Market-Consistent Replication of Insurance Liabilities in a Multiple Risk Economy

20th Int. AFIR Coll., Madrid, June 19-22, 2011
Agenda

- Concept of Valuation Portfolio (VaPo)
- VaPo & Market-Consistent Actuarial Valuation (MCAV) with state-price deflator
- State-Price Deflators
- Black-Scholes-Vasicek (BSV) deflator
- Index Linked Inflation Protected Life Contracts
- Inflation Protected Non-Life Contracts
Concept of Valuation Portfolio (VaPo)

**VaPo construction**

**Idea:** perfect replication of insurance contracts based on financial instruments (Wüthrich et al. (2010), Sections 3 & 5, Sandström(2011), Section 8.3)

**VaPo** = list of financial instruments with a specification of the number of units of each instrument that are needed to replicate the insurance liabilities

**Given**
- random vector \( C = (C_0, C_1, \ldots, C_n) \) of liability cash-flows over time horizon \( n \)
- series of replicating zero-coupon bonds \( Z^{(k)} \) that pay one unit at time \( k = 0, \ldots, n \), with price structure \( P(s,k), 0 \leq s < k \)
- series of replicating risky units \( U^{(\ell)} = U^{(\ell)}(I_1, \ldots, I_m), \ell = 1, \ldots, p \)

(financial instruments derived from \( m \) basic economic risks mapped by risky financial instruments \( I_k, k = 1, \ldots, m \) )
- **Number of units** used to replicate the liability cash-flows $C_{(j)} = (0, ..., 0, C_j, ..., C_n)$ with the series $Z^{(k)}$ respectively $U^{(\ell)} : \lambda^{(j)}_k, \eta^{(j)}_{\ell} , j = 0, ..., n, k = 0, ..., n, \ell = 1, ..., p$

**VaPo**: set of linear combinations in the replicating financial instruments defined by

$$VaPo(C_{(j)}) = \sum_{k=0}^{n} \lambda^{(j)}_k \cdot Z^{(k)} + \sum_{\ell=1}^{p} \eta^{(j)}_{\ell} \cdot U^{(\ell)} , \quad j = 0, ..., n$$

**MCAV at initial time** (market value principle using state-price deflator $\{D_t\}_{0 \leq t \leq n}$)

$$M_0(VaPo(C_{(j)})) = \sum_{k=0}^{n} \lambda^{(j)}_k \cdot M_0(Z^{(k)}) + \sum_{\ell=1}^{p} \eta^{(j)}_{\ell} \cdot M_0(U^{(\ell)})$$

$$= \sum_{k=0}^{n} \lambda^{(j)}_k \cdot P(0, k) + \sum_{\ell=1}^{p} \eta^{(j)}_{\ell} \cdot E_0 \left[ \sum_{r=j}^{n} D_r \overline{U}^{(\ell)} S^{(1)}, ..., S^{(m)} \right] , \quad j = 0, ..., n$$

with $\overline{U}^{(\ell)} = \overline{U}^{(\ell)}(S^{(1)}, ..., S^{(m)}), \ell = 0, ..., p$ the vectors of cash-flows associated to the risky units $U^{(\ell)}$, and $S^{(k)} = \{S^{(k)}_t\}_{0 \leq t \leq n}, k = 1, ..., m$ are the real-world prices of the risky financial instruments $I_k, k = 1, ..., m$
MCAV of current and future accounting years

\[
M_s \left( \text{VaPo}(C_{(j)}) \right) = \sum_{k=0}^{n} \lambda_{k}^{(j)} \cdot M_s (Z^{(k)}) + \sum_{\ell=1}^{p} \eta_{\ell}^{(j)} \cdot M_s (U^{(\ell)})
\]

\[
= \sum_{k=0}^{n} \lambda_{k}^{(j)} \cdot P(s, k) + \sum_{\ell=1}^{p} \eta_{\ell}^{(j)} \cdot D_s^{-1} \cdot E_s \left[ \sum_{r=j}^{n} D_r \bar{U}_{r}^{(\ell)} \left( S^{(1)}, ..., S^{(m)} \right) \right],
\]

\[0 \leq s \leq j - 1, \quad j = 1, ..., n\]

Fair value future liabilities at time \( s \leq j - 1 \) (after premium payment at time \( j - 1 \))

**State-Price Deflators**

Consider a price process \( S = \{S_t\}_{0 \leq t \leq n} \) such that \( S_t \) represents the random value at time \( t \) of a financial instrument. A state-price deflator \( D = \{D_t\}_{0 \leq t \leq n} \) is a strictly positive process such that the market value of \( S_t \) at time \( s < t \) is given by \( S_s = D_s^{-1} \cdot E_s [D_t S_t] \), \( 0 \leq s < t \). In other words, the deflated or discounted price process \( DS = \{D_t S_t\}_{0 \leq t \leq n} \) is a martingale.
Black-Scholes-Vasicek (BSV) Deflator (1)

Black-Scholes-Vasicek (BSV) Deflator (see Hürlimann (2011a))

Consider a **multiple risk economy** with \( m \geq 1 \) risky assets, whose real-world prices follow lognormal distributions. Given the current prices of these risky assets at time \( s \geq 0 \) the future prices at time \( t > s \) are described by

\[
S_t^{(k)} = S_s^{(k)} \exp \left\{ (m_k(s,t) - \frac{1}{2} \sigma_k^2)(t-s) + \nu_k \sqrt{t-s} \cdot W_{t-s}^{(k)} \right\}, \quad 0 \leq s < t, \quad k = 1, \ldots, m
\]

where the \( W_{t-s}^{(k)} \)’s are **correlated** standard Wiener processes such that

\[
E[dW_{t-s}^{(i)}dW_{t-s}^{(j)}] = \rho_{ij} dt, \quad m_k(s,t), \nu_k(s,t) \text{ are the mean and standard deviation per time unit of the return differences on these risky assets, and } \sigma_k \text{ is a volatility.}
\]

This representation includes two popular return models:

**Black-Scholes return model:**

\[
dr_t^{(k)} = \mu_k dt + \sigma_k dW_t^{(k)}
\]

\[
m_k(s,t) = \mu_k \quad \nu_k(s,t) = \sigma_k, \quad 0 \leq s < t
\]
Vasicek (Ornstein-Uhlenbeck) return model:

\[
dr_t^{(k)} = a_k (b_k - r_t^{(k)}) dt + \sigma_k dW_t^{(k)}
\]

\[
m_k(s,t) = \frac{(b_k - r_s^{(k)})(1-e^{-a_k(t-s)})}{t-s}
\]

\[
v_k(s,t) = \sigma_k \sqrt{\frac{1-e^{-2a_k(t-s)}}{2a_k(t-s)}}
\]

The economic model contains also a deterministic money market account with value \( M_t = M_s \exp((t-s)R(s,t)) \), \( 0 \leq s < t \), where \( R(s,t), 0 \leq s < t \), are the deterministic continuously-compounded spot rates. The zero-coupon bond prices are denoted by \( P(s,t) = \exp(-(t-s)R(s,t)) \), \( 0 \leq s < t \). The BSV deflator of dimension \( m \) has the same form as the price processes of the risky assets:

\[
D_t^{(m)} = D_s^{(m)} \exp\{\alpha^{(m)}(s,t)(t-s) - \beta^{(m)}(s,t)^T \sqrt{t-s} \cdot W_{t-s}\}, \quad 0 \leq s < t,
\]

**Theorem 1.** Given is a financial market with a risk-free money market account and \( m \) risky assets with the defined real-world prices. Assume a non-singular positive semi-definite correlation matrix \( C = (\rho_{ij}) \). Then, the BSV deflator is determined by
Black-Scholes-Vasicek (BSV) Deflator (3)

\[
D_t^{(m)} = D_s^{(m)} \exp \left\{ -R(s,t)(t-s) - \frac{1}{2} \sum_{j=1}^{m} \beta_j^{(m)}(s,t)^2 (t-s) - \sum_{1 \leq i < j \leq m} \rho_{ij} \beta_i^{(m)}(s,t) \beta_j^{(m)}(s,t)(t-s) - \sum_{j=1}^{m} \beta_j^{(m)}(s,t) \sqrt{t-s} \cdot W_{t-s}^{(j)} \right\}, \quad 0 \leq s < t,
\]

with

\[
\beta_j^{(m)}(s,t) = \det(C)^{-1} \cdot \sum_{i=1}^{m} (-1)^{i+j} \det(C_j^{(i)}) \cdot \lambda_i(s,t),
\]

\[
\lambda_i(s,t) = \frac{m_i(s,t) - R(s,t) - \frac{1}{2} \left( \sigma_i^2 - v_i^2(s,t) \right)}{v_i(s,t)}, \quad 0 \leq s < t,
\]

where \( C_j^{(i)} \) is the matrix formed by deleting the i-th row and k-th column of \( C \).

The quantity \( \lambda_i(s,t) \) is called market price of the i-th risky asset.

**Proof.** One must satisfy the martingale conditions (equivalent to a system of linear equations):

\[
E_s \left[ D_t^{(m)} \right] = D_s^{(m)} P(s,t) = D_s^{(m)} e^{-(t-s)R(s,t)} ,
\]

\[
E_s \left[ D_t^{(m)} S_t^{(k)} \right] = D_s^{(m)} S_s^{(k)} , \quad 0 \leq s < t, \quad k = 1,\ldots,m.
\]
Index Linked Inflation Protected Life Contracts (1)

**Example:** index linked cohort of \( n \)-year endowment contracts

**Notations**
- \( n \): contract term
- \( x \): age of an insured live at contract issue
- \( \ell_x \): number of insured lives in the cohort at contract issue
- \( d_x = q_x \ell_x \): number of insured lives aged \( x \) who exit within one year
- \( \ell_{x+k} = \ell_{x+k-1} - d_{x+k-1}, k = 1,\ldots, n-1 \): recursion for number of insured lives
- \( i \): technical interest rate, \( r = 1 + i \)
- \( \pi \): market-consistent (or fair) pure risk premium per contract

MCAV follows a **three steps algorithm.**

**Case 1:** minimum interest guarantee only

**Step 1:** set of replicating financial instruments
- \( Z^{(k)}, k = 0,\ldots, n-1 \): zero-coupon bonds
- \( U^{(1)} = S \): indexed fund with price process \( S_t, S_0 = 1 \)
Index Linked Inflation Protected Life Contracts (2)

\[ U^{(\ell)} = P^{(\ell-1)}, \ell = 2,...,n+1 \] , with \[ P^{(k)} = P^{(k)}(S,r^k), k = 1,...,n \] :

European put option on indexed fund with strike time \( k \) and strike price \( r^k \)

Step 2: number of units

\[ \lambda_k^{(j)} = \begin{cases} 
-\pi \cdot \ell_{x+k}, & k = j,...,n-1, \\
0, & \text{else}
\end{cases} \]

\[ \eta_1^{(j)} = \begin{cases} 
\ell_x, & j = 0, \\
\ell_{x+j-1}, & j = 1,...,n,
\end{cases} \]

\[ \eta^{(j)} = d_{x+\ell-2}, \quad \ell = 2,...,n+1, \quad j = 0,...,n \]

Step 3: determination of market values (Black-Scholes formula)

a) Determine the fair premium

Applying the fair premium equivalence principle (law of one price) solve the equation

\[
M_0\left(VaPo(C_{(0)})\right) = -\pi \cdot \sum_{k=1}^{n} \ell_{x+k-1} P(0,k-1) + \ell_x \\
+ \sum_{k=1}^{n} d_{x+k-1} \left(r^k P(0,k) \cdot \Phi(d_2(0,k)) - \Phi(d_1(0,k))\right) = 0
\]
b) Determine the market-consistent value of future liabilities

$$M_s \left( \text{VaPo}(C_{(j)}) \right) = -\pi \cdot \sum_{k=j}^{n-1} \ell_{x+k} P(s, k) + \ell_{x+j-1} S_s$$

$$+ \sum_{k=j}^{n} d_{x+k-1} \left( r^k P(s, k) \cdot \Phi(d_2(s, k)) - S_s \cdot \Phi(d_1(s, k)) \right),$$

$$d_1(s, k) = \frac{-\ln \left\{ r^k P(s, k) / S_s \right\} + \frac{1}{2} \sigma^2 (k - s)}{\sigma \sqrt{k - s}},$$

$$d_2(s, k) = d_1(s, k) - \sigma \sqrt{k - s}, \quad 0 \leq s \leq j - 1, \ j = 1, ..., n,$$

**Case 2**: inflation protection only

**Step 1**: set of replicating financial instruments

The European put options on the indexed fund are replaced by European exchange options $EX^{(k)} = EX^{(k)} (I, S), k = 1, ..., n$ to exchange the inflation index with the index fund

**Step 3**: determination of market values
Instead of the Black-Scholes formula one applies an extended version of Margrabe’s formula in a multiple risk economy (e.g. Hürlimann (2011a), Theorem 2):

\[ M_s \left( \text{EX}^{(k)} \right) = D_s^{-1} \cdot E_s \left[ D_k \left( I_k - S_k \right)_+ \right] \]

\[ = I_s \cdot \Phi \left( \frac{\ln \left( I_s / S_s \right) + \frac{1}{2} \nu^2(s,k)(k-s)}{\nu(s,k)\sqrt{k-s}} \right) - S_s \cdot \Phi \left( \frac{\ln \left( I_s / S_s \right) - \frac{1}{2} \nu^2(s,k)(k-s)}{\nu(s,k)\sqrt{k-s}} \right), \]

\[ \nu^2(s,k) = \sigma_S^2 + \nu_I^2(s,k) - 2 \rho_{SI} \sigma_S \nu_I(s,k), \quad 0 \leq s \leq k-1, \quad k = 1, \ldots, n. \]

**Case 3**: combined minimum interest guarantee and inflation protection

It is possible to combine the inflation protection and a guaranteed minimum death benefit, say \( T_k \) at time \( k \) (generalizing the preceding guarantee \( T_k = r^k \)). The required double-trigger option at time \( k \) has the contingent financial payoff

\[ (I_k - S_k)_+ \cdot 1\{I_k > T_k\} + (T_k - S_k)_+ \cdot 1\{I_k \leq T_k\} \]

Its market value is determined in Hürlimann (2011b).
Inflation Protected Non-Life Contracts (1)

**Example:** index linked cohort of non-life contracts (single accident year)

**Notations** (claims development model from Walhin et al.(2001))

- $n$ : run-off time horizon
- $c_j$, $j = 1, \ldots, n$ : claims payment pattern
- $d_j$, $j = 1, \ldots, n$ : reserve deviation pattern ($d_0 = 0$ by convention)
- $f_j$, $j = 0, \ldots, n$, $f_0 = 1$ : expected inflation pattern
- $\pi$ : fair pure risk premium of the cohort per unit of expected ultimate nominal aggregate paid claims

**Case 1:** replication of nominal values only

**Step 1:** set of replicating financial instruments

$Z^{(k)}$, $k = 0, \ldots, n$ : zero-coupon bonds

**Step 2:** number of units

$$\lambda_0^{(0)} = -\pi, \quad \lambda_k^{(j)} = \begin{cases} (1 - d_{j-1}) \cdot c_j, & k = j, \\ \Delta d_j \cdot c_k, & k = j + 1, \ldots, n, \\ 0, & else \end{cases}, \quad j = 1, \ldots, n$$
Inflation Protected Non-Life Contracts (2)

Step 3: determination of market values

a) Determine the fair premium

Solve the equation \( M_0(VaPo(C_{(0)})) = 0 \) to get

\[
\pi = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \lambda_k^{(j)} \right) P(0,k)
\]

b) Determine the market-consistent value of future liabilities

\[
M_s(VaPo(C_{(j)})) = \sum_{k=j}^{n} \left( \sum_{j=1}^{n} \lambda_k^{(j)} \right) P(s,k) \quad 0 \leq s \leq j - 1, \quad j = 1, \ldots, n
\]

Case 2: replication with inflation protection

Step 1: set of replicating financial instruments

(i) zero-coupon bonds \( Z^{(k)}, k = 0, \ldots, n \)

(ii) replicating call options \( U^{(\ell)} = C^{(\ell)}(I, f_{\ell}), \ell = 1, \ldots, n \) on the inflation index \( I \)

with future random values \( I_k, k = 1, \ldots, n \) and initial value \( I_0 = 1 \)

Step 2: number of units

\[
\lambda_0^{(0)} = -\pi
\]

\[
\lambda_k^{(j)} = \begin{cases} 
(1-d_{j-1}) \cdot c_j \cdot f_j, & k = j, \\
\Delta d_j \cdot c_k \cdot f_k, & k = j+1, \ldots, n, \\
0, & \text{else} 
\end{cases} \quad j = 1, \ldots, n
\]
I Inflation Protected Non-Life Contracts (3)

\[
\eta_{\ell}^{(j)} = \begin{cases} 
(1-d_{j-1}) \cdot c_j, & \ell = j, \\
\Delta d_j \cdot c_\ell, & \ell = j+1, \ldots, n, \\
0, & \text{else} 
\end{cases} 
\]

Step 3: determination of market values

a) Determine the fair premium: solve the equation \( M_0(VaPo(C_{(0)})) = 0 \) to get

\[
\pi = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \lambda_k^{(j)} \right) P(0, k) + \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \eta_k^{(j)} \right) M_0(C^{(k)})
\]

b) Determine the market-consistent value of future liabilities

\[
M_s(VaPo(C_{(j)})) = \sum_{k=j}^{n} \left( \sum_{j=1}^{n} \lambda_k^{(j)} \right) P(s, k) + \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \eta_k^{(j)} \right) M_s(C^{(k)}) \quad 0 \leq s \leq j-1, \quad j = 1, \ldots, n
\]

with

\[
M_s(C^{(k)}) = D_s^{-1} \cdot E_s [D_k (I_k - f_k) + ] = I_s \cdot \Phi(d_1(s,k)) - f_k P(s,k) \cdot \Phi(d_2(s,k)),
\]

\[
d_1(s,k) = \frac{\ln(I_s/f_k) + \left(R(s,k) + \frac{1}{2} v_j^2(s,k)\right)(k-s)}{v_j(s,k)\sqrt{k-s}},
\]

\[
d_2(s,k) = d_1(s,k) - v_j(s,k)\sqrt{k-s}, \quad 0 \leq s \leq k-1, \quad k = 1, \ldots, n.
\]
Some references (Presentation only)


