SOLVENCY MEASURE FOR PENSION LIABILITIES: time, inflation and longevity

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PRM = PENSION + ERM
The Solvency Issue

INSURANCE → SOLVENCY 2
BANKS → BASEL 2 / 3
PENSION FUNDS → ???

**IAS Norms** : Best estimate approach ; not risk based!
Outline

• 1. Pension and solvency
• 2. Market risk Model
• 3. Risk Measure Approach
• 4. Probability of ruin Approach
• 5. Introduction of the Longevity Risk
• 6. Extension to DB schemes
1. Pension and solvency
Risk mapping of pension liabilities:

- **MARKET RISKS:**
  - discount in the Actuarial Liabilities
  - return on the Assets

- **ECONOMICAL RISKS:**
  - inflation and salary increase

- **DEMOGRAPHIC RISKS:**
  - longevity risk
  - withdrawal risk

+ Long term aspect !!!
**DB versus DC Pension schemes:**

In **DC Schemes** (*Defined Contribution Pension Schemes where the only liability of the sponsor is to pay a well defined level of contribution*), risks are essentially transferred to the affiliates.

Market risk can be present in case of guarantee of a minimum rate of return on the contributions.

In **DB Schemes** (*Defined Benefit Pension Schemes where the sponsor promises a well defined level of benefit at retirement*), risks are clearly the problem of the sponsor!!
We consider a **pension investment product** (DC plan) with a fixed minimum guaranteed return given at retirement.

**Notations:**

- P=1: initial pension account at time t=0
- N: maturity of the pension account
- i: fixed guaranteed return
- r: risk free rate
- L(N): liability at retirement age
On the asset side we assume the insurer invests the initial contribution in a risky asset.

\[ A(t) = \text{asset at time } t \ (\text{stochastic evolution}) \]

\[ A(N) \text{ is a random variable ...} \]
3 fundamental issues in DC / Funding for the affiliates:

1°) the optimal investment strategy of the contributions: risk / return

2°) An eventual minimum guaranteed return

3°) the payment of the benefits: lump sum or annuity
**Problem 1**: Solvency at maturity:

Ability of the fund to fulfill the guarantee only *at maturity* and *without any additional equity*.

**Probability of no default at maturity**: 

Assets must be enough only at maturity to match the liability:

\[ P(A(N) \geq L(N)) \]
**Problem 2**: Risk measure at maturity:

Ability of the fund to fulfill the pension guarantee only
at maturity *with eventual additional equity*

**EXAMPLE**: Static value at risk:

Initial amount $C$ to put in the fund in order to guarantee
with a high probability the liability at maturity:

$$P(A(N) + C e^{rN} \geq L(N)) \geq \alpha \quad (99\%)$$
**Problem 3**: Ruin theory approach:

Ability of the fund to fulfill the guarantee *at any time* with eventual additional equity (provision approach)

**EXAMPLE**: Probability of ruin:

Reserve process:

\[ R(s) = A(s) + C \cdot e^{rs} - L(s) \]

for: \( 0 \leq s \leq N \)

with: \( A(s) = \) market value of Assets at time \( s \)
\( C = \) initial solvency Capital at time \( 0 \)
\( L(s) = \) value of Liabilities at time \( s \)
Initial own funds $C$ to put in the pension fund in order to guarantee with a high probability the solvency at any time before maturity:

$$
\phi(C) = P(R(s) \geq 0, \forall s \in [0, N])
$$

The initial capital $C$ is then chosen such that:

$$
\phi(C) \geq \alpha
$$
2. Market Risk Model
Assumptions:

Existing amount at time $t=0$
$S(0)=1$

$x=30$  $x=50$  $x=65$

time $t$

$S(t, N)$ = value of the asset at time $t$ for a maturity $N$

t=N
Maturity (retirement age)
Model:

Risky asset:

\[ dX(t) = \delta X(t) dt + \sigma X(t) dw(t) \]

Riskless asset:

\[ dB(t) = r B(t) dt \]

Mixing:

\[ \alpha(t, N) = \text{part invested at time } t \text{ in the risky asset for a maturity } N \text{ (measurable process)} \]
Stochastic differential equation of the mixed asset:

\[ dS(t, N) = (\alpha(t, N) \cdot \delta + (1 - \alpha(t, N)) \cdot r) \cdot S(t, N) \, dt + \alpha(t, N) \cdot \sigma \cdot S(t, N) \, dw(t) \]

The solution of this equation is given by:

\[ S(t) = S(0) \cdot \exp((r \cdot t + (\delta - r) \int_0^t \alpha(s) \, ds - \sigma^2 / 2 \int_0^t \alpha^2(s) \, ds + \sigma \int_0^t \alpha(s) \, dw(s))) \]

Investment strategy = how to choose the process \( \alpha(t) \) ?
Mixed strategy

We consider 3 deterministic investment strategies mixing:

- a risky asset (stocks, options,…)
- a riskless asset (bonds)

Strategy 1: **Constant proportion portfolio**
Strategy 2: **Linear decreasing strategy**
Strategy 3: **Lifecycle strategy**
**Constant proportion portfolio**

Fixed constant proportion in the risky asset during all the horizon

(MERTON: *optimal strategy using utility function arguments*)

1 parameter: - constant proportion
Linear decreasing strategy

Linear decreasing proportion in the risky asset during all the horizon

$\alpha(t)$

1 parameter : - initial proportion
Lifecycle strategy

Fixed constant proportion in the risky asset during a first period (accumulation phase) and then linearly decreasing until retirement (consolidation phase).

2 parameters: - initial proportion
- consolidation period M
**Constant proportion portfolio**

\[ \alpha(t) = \gamma \quad \forall 0 \leq t \leq N \]

For instance: \( \gamma = 50\% \)

**Solution:**

\[ S(t, N) = S(0).\exp((r + \gamma(\delta - r))t - \gamma^2\sigma^2t / 2 + \gamma\sigma.w(t)) \]

*Independent of the maturity for a fixed time t!*

**Mean:**

\[ ES(t, N) = S(0).\exp((r + \gamma(\delta - r))t) \]
Constant proportion portfolio

Amount at retirement age (maturity $t=N$):

\[ S(N, N) = S(0) \cdot \exp\{N(m(N), V^2(N))\} \]

with:

\[ m(N) = r \cdot N + \gamma(\delta - r) \cdot N - \sigma^2 \gamma^2 N / 2 \]
\[ V^2(N) = \sigma^2 \gamma^2 N \]

Mean:

\[ ES(N, N) = S(0) \cdot \exp(r \cdot N + \gamma(\delta - r)N) \]
Linear decreasing strategy

\[ \alpha(t) = \beta \cdot (1 - \frac{t}{N}) \quad \forall 0 \leq t \leq N \]

For instance: \( N = 20 \)
\( \beta = 70\% \)

\[ \begin{align*}
\alpha(0) &= 70\% \\
\alpha(5) &= 52.5\% \\
\alpha(10) &= 35\% \\
\alpha(15) &= 17.5\% \\
\alpha(20) &= 0\%
\end{align*} \]
Linear decreasing strategy

Amount at retirement age (t=N):

\[ S(N, N) = S(0) \cdot \exp\{N(m(N), V^2(N))\} \]

with:

\[ m(N) = r \cdot N + \beta \cdot (\delta - r) \cdot N / 2 - \sigma^2 \beta^2 N / 6 \]
\[ V^2(N) = \sigma^2 \beta^2 N / 3 \]

Mean:

\[ ES(N, N) = S(0) \cdot \exp(r \cdot N + \beta \cdot (\delta - r) N / 2) \]
Lifecycle strategy

\[ \alpha(t) = \alpha \quad \text{for} \quad 0 \leq t \leq N - M \]

\[ \frac{\alpha}{M} (N - t) \quad \text{for} \quad N - M \leq t \leq N \]

For instance:

\[
\begin{align*}
N &= 30 \\
M &= 15 \\
\alpha &= 70\%
\end{align*}
\]

\[
\begin{align*}
\alpha(0) &= 70\% \\
\alpha(5) &= 70\% \\
\alpha(10) &= 70\% \\
\alpha(15) &= 70\% \\
\alpha(20) &= 47\% \\
\alpha(25) &= 23\% \\
\alpha(30) &= 0\%
\end{align*}
\]
Lifecycle strategy

Amount at retirement age \( t=N \) : first case: \( N > M \)
(accumulation phase)

\[
S(N, N) = S(0).\exp\{N(m(N), V^2(N))\}
\]

with:

\[
m(N) = r \cdot N + \alpha \cdot (\delta - r) \cdot (N - M/2) - \sigma^2 \alpha^2 (N - 2M/3)/2
\]
\[
V^2(N) = \sigma^2 \alpha^2 (N - 2M/3)
\]

Mean:

\[
ES(N, N) = S(0).\exp(r \cdot N + \alpha \cdot (\delta - r) (N - M/2))
\]
Lifecycle strategy

Amount at retirement age (t=N): second case: \( N < M \)

(consolidation phase)

\[
S(N, N) = S(0) \cdot \exp\{N(m(N), V^2(N))\}
\]

with:

\[
m(N) = r \cdot N + \alpha(\delta - r) \cdot N^2 / 2M - \sigma^2 \alpha^2 N^3 / 6M^2
\]

\[
V^2(N) = \sigma^2 \alpha^2 N^3 / 3M^2
\]

Mean:

\[
ES(N, N) = S(0) \cdot \exp(r \cdot N + \alpha(\delta - r)N^2 / 2M)
\]
3. Risk Measure Approach
Maturity guarantee

Comparison of these 3 strategies:

Matching between the final asset at maturity $S(N,N)$ and the minimum maturity guarantee $L(N)$

\[
\begin{align*}
\text{Asset:} & \quad S(N,N) \\
\downarrow & \\
\text{Liability:} & \quad L(N) = e^{r_G \cdot N}
\end{align*}
\]

($r_G =$ guaranteed rate at maturity)
Probability of default

In the 3 considered strategies, the asset at maturity can be written as:

\[ S(N, N) = S(0) e^{N(m(N), V^2(N))} \]

Then the probability of default at maturity becomes:

\[ \Psi(N) = P(S(N, N) < L(N)) \]
\[ = P(e^{N(m(N), V^2(N))} < e^{r_G N}) \]
\[ = \Phi\left( \frac{r_G N - m(N))}{V(N)} \right) = \text{NORMSDIST}\left( \frac{r_G N - m(N))}{V(N)} \right) \]
Probability of default - constant

Strategy 1: constant proportion portfolio:

\[
\Psi_1(N) = \Phi(\sqrt{N} \cdot \frac{r_g - (r + \gamma(\delta - r) - \gamma^2 \sigma^2 / 2)}{\gamma \sigma})
\]

\[
= \Phi(\sqrt{N} \cdot a(\gamma))
\]

The number \( a(\gamma) \) is in general negative.
So this probability decreases (deeply!) with the maturity \( N \).
Probability of default - constant

**EXAMPLE:**

\[ \alpha(t) = 70\% \]
\[ \delta = 7\% \quad \sigma = 15\% \]
\[ r = 3\% \quad r_G = 1\% \]

![Graph showing constant proportion portfolio probability](attachment:image.png)
Strategy 2 : linear decreasing strategy :

\[
\Psi_2(N) = \Phi(\sqrt{3N} \cdot \frac{r_G - (r + \beta.(\delta - r)/2 - \beta^2 \sigma^2 / 6)}{\beta \sigma}) \\
= \Phi(\sqrt{N} \cdot b(\beta))
\]

The number \( b(\beta) \) is in general negative .
So this probability decreases also with the maturity \( N \).
Probability of default - linear

EXAMPLE:

\[ \alpha(t) = 70\% \]
\[ \delta = 7\% \quad \sigma = 15\% \]
\[ r = 3\% \quad r_G = 1\% \]
Probability of default - lifecycle

Strategy 3 : life cycle strategy :

\[ \Psi_3(N) = \Phi\left( M\sqrt{3} \left( \frac{r_G N - (rN + \alpha(\delta - r)N^2 / 2M - \sigma^2 \alpha^2 N^3 / 6M^2)}{\alpha\sigma\sqrt{N^3}} \right) \right) \]

\[ \Psi_3(N) = \Phi\left( \frac{r_G N - (rN + \alpha(\delta - r)(N - M / 2) - \sigma^2 \alpha^2 (N - 2M / 3) / 2)}{\alpha\sigma\sqrt{N - 2M / 3}} \right) \]
Probability of default – life cycle

EXAMPLE:

\[ \alpha(t) = 70\% \]
\[ \delta = 7\% \quad \sigma = 15\% \]
\[ r = 3\% \quad r_G = 1\% \quad M = 15 \]

Life cycle probabilities
Constant proportion portfolio - probability
(for different levels of guarantee)

![Graph showing probability of default for different levels of guarantee]
Value at risk - constant

We define the solvency level \( C(N) \) as the initial amount of equity invested in the riskless asset such that at maturity:

\[
P(S(N, N) + C(N)e^{rN} < L(N)) = \varepsilon_N
\]

where:

\[
S(N, N) = e^{N(m(N), V^2(N))}
\]

\[
L(N) = e^{rG_N}
\]

\[
\varepsilon_N = \text{safety level (for instance: 0.5%)}
\]
Value at risk-constant

The solvency capital is then given explicitly by:

\[
C(N) = e^{(r_G-r)N} - e^{-rN} \cdot e^{m(N)+z\epsilon_N \cdot V(N)}
\]

( where : \( P(N(0,1) < z\epsilon) = \epsilon \) )

For instance for the constant proportion portfolio:

\[
C(N) = e^{(r_G-r)N} - e^{-rN} \cdot e^{rN+\gamma(\delta-r)N-\gamma^2\sigma^2N/2+z\epsilon_N \cdot \gamma \sigma \sqrt{N}}
\]
\[ 1 - \varepsilon_N = (0.995)^N \]

\[ r_G = 1\% \quad r = 3\% \]
\[ \delta = 7\% \quad \gamma = 70\% \]
\[ \sigma = 15\% \]
Constant proportion portfolio - solvency
(for different levels of guarantee)
4. Probability of Ruin Approach
4.1. Probability of ruin without capital

We try to compute the probability of non ruin without any extra capital:

\[ P(N) = P(A(t) \geq L(t), \forall t \in [0, N]) \]

With a pure equity strategy first:

\[ A(t) = \exp(\delta t + \sigma w(t) - \sigma^2 t / 2) \]
\[ L(t) = e^{r_G t} \]
This probability can be written using a minimum condition on a geometric Brownian motion:

\[
P(A(t) - L(t) \geq 0, \forall t \in [0, N])
\]

\[
= P( e^{(\delta - \sigma^2/2)t + \sigma w(t)} \geq e^{r_G t}, \forall t \in [0, N])
\]

\[
= P( \min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} \geq 1)
\]

\[
= 1 - P( \min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} < 1)
\]

Distribution function of the minimum of a geometric Brownian motion
Theorem:

**Law of the minimum of a geometric Brownian motion.**

Let:

\[ S(t) = e^{(\mu - \sigma^2/2)t + \sigma \cdot w(t)} \]

\[ z = \min_{0 \leq s \leq t} \{ S(s) \} \]

\[ 0 < L \leq S(0) \]

Then:

\[ \Pr(z \leq L) = \Phi(d_1) + L^{\frac{2\mu}{\sigma^2} - 1} \cdot \Phi(d_2) \]

with:

\[ d_{1,2} = \frac{\ln L + (\mu - \sigma^2/2)t}{\sigma \sqrt{t}} \]
Computation of the probability of non ruin:

\[ P(N) = P(A(t) - L(t) \geq 0, \forall t \in [0, N]) \]
\[ = 1 - P(\min_{0 \leq s \leq N} e^{(\delta - r_G - \sigma^2/2)s + \sigma w(s)} < 1) \]
\[ = 1 - (\Phi\left(\frac{- (\delta - r_G - \sigma^2/2)N}{\sigma \sqrt{N}}\right) + \Phi\left(\frac{(\delta - r_G - \sigma^2/2)N}{\sigma \sqrt{N}}\right) \]
\[ = 0 \]

Under these assumptions the ruin is certain!!!
Variant 1: partial solvency requirement before maturity (surrender charges in life insurance)

\[
P(A(t) \geq e^{r_g t}; \forall t \in [0, N])
\]

Example:

\[
\rho(s) = e^{-\lambda(N-s)} \\
(\lambda > 0)
\]
The solvency requirement becomes then:

\[
P(N, \lambda) = P(A(t) \geq e^{-\lambda N} \cdot e^{(r_G + \lambda)t}; \forall t \in [0, N])
\]

Introducing the minimum:

\[
P(N, \lambda) = P\left( \min_{0 \leq s \leq N} e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)} \geq e^{-\lambda N} \right)
\]

\[
= 1 - P\left( \min_{0 \leq s \leq N} e^{(\delta - r_G - \lambda - \sigma^2/2)s + \sigma w(s)} < e^{-\lambda N} \right)
\]

\[
L < 1
\]
The probability of non-ruin becomes now:

\[
P(N, \lambda) = P(A(t) \geq e^{-\lambda(N-t)}L(t)), \quad \forall t \in [0, N]
\]

\[
= 1 - \{ \Phi\left( \frac{-\lambda N - (\delta - r_G - \lambda - \sigma^2 / 2)N}{\sigma \sqrt{N}} \right) \\
+ \left(e^{-\lambda N} \right)^{2(\delta - r_G - \lambda)} \cdot \Phi\left( \frac{-\lambda N + (\delta - r_G - \lambda - \sigma^2 / 2)N}{\sigma \sqrt{N}} \right) \}
\]
The probability of ruin can be written:

$$\Psi(N, \lambda) = 1 - P(N, \lambda)$$

$$= \Phi\left(-\frac{(\delta - r_G - \sigma^2 / 2)N}{\sigma \sqrt{N}}\right)$$

$$+ \left(e^{-\lambda N}\right)^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1} \Phi\left(\frac{(\delta - r_G - 2\lambda - \sigma^2 / 2)N}{\sigma \sqrt{N}}\right)$$

$$= \Phi(a\sqrt{N}) + \left(e^{-\lambda N}\right)^{\frac{2(\delta - r_G - \lambda)}{\sigma^2} - 1} \Phi\left((-a - 2\lambda / \sigma)\sqrt{N}\right)$$

- Probability of default at maturity
- Price of non ruin before maturity
**Constant portfolio allocation:**

We can also analyze the influence of the asset allocation on this last probability of non ruin:

\[
\delta \rightarrow \gamma \delta + (1 - \gamma)r \\
\sigma \rightarrow \gamma \sigma
\]

\[
P(N, \lambda, \gamma) = P(A(t) \geq e^{-\lambda(N-t)}L(t), \ \forall t \in [0, N])
\]

\[
= 1 - \{ \Phi\left( \frac{-((\gamma \delta + (1 - \gamma)r - r_g) - \gamma^2 \sigma^2 / 2)N}{\gamma \sigma \sqrt{N}} \right) \\
+ \left(e^{-\lambda N} \right) \frac{2((\gamma \delta + (1 - \gamma)r - r_g - \lambda)}{\gamma^2 \sigma^2}^{-1} \Phi\left( \frac{((\gamma \delta + (1 - \gamma)r - r_g) - 2\lambda - \gamma^2 \sigma^2 / 2)N}{\gamma \sigma \sqrt{N}} \right) \}
\]
**Variant 2 : Fair Valuation approach**

Another possibility for the liability function is to introduce the fair value of the final liability (present value at the risk free rate):

\[
L(N) = e^{r_G N} \\
L(t) = e^{-r(N-t)}.L(N) = e^{-r(N-t)}.e^{r_G N} \\
= e^{-(r-r_G)(N-t)}.e^{r_G t} \\
= e^{-\lambda^*(N-t)}.e^{r_G t}
\]

This is a particular case of variant 1 with exponential penalty parameter given by:

\[
\lambda^* = r - r_G
\]

( spread of rates )
This coefficient is positive if the risk free rate is strictly higher than the guaranteed rate.

Then the probability of ruin becomes for a constant strategy:

\[
\Psi(N, \gamma) = \Phi(a(\gamma)\sqrt{N})
\]

\[
+ \left( e^{-(r-r_G)N} \right)^{\frac{2\gamma(\delta-r)}{\gamma^2\sigma^2}-1} \Phi\left(\frac{-a(\gamma) - 2(r - r_G)}{\gamma\sigma}\sqrt{N}\right)
\]

If the two rates are close, the coefficient \( \lambda \) is nearly zero and this probability is nearly equal to one (for every time horizon).
Probability of ruin – constant
\textit{(fair valuation approach)}

\textbf{EXAMPLE:}

\[ \alpha(t) = 70\% \quad \lambda = 2\% \]
\[ \delta = 7\% \quad \sigma = 15\% \]
\[ r = 3\% \quad r_G = 1\% \]
4.2. Probability of ruin with riskless capital

We consider now the probability of non ruin, starting with a positive solvency capital assumed to be invested in a riskless asset and the contribution invested in the risky asset:

\[ P(N, C) = P(A(t) + C.e^{rt} \geq L(t), \forall t \in [0, N]) \]

Or with a partial solvency requirement:

\[ P(N, \lambda, C) = P(A(t) + C.e^{rt} \geq e^{-\lambda(N-t)}L(t), \forall t \in [0, N]) \]
**Computation of the probability of non ruin:**

In general the problem is not so trivial and can no more be written as the law of the minimum of a geometric Brownian motion ...

\[
P(N, \lambda, C) = P(A(t) + C.e^{rt} \geq e^{-\lambda(N-t)}L(t), \ \forall t \in [0, N])
\]

\[
= P(e^{(\delta-\sigma^2/2)t+\sigma w(t)} \geq e^{-\lambda N}e^{(r_G+\lambda)t} - C.e^{rt}, \ \forall t \in [0, N])
\]

We can obtain an explicit form in the fair value case:

\[
\lambda^* = r - r_G
\]

Then we can write:

\[
P(N, \lambda^*, C) = P(e^{(\delta-r_G-\lambda^*-\sigma^2/2)t+\sigma w(t)} \geq e^{-\lambda^* N} - C, \ \forall t \in [0, N])
\]
Using the previous results we obtain for the probability of ruin:

\[
\Psi(N, \lambda^*, C) = 1 - P\left(N, \lambda^*, C\right) = 1 - P\left(A(t) \geq e^{-\lambda^*(N-t)} \cdot L(t), \forall t \in [0, N]\right)
\]

\[
= \Phi\left(\frac{\ln\left(e^{-\left(r-r_G\right)N} - C\right) - (\delta - r - \sigma^2 / 2)N}{\sigma\sqrt{N}}\right) \\
+ \left(e^{-\left(r-r_G\right)N} - C\right)^{\frac{2(\delta-r)}{\sigma^2}} \cdot \Phi\left(\frac{\ln\left(e^{-\left(r-r_G\right)N} - C\right) + (\delta - r - \sigma^2 / 2)N}{\sigma\sqrt{N}}\right)
\]

With: \(\Psi(N,0,0) = 1\)

Without capital and without spread you are sure to be ruined !!!
5. Introduction of the Longevity Risk
Introduction of a longevity condition:

The pension liability is paid at retirement age only in case of survival.

The guarantee takes now into account a pre defined life table based on a deterministic mortality intensity:

\[ \mathbb{N} p_x = \text{Probability to survive } N \text{ years between } x \text{ and } x+N \]

\[ = \exp\left(-\int_{0}^{N} \mu_{x+s} \, ds \right) \]
The minimum pension benefit to pay in case of survival at retirement age is now given by:

\[
B(N) = \frac{e^{r_G N}}{N P_x} = e^0 \int_0^N (r_G + \mu_{x+s}) ds
\]

For this “a priori” life table we will use a classical GOMPERTZ law:

\[
\mu_{x+s} = \mu_x e^{\beta s}
\]
On the other hand we assume that the real death intensity will follow an Ornstein-Uhlenbeck stochastic process given by:

$$d\lambda_x(t) = \alpha \lambda_x(t) \, dt + \rho \, dw_x(t)$$

The solution of this classical SDE is given by:

$$\lambda_x(t) = \lambda_x e^{\alpha t} + \rho \int_0^t e^{\alpha (t-s)} \, dw_x(s)$$
Assuming as before an initial contribution of 1 € paid at time t=0, the asset–liability structure for a pure equity strategy (or a constant proportion portfolio), is now given by:

\[
A(N) = e^{(\delta - \sigma^2/2)N + \sigma \cdot w(N)} \\
L(N) = B(N) \cdot e^{-\int_0^N \lambda_x(s) \, ds} = e^{r_G N} \cdot e^{\int_0^N (\mu_{x+s} - \lambda_x(s)) \, ds}
\]

Asset and Liability are now both stochastic !!!
Probability of default at maturity
(taking into account market and longevity risks)

\[ P(\mathcal{A}(N) < \mathcal{L}(N)) = P\left(e^{\left(\delta - \frac{\sigma^2}{2}\right)N + \sigma w(N)} < e^{\left(\frac{r_G N + \int_0^N (\mu_{x+s} - \lambda_x(s))ds}{\sqrt{2\pi}\sigma}\right)} \right) \]

\[ = P(\sigma w(N) + \int_0^N \lambda_x(s)ds < (r_G - \delta + \frac{\sigma^2}{2})N + \int_0^N \mu_{x+s}ds) \]

\[ = P(\Theta(N) < f(N)) \]

Where

\[ \Theta(N) = \sigma w(N) + \int_0^N \lambda_x(t)dt \]

\[ = \sigma w(N) + \int_0^N \{\lambda_x e^{at} + \rho \int_0^{t-s} e^{a(t-s)}dw_x(s)\} dt \]}
\( \Theta(N) \) is normally distributed:

\[
\Theta(N) \sim N(\mu(N), \sigma^2(N))
\]

The probability of joined default becomes:

\[
P(A(N)) < L(N)) = \Phi\left(\frac{(r_G - \delta + \sigma^2 / 2)N + \int_0^N \mu_{x+s} ds - \mu(N)}{\sigma(N)}\right)
\]
Value of the 2 moments:

1° mean:

\[ m(N) = E\Theta(N) = \int_0^N \lambda_x e^{at} \, dt = \lambda_x \frac{e^{aN} - 1}{a} \]

\[ \int_0^N \mu_{x+s} ds = \int_0^N \mu_x e^{\beta s} ds = \mu_x \frac{e^{\beta N} - 1}{\beta} \]

So the probability of default becomes:

\[ P = \Phi(\frac{(r_G - \delta + \sigma^2/2)N + (\mu_x \frac{e^{\beta N} - 1}{N} - \lambda_x \frac{e^{aN} - 1}{N})}{\sigma(N)}) \]

Spread of rates

Spread of mortality
Value of the 2 moments:

$2^\text{nd}$ variance:

Assumption: independence between financial and longevity risks

\[
\sigma^2(N) = \text{var } \Theta(N) \\
= \sigma^2 N + \rho^2 \left( \int_0^N \int_0^t e^{a(t-s)} dw_x(s) \right)^2 \\
= \sigma^2 N + \frac{\rho^2}{2a^3} \left( 2aN + e^{2aN} - 4e^{aN} + 3 \right)
\]
6. Extension to DB schemes
Model for a DB Scheme

We consider a Final Salary DB Scheme

At retirement age: lump sum expressed as a multiple of the last salary.

For instance: at 65: Liability side:

\[ C = \frac{N}{40} \cdot k \cdot S(65) \]

where: \( N \) = years of service at retirement age
\( S(65) \) = last salary before retirement
Following the IAS 19 norms, we use as funding method the *Projected Unit Credit Cost*.

In order to compute the contribution ("Normal Cost") we need the following assumptions:

1° *discount rate* (risk free rate): $r$

2° *life table*: we will assume here no mortality before retirement (no longevity risk considered here)

3° *salary scale projection*: stochastic model:

\[ S(t) = S(0).e^{\mu t + \eta z(t) - \eta^2 t/2} \]
\[ z(t) = \text{standard brownian motion} \]
\[ \mu = \text{average salary increase} \]
\[ \eta = \text{volatility on salary} \]

The contribution for one year of service in a best estimate pricing is then (assuming \( S(0) = 1 \)):

\[ NC_0 = \frac{1}{40} k e^{(\mu-r)T} \]

where: \( T = \text{residual time to retirement} \)
Possible loading on contribution (Normal Cost):

We can introduce an eventual loading on contribution:

\[ NC_0 = \frac{1}{40} k e^{(\mu - r)T} (1 + \beta) \]
Asset side:

The contribution is assumed to be invested in a geometric Brownian asset:

\[ A(t) = A(0).e^{\delta t + \sigma w(t) - \sigma^2 t/2} \]

\[ w(t) = \text{standard brownian motion} \]
\[ \text{corr}(w(t), z(t)) = \rho t \]
\[ \delta = \text{mean return of the risky asset} \]
\[ \sigma = \text{volatility} \]
\[ A(0) = NC_0 \]

( correlation between inflation and financial returns !!! )
Asset Liability risk management:

We compare at retirement age for this specific year of service the corresponding Asset and Liability:

**Liability at maturity:**

\[ L(T) = \frac{1}{40^2} k e^{\mu T + \eta z(T) - \eta^2 T/2} \]

**Asset at maturity:**

\[ A(T) = \frac{1}{40^2} k (1 + \beta) e^{(\mu - r)T} e^{\delta T + \sigma w(T) - \sigma^2 T/2} \]

Asset and Liability are again both stochastic!!!
Probability of default at maturity:

$$\Psi(T) = P(A(T) < L(T))$$

$$\Psi(T) = P((1 + \beta)e^{(\mu - r)T} \cdot e^{\delta T} \cdot e^{\sigma w(T) - \sigma^2 T/2} < e^{\mu T} \cdot e^{\eta z(T) - \eta^2 T/2})$$

$$= P(Y(T) < M)$$

With:

$$M = (r - \delta + \sigma^2 / 2 - \eta^2 / 2)T - \ln(1 + \beta)$$

$$Y(T) = \sigma w(T) - \eta z(T)$$

$$= N(0, \bar{\sigma}^2)$$

$$\bar{\sigma}^2 = (\sigma^2 + \eta^2 - 2\rho \sigma \eta)$$
Probability of default at maturity:

So finally:

$$\Psi(T) = \Phi(d(T))$$

With:

$$\Phi = \text{distribution function } N(0,1)$$

$$d(T) = \frac{1}{\sigma} \left( (r - \delta + \sigma^2 / 2 - \eta^2 / 2) \sqrt{T} - \ln(1 + \beta) / \sqrt{T} \right)$$

Rem.: independent of salary increase!
Probability of default: correlation impact

\[ r = 3\% \quad \delta = 7\% \]
\[ \mu = 5\% \quad \sigma = 15\% \]
\[ \eta = 10\% \quad \beta = 5\% \]
Next research

• Introduction of jumps
• Mean reverting inflation
• Generalization to LEVY processes
• Guarantees on periodical contributions
• Adaptation to PAYG schemes (NDC technique) with a demographic stochastic entrance process
References

Thank you !!!

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