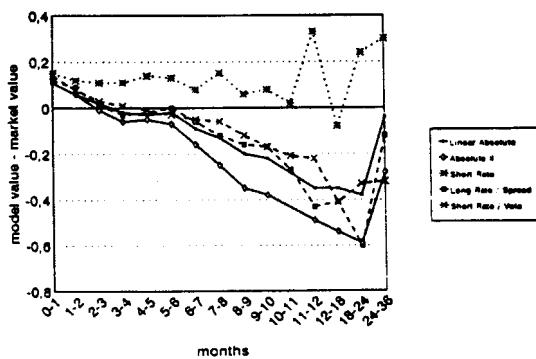
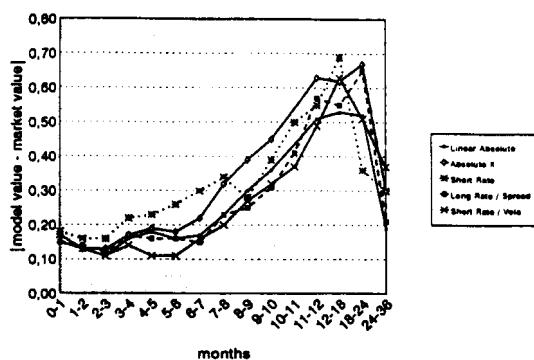


Figure 12
Average Absolute Pricing Error by Time to Maturity

Figure 12
Average Absolute Pricing Error by Time to Maturity



were underpriced. In general there is a tendency for the mean absolute pricing error and the mean absolute relative pricing error to increase with time to maturity.

Figure 13 shows the absolute pricing errors for different subperiods of the valuation period. The study of the pricing error over time offers some interesting insights. Whereas the absolute pricing errors were very high in 1990, the error reduced significantly over time for all the models. From 1990 to 1993 the absolute pricing errors decreased by at least 50% and for some of the models by more than 70%. The evolution of the average pricing error in Figure 14 is also interesting: In 1990 all but the one-factor inversion model underpriced the options significantly. However underpricing reduced over time and in 1993 the pricing error was on average close to zero. With these results in mind there are good reasons for the supposition that the issuers of the warrants quoted (too) high prices in the beginning of the sample period, which also coincides with the beginning of trading in interest rate warrants,

Figure 13
Average Absolute Pricing Error over Time

Figure 13
Average Absolute Pricing Error over Time

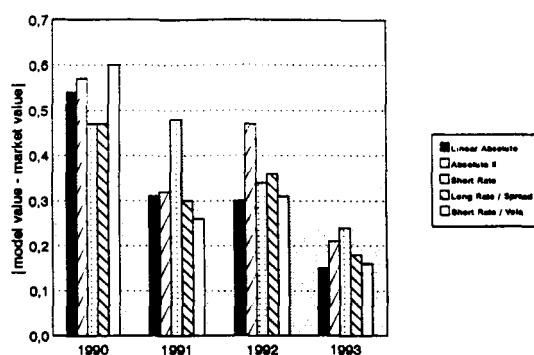
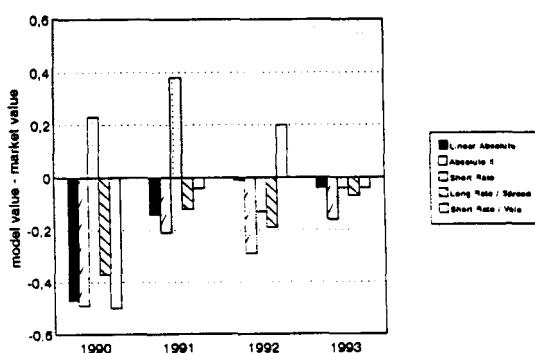


Figure 14
Average Pricing Error over Time

Figure 14
Average Pricing Error over Time



and that the market was willing to accept such high prices at first.

6 Assessment of the Models

As pointed out in the Introduction is is not sufficient to assess the valuation models only by their empirical quality as discussed in Section 5.3. In addition, differences in estimating the input data and in numerically valuing the warrants should be reflected in the overall assessment. Therefore the following four criteria are applied in order to come up with a final recommendation.

1. Estimation Problems
2. Fitting Problems
3. Valuation Problems
4. Empirical Quality

6.1 Estimation Problem

The different valuation approaches exhibit large differences with regard to the estimation problems of the parameters. The historical estimation of parameters using forward rate changes for the HJM models is very sensitive to small changes when estimating forward rate curves. Furthermore, reasonable estimates for the volatility parameters resulted only if the forward rate curves were smoothed by splines with a small number of nodes. For the two-factor models parameter estimation is in addition quite costly as a factor analysis must firstly be carried out.

With the exception of the mean reverting parameter κ the parameters of the one-factor inversion model were obtained by a standard maximum likelihood estimation. The maximum likelihood estimate of κ proved to be

In general we found that the current term structure of interest rates has to be sufficiently smooth in order to avoid strong variations of the time-dependent functions. A fitting function with large variance sometimes results in options prices with unexplained collapses, this is particularly the case for the one-factor inversion model.

6.3 Valuation Problem

In contrast to the inversion models the HJM models result in non-markovian models for some specifications of the volatility functions. In these cases the valuation of the interest rate warrants using non-combining trees is quite costly. Naturally the expenditure for valuing interest rate warrants within one-factor models is lower than within two-factor models. Within the two-factor inversion models the valuation problem is slightly more difficult for the two-factor model with long-rate and spread than for the two-factor model with stochastic volatility. This is due to the fact that the choice of an Ornstein-Uhlenbeck process results in difficulties concerning the treatment of the boundaries within the implicit scheme.

6.4 Empirical Quality

As far as the empirical quality is concerned the valuation results make also some contribution to the differentiation of the models. The one-factor HJM models show fairly similar patterns of mispricing. However, if one had to choose one of the one-factor HJM models, the model with the linear absolute volatility shows the best performance. It is rather surprising that compared to the one-factor HJM models the two-factor HJM models do not perform as well. This may be due to the fact that we did not succeed in estimating the realizations of the two factors in the correct manner. For the inversion models the reverse situation is found. The one-factor inversion model often

Table 10
Ranking of the Models

Model	estimation	fitting	valuation	empirical quality
1-Factor HJM- Models	4	1	4	3
2-Factor HJM- Models	5	1	4	5
Short-Term Rate	2	5	1	4
Long-Term Rate and Spread	1	3	3	1
Short-Term Rate and Volatility	3	4	2	2

performs very poorly compared to the results of the two two-factor inversion models.

6.5 Conclusion

To conclude we present a ranking of the models according to the four aforementioned different criteria. We have treated all the one-factor HJM models as one approach and the two two-factor HJM models as another, therefore their ranking should be interpreted as a sort of average assessment. Our experiences have shown that there are some difficulties concerning the practical implementation of the HJM approaches as well as the one-factor inversion model. Because of the easy applicability of the two-factor inversion model with long rate and spread among other things this model comes out on top.

7 Summary

This paper presents the first comprehensive comparative study of alternative models for valuing interest rate options. The valuation models were assessed by different criteria that are of considerable importance for the

practical use of the models. Not only the empirical quality but also the problems in implementing the different approaches make contribution to the differentiation of the models. Finally we give a clear recommendation which model should be used.

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