Pension Fund Investments: Stocks or Bonds? *

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Very preliminary! Please do not quote.

Abstract: This paper reviews the investment policy of collective pension plans. The Defined Benefit nature of these plans would call for an investment portfolio that fully consists of long term, index linked bonds. In practice, pension plans have substantial investments in equities and real estate. We suggest two reasons to invest in equities: the lack of a well-developed market in index linked bonds, and deliberate deviations from the Defined Benefit nature of the plan. Furthermore, this paper assesses the value and optimality of conditional indexation rules that are found in many plans.

Keywords: High frequency data, microstructure, structural time series models.

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"Even if only nominal bonds are available, conservative long-term investors should hold large positions in long-term bonds if they believe that inflation risk is low, as we have estimated it to be in the USA in the period 1983-99. In this sense, the message of this chapter might be summarized as 'Bonds, James, Bonds'. Inflation risk is however a serious caveat." John Campbell and Luis Viceira, *Strategic Asset Allocation*, p.87.

1 Introduction

Several countries in Europe have organized their pension system by means of fully funded pension plans. The nature of the plans is typically Defined Benefit, i.e. the participants are promised a pension that is linked to their average or last earned wage before retirement. From a financial investments perspective, it is important that in such a scheme there is no direct link between the benefits (pensions) and the returns earned on the investment portfolio. Exley (2001) and other have argued that this feature implied that pension funds should invest in a portfolio that exactly matches this liability. Essentially, this portfolio would consists of 100% long term index linked bonds. The quote from Campbell and Viceira (2002) also states that long-term investors, such as pension funds, should take substantial positions in bonds.

In practice, we do see quite a different composition of pension fund portfolio's. Many funds hold between 40% and 60%, and sometimes even more especially in the UK, of their wealth in equities and real estate. The remainder is often invested to a large extent in medium-term nominal bonds, and only a small fraction of pension fund assets is index linked bonds. The risk-return profile of this investment portfolio is therefore quite different from the risk and return profile of the pension plan's liabilities. The expected return on the actual portfolio is higher than the expected return on index linked bonds, if there is a positive equity premium. But the risk is also larger and especially the inflation hedge of the portfolio is rather weak [numbers?]. This has lead to a lot of criticism on pension fund investment managers that they take too many risks, at the expense of the pension fund participants who see their benefits endangered by the risk of low returns. The recent melt-down of the stock market has made this point very clear, with many funds that are technically underfunded.[1] In this paper I assess

[1] Estimates in the beginning of 2003 of the pension fund supervisor in the Netherlands show that one quarter of pension funds has assets that are smaller then the present value of liabilities.

the validity of this critcism and review the case for investing in equities and real estate.

There are two main arguments for large stock investments. The first argument is that the market for index linked bonds is severely underdeveloped, effectively preventing pension funds from investing large fractions of their wealth in ILB's. Instead, they invest in a second best portfolio of more liquid assets that replicate the risk profile of ILB's as closely as possible. It is argued that equities and real estate are an important part of this portfolio. A second reason for a substantial investment in equity is that pension funds deliberately take more risks than a pure Defined Benefit scheme would impose. They invest more in equities to reap the equity premium. This gain is balanced against the larger risk, but leads to a higher fraction of equity investments.

1.1 Replication of Index Linked Bonds

One could argue that a Defined Benefit pension plan is like a long term riskless investment. So, it provides the hedge part of the optimal portfolio. The optimal investment for the pension fund is therefore 100% in index linked bonds. In practice, ILB's may be unavailable to the investor. Then the second best strategy of the fund is to match the long term risks as closely as possible by choosing assets that have the highest correlation with the long term risk exposures, i.e. real interest rate risk and inflation. This is exactly the second part of expression (10), the least squares hedge.

What is the composition of this least squares hedge? Of course, this depends on the exposures of the asset returns to the risk factors σ and the correlations between the risk factors ρ, and the correlation of the assets with long term real interest rate risk (b) and unexpected inflation risk (ξ). To get to a quantitative answer one needs to calibrate the model, but a few intuitive results can be established

- Nominal bonds provide a good hedge against interest rate risk, but no hedge against unexpected inflation risk.
- A roll over of short term bonds provides a good hedge against unexpected inflation risk, but almost no hedge against interest rate risk
- For stocks (and other risky assets) everything depends on their correlation with interest rates and inflation. Empirically, these correlations are weak and sometimes

even negative.[2] Moreover, stock returns have a fairly high variance. All this suggest that the role of stocks in the hedge portfolio will at best be very limited.

- A more useful risky asset may be real estate. Theebe (2002) shows that in the long run the value of real estate is strongly correlated with the price level. However, like stocks real estate returns have a high variance and that will limit the holdings in the hedge portfolio.

So, in the absence of ILB's the composition of the hedge portfolio has to strike a balance between the inflation hedge of short term bonds and the interest rate hedge of long term bonds, with a very limited role for stocks and other risky assets. The optimal portfolio will be a medium term bond portfolio, with duration a function of the relative magnitude of interest rate risk to inflation risk.

1.2 Deviations from Defined Benefit

The previous discussion assumed that Defined Benefit plans basically provide the hedge part of the optimal investment portfolio. Now for many individuals, the accumulated pension rights are a substantial part of their total wealth, larger than private savings and investments, and second only to the value of the housing. Especially in countries with a relatively generous pension system this will be the case. For example, in the Netherlands the pension benefit of a retiree with 40 years of labor history is 70% of the final wage. It is estimated that the value of this benefit amounts to 30 to 40 percent of total earnings during the working life of the individual. Of course, if the individual is not overly risk averse, he might be interested in investing a part of this wealth in the speculative portfolio, to reap the benefits of the risk premia on risky assets.

In a related study, Blake et al. (2002) consider a similar setting for a retiree. They compare the utility of investing in traditional nominally risk free annuities and equity linked annuities. Such ELA's are more risky that traditional annuities but also provide a higher expected payoff. Blake et al. show that the utility gains of investing in ELA's can be substantial.

The same line of reasoning suggests that participants may be interested in giving up some of the safe part of the pension fund portfolio to invest in risky assets. However, the

[2] These estimates are for annual data. For longer horizons, the correlations may be stronger. This aspect has to be studied in more detail.

flip side of this is that the pension plan is not Defined Benefit any more in the strict sense, but more like a mixture of DB and DC. This provides us with a second reason to invest in equities, as a part of the speculative portfolio. Of course, one could question whether collective pension plans should provide this part of the total portfolio or whether this is best left to the individuals. If we opt for the latter, the Dutch pension plans will have to become much smaller and more pure DB oriented (Boot, van Ewijk, 2003). The raison d'etre for pension funds then is to provide DB guarantees that individuals cannot buy in the marketplace, because there are no ILB's and certainly not wage-indexed products.

Pension plans also give an automatic annuitization of cash flows, but in principle any insurance company could do this. But again, when indexed annuities are unavialable in the marketplace, pension funds have a useful role in effectively providing these. Lopes (2002) emphasizes the benefits of index linked annuities for retired individuals.

1.3 Structure of the paper

In this paper, I shed my light on the validity of these arguments. I will look both at theoretical underpinnings of the argument and some relevant empirical evidence. The structure of the paper is as follows. First, I will give a formal theoretical structure for the optimal investment portfolio of a long term investor. Given the insights of this model, I discuss the two arguments for equity investments in more detail. I then turn to a calibration of the model to get some numerical results.

2 Long term investments: a review of the theory

In this section I review the most important results of the recent long term investments literature, which is summarized in the book by Campbell and Viceira (2002). The exact model structure presented here will be based on Brennan and Xia (2002), but other models give very similar results. I picked the Brennan-Xia model because of the explicit solutions for optimal portfolio's for investors with a long but $f$inite investment horizon. This model considers an investor with investment horizon T, which one could think of as the retirement date. The objective of the investor is to maximize the utility of end-of-period real wealth. The investor is assumed to have a CRRA utility function with relative risk aversion parameter γ. Formally, we write the problem of the investor as, with

$$\max \mathsf{E}\left[U\left(W_T/\Pi_T\right)\right], \qquad U(w)=\frac{w^{1-\gamma}}{1-\gamma} \tag{1}$$

where W_T denotes end-of-period nominal wealth, and Π_T the price level. The budget constraint of the investor is given by the initial wealth W_0 and the nominal wealth dynamics

$$dW/W=[x'\mu+(1-\iota'x)R_f]+x'\sigma dZ \tag{2}$$

where x is the vector of portfolio weights on the risky assets, with expected return μ, and $1-\iota'x$ the weight on the nominally riskless asset, with return R_f. The matrix σ denotes the exposure of the asset returns to the risk factors dZ, which will be specified shortly. Notice that the wealth dynamics can also be written as

$$dW/W=[R_f+x'\sigma\lambda]+x'\sigma dZ \tag{3}$$

where λ is the vector of market prices of factor risk and $\sigma\lambda=\mu-R_f$ is the vector of asset risk premia. The risk factor dynamics in the Brennan-Xia model can be summarized in the following state variables: the stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π and the price level Π. The equations driving these state variables are

$$dS/S = \mu_S dt+\sigma_S dZ_S \tag{4a}$$
$$dr = \kappa(\bar{r}-r)dt+\sigma_r dZ_r \tag{4b}$$
$$d\pi = \alpha(\bar{\pi}-\pi)dt+\sigma_\pi dZ_\pi \tag{4c}$$
$$d\Pi/\Pi = \pi dt+\sigma_\Pi dZ_\Pi \tag{4d}$$

It is sometimes useful to orthogonalize the equation for unexpected inflation

$$d\Pi/\Pi=\pi dt+\xi_S dZ_S+\xi_r dZ_r+\xi_\pi dZ_\pi+\xi_u dZ_u=\xi' dZ \tag{5}$$

where dZ_u is the part of dZ_Π orthogonal to (dZ_S, dZ_r, dZ_π).[3]

The investment vehicles in this model are stocks, nominal bonds and index linked bonds. The price dynamics of the stock are given by the first equation of this system. The bond price dynamics follow from the Vasicek (1977) model. The price dynamics of a nominal zero-coupon bond is given by

$$dP/P=[R_f-B(\tau)\sigma_r\lambda_r-C(\tau)\sigma_\pi\lambda_\pi]dt-B(\tau)\sigma_r dZ_r-C(\tau)\sigma_\pi dZ_\pi \tag{6}$$

[3] In my notation, σ denotes the exposure to all the risk factors, including the unexpected inflation risk, and ρ denotes the correlation matrix of $dZ=(dZ_S, dZ_r, dZ_\pi, dZ_\Pi)'$.

and the *nominal* price dynamics for an Index Linked Bond (ILB) are

$$dILB/ILB = [r + \pi - B(\tau)\sigma_r\lambda_r]dt - B(\tau)\sigma_r dZ_r + \sigma_\Pi dZ_\Pi \tag{7}$$

where τ is the time-to-maturity of the bond and

$$B(\tau) = \frac{1 - e^{\kappa\tau}}{\kappa}, \quad C(\tau) = \frac{1 - e^{a\tau}}{a}$$

These equations show that a nominal bond provides a hedge against real interest rate and expected inflation risk, but not against unexpected inflation risk. The ILB provides a hedge against real interest rate risk and *u*nexpected inflation risk. By construction, the risk premia in this model are time-invariant. The risk premium on stocks is given by $\mu_S - R_f = \lambda_S\sigma_S$. From the model parameter values we also estimate the risk premium on long term bonds from the drift of equation (6),

$$\mu_B - R_f = -B(\tau)\sigma_r\lambda_r - C(\tau)\sigma_\pi\lambda_\pi \tag{8}$$

where τ denotes the maturity of the bond. The risk premium on an index linked bond is from equation (7) and using $R_f = r + \pi + \lambda_\Pi\sigma_\Pi$:

$$\mu_{ILB} - R_f = -B(\tau)\sigma_r\lambda_r - \lambda_\Pi\sigma_\Pi \tag{9}$$

Notice that ILB's don't have a risk premium for expected inflation risk, but they do have a risk premium for unexpected inflation.

Brennan and Xia (2002) and the appendix to this paper show that the optimal portfolio composition for this investor is

$$x^{opt} = \frac{1}{\gamma}(\sigma\rho\sigma')^{-1}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)(\sigma\rho\sigma')^{-1}\sigma\rho(b + \xi)' \tag{10}$$

where $b = (0, -B(T)\sigma_r, 0, 0)'$ and $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_u)'$. The intuition for this portfolio is as follows. The portfolio consists of a speculative part and a hedge part. How much of each is determined by the risk aversion parameter $1/\gamma$. The speculative part is equal to the optimal portfolio of Merton's (1969) investment problem, and gives the usual mean-variance tradeoff between risk and risk premium. The hedge part gives the minimum variance (least squares) hedge against the long term risks for the investor: b is the long term real interest rate risk, and ξ is the unexpected inflation risk exposure. The long term risk of the investor is exactly the risk exposure of an ILB with maturity equal to the investment horizon T. The proof is simple: we can write the nominal price dynamics of an ILB as

$$dILB/ILB = [..]dt + (b+\xi)'dZ \tag{11}$$

We can express the effectiveness of this hedge in a coefficient of determination measure

$$R^2_{hedge} = \frac{(b+\xi)'\rho\sigma'(\sigma\rho\sigma')^{-1}\sigma\rho(b+\xi)}{(b+\xi)'(b+\xi)} \tag{12}$$

This insight also implies that if there is an ILB, it is an ideal hedge instrument. Suppose that there are n assets and let the ILB be the first in the portfolio. Then one can write its row in the return exposure matrix σ as $(b+\xi)'$. From this it follows

$$b+\xi = \sigma' \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}$$

and the optimal portfolio simplifies to

$$\begin{pmatrix} x_{ILB} \\ x_{rest} \end{pmatrix} = \frac{1}{\gamma}(\sigma\rho\sigma')^{-1}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \tag{13}$$

Hence, the hedge part of the portfolio is completely dominated by T-maturity ILB, which is the perfect hedge instrument, with an R^2_{hedge} equal to 1.

This optimal portfolio equation also highlights the two reasons to hold equity (or any other asset for that matter): (i) as an element of the speculative part of the portfolio. Here the key determinants of the amount of equity are the risk (variance) and the risk premium or Sharpe ratio; (ii) as an element of the hedge portfolio. Here the key determinants of the amount of equity are the correlation with long-term real interest rate risk and unexpected inflation risk. The stronger the correlation, the better the hedge and the higher the portfolio weight. We now go into more detail on each of these two points.

3 Calibration

We now investigate in more detail the intuition developed in the previous section. In order to do this we have to calibrate the model, i.e. impose parameter values and calculate the associated optimal portfolio's. We pick the parameters reported in Brennan and Xia (2002), with two modifications. First, Brennan and Xia report a high and a low value for the mean reversion parameter of the real interest rate. With the high value,

Table 1: Parameters for Brennan and Xia model

	Brennan-Xia	alternative
α	0.027	
κ	0.630	0.105
σ_S	0.158	
σ_r	0.026	0.013
σ_π	0.014	
σ_Π	0.013	
ρ_{Sr}	-0.129	
$\rho_{S\pi}$	-0.024	
$\rho_{r\pi}$	-0.061	
λ_S	0.343	0.200
λ_r	-0.209	-0.100
λ_π	-0.105	-0.050
λ_Π	NA	0

the interest rate hedge component is very small. We therefore select the more realistic low mean reversion parameter. The second modification is in the market prices of risk. Brennan and Xia report values that imply fairly high risk premia on stocks and bonds (5.5% per annum for stocks and 3% per annum for 10-year bonds). In the current market circumstances, these values seem unrealistically high. We therefore pick lower values for the market prices of risk. The values are summarized in Table 1.

Brennan and Xia don't give an estimate of the market price of risk for unexpected inflation. We take a conservative approach (i.e. biased against holding ILB's in the portfolio) by assuming that $\lambda_\Pi = 0$. We also assume that unexpected inflation is uncorrelated with stock returns, expected inflation and the real interest rate.[4] So, stocks and nominal bonds provide no hedge against unexpected inflation.

We now consider two situations, one with nomoinal bonds only and one with index linked bonds. For the first situation, we assume that the investor can invest in cash, one nominal long bond and stocks. In the Brennan-Xia model, a linear combination of two nominal bonds can hedge perfectly against expected inflation risk. However, this

[4]In Brennan and Xia's notation, we assume $\xi = 0$.

combination requires a short position in one of the bonds. This makes the strategy infeasible for a typical pension fund that is restricted to long positions. Therefore, we do not consider the case with two nominal bonds. Instead, we consider the optimal choice of maturity for the single bond that the fund can buy, with the restriction that the position in this bond does not exceed 100% of the invested wealth.

The optimal portfolio is a weighted average of the speculative portfolio and the hedge portfolio. Table 2 shows the composition of the optimal portfolio for an investment horizon of 20 years and a bond maturity of 5 years. We see that the speculative portfolio is highly leveraged, but has stocks and bonds in almost 50-50 proportions. The hedge portfolio is tilted towards bonds and cash, with only very small position in stocks because of the positive correlation between stock and bond returns. Recall here that we assumed that stocks don't provide a hedge against unexpected inflation and the correlation between expected inflation and stock returns in the data is extremely weak. Stocks therefore don't provide an inflation hedge in this model. The optimal composition of the hedge portfolio may change of we include risky assets that correlate more with inflation, e.g. real estate. The effectiveness of the hedge, i.e. the squared correlation between the long term risk and the hedge portfolio return, is small, it takes the value 0.337.

Of course, other maturities for the bond can be chosen. Figure 1 shows the optimal position in the nominal bond and the hedge effectiveness as a function of the bond maturity. The figure shows that for longer maturities the optimal position in bonds is smaller and that the hedge effectiveness decreases. This is an immediate result of the high expected inflation risk of long term nominal bonds. This can be seen by writing the hedge effectiveness for the simplified case where $\rho_{Sr} = \rho_{S\pi} = 0$ and also $\rho_{r\pi} = 0$, which is approximately true in the data. In that case, the optimal position in the bond is

$$x_B^{hedge} = \frac{B(\tau)B(T)\sigma_r^2}{B(\tau)^2\sigma_r^2 + C(\tau)^2\sigma_\pi^2 + \sigma_\Pi^2} \tag{14}$$

The hedge effectiveness is

$$R^2_{hedge} = \frac{B(\tau)^2 B(T)^2 \sigma_r^4}{(B(\tau)^2\sigma_r^2 + C(\tau)^2\sigma_\pi^2 + \sigma_\Pi^2)(B(T)^2\sigma_r^2 + \sigma_\Pi^2)} \tag{15}$$

Given our parameter values, $C(\tau)$ converges much slower to its maximum value than $B(\tau)$ because of the slow mean reversion in expected inflation. The R^2_{hedge} is therefore decreasing in τ and the model suggests to invest in short-term nominal bonds. However, to get the right exposure to the long-term real interest rate risk a highly leveraged posi-

Table 2: Optimal portfolio of stocks and nominal bonds

	stock	5yr	cash	stock	20yr	cash
risk premium	3.16	0.83		3.16	2.17	
st. dev.	15.80	8.03		15.80	23.61	
correlation		0.101			0.081	
Sharpe ratio	0.200	0.104		0.200	0.092	
x^{spec}	1.21	1.05	-1.26	1.23	0.32	-0.55
Sharpe ratio			0.217			0.214
x^{hedge}	0.05	0.78	0.17	0.07	0.18	0.75
R^2_{hedge}			0.337			0.170
$x^{opt}(\gamma = 1)$	1.21	1.05	-1.26	1.23	0.32	-0.55
$x^{opt}(\gamma = 2)$	0.63	0.91	-0.54	0.65	0.25	0.10
$x^{opt}(\gamma = 5)$	0.28	0.83	-0.11	0.30	0.21	0.49
$x^{opt}(\gamma = 10)$	0.17	0.80	0.03	0.18	0.20	0.62
$x^*(\gamma = 1)$	0.99	0.01	0	0.84	0.16	0
$x^*(\gamma = 2)$	0.53	0.47	0	0.65	0.25	0.10
$x^*(\gamma = 5)$	0.26	0.74	0	0.30	0.31	0.49
$x^*(\gamma = 10)$	0.17	0.80	0.03	0.18	0.20	0.62

tion is needed. Only for bond maturities of approximately 4 years or higher, no leverage is needed in the hedge portfolio. The hedge effectiveness is therefore also bounded to around $R^2_{hedge} = 0.35$.

The second situation we consider is cash, an index linked bond and stock. Table 3 shows the relevant data for this situation. In the table, it is assumed that the ILB has a maturity equal to the investment horizon, 20 years. Obviously, this ILB provides a perfect hedge to the long term risk and therefore completely dominates the hedge portfolio. The ILB also enters the speculative portfolio because of its positive risk premium, caused by the real interest rate risk premium. The speculative investment in ILB's is smaller than the speculative investment in nominal bonds, mainly because the risk premium on nominal bonds is higher due to the unexpected inflation risk that these carry.

In practice, ILB's with a very long maturity may be unavailable and the fund has to

Table 3: Optimal portfolio of stocks and 20 year index linked bond

	stock	ILB	cash
risk premium	3.16	1.09	
st. dev.	15.80	10.94	
correlation		0.128	
Sharpe ratio	0.200	0.100	
x^{spec}	1.21	0.68	-0.89
Sharpe ratio			0.213
x^{hedge}	0.00	1.00	0.00
R^2_{hedge}			1.000
$x^{opt}(\gamma = 1)$	1.21	0.68	-0.89
$x^{opt}(\gamma = 2)$	0.60	0.84	-0.44
$x^{opt}(\gamma = 5)$	0.24	0.94	-0.18
$x^{opt}(\gamma = 10)$	0.12	0.97	-0.09
$x^*(\gamma = 1)$	0.94	0.06	0
$x^*(\gamma = 2)$	0.47	0.53	0
$x^*(\gamma = 5)$	0.19	0.81	0
$x^*(\gamma = 10)$	0.09	0.91	0

invest in shorter maturity ILB's. Figure 2 shows the optimal fraction of ILB's and the hedge effectiveness for shorter maturities. It turns out that even with reasonable short maturities, around 5 years, the hedge effectiveness is close to 1. This is a result of the relatively quick mean reversion of the real interest rate. The portfolio has substantial leverage, however.

We now turn to the overall optimal portfolio composition. The overall optimal portfolio depends of course on the mix between the speculative and the hedge portfolio. Theoretically, this depends on the investor's risk aversion. A pension fund invests on behalf of its participants and if it invests all their wealth, it will inherit the participant's risk aversion. We proxy this by the $\gamma = 2$ scenario in the tables. On the other hand, if the pension fund invests only a fraction of the investor's wealth, its target is more likely to be a replication of defined benefits, with the portfolio tilted towards the hedge portfolio. We proxy this by the $\gamma = 10$ scenario in the tables. The optimal portfolio for

$\gamma = 2$ shows a substantial investment in stocks, around 60%, and a substantial leverage. Notice that this holds even with our relatively conservative values for equity and bond risk premiums (3.16% and around 1%, respectively). The optimal portfolio also has a substantial leverage, around 50% borrowed cash. The more conservative assumption $\gamma = 10$ generates portfolio's with around 15% in stocks (more when the bond is nominal), between 80% and 100% in bonds (more when the bond is index linked), almost no stock and a small amount of leverage.

4 The value of conditional indexation

In further work, we want to assess the riskiness of optimal strategy in terms of the probability of underfunding, i.e. the probability of not covering the Defined Benefits guarantees. Based on this work we design and calculate the costs of option strategies to cover that risk. This is not a trivial exercise, especially in the incomplete markets case where there is no instrument to directly hedge the inflation risk. The Brennan and Xia (2002) model is very suitable for pricing such claims because the pricing kernel takes a very convenient form.

The (real) pricing kernel in the Brennan-Xia model is given by

$$M_T^* = \exp\left\{\int_0^T (-r(s) - \tfrac{1}{2}\phi'\rho\phi)ds + \int_0^T \phi' dZ\right\} \tag{16}$$

where ϕ is related to the (real) market prices of risk by $\lambda^* = -\rho\phi$. The price level is given by equation 5 and equals

$$\Pi_T = \exp\left\{\int_0^T (\pi(s) - \tfrac{1}{2}\xi'\rho\xi)ds + \int_0^T \xi' dZ\right\} \tag{17}$$

The nominal pricing kernel is therefore given by

$$M_T = M_T^*/\Pi_T = \exp\left\{\int_0^T (-r(s) - \pi(s) - \tfrac{1}{2}\phi'\rho\phi + \tfrac{1}{2}\xi'\rho\xi)ds + \int_0^T (\phi - \xi)' dZ\right\} \tag{18}$$

where we normalized $M_0 = \Pi_0 = 0$. The nominal pricing kernel can also be written as

$$M_T = \exp\left\{\int_0^T (-R_f(s) - \tfrac{1}{2}(\phi - \xi)'\rho(\phi - \xi))ds + \int_0^T (\phi - \xi)' dZ\right\} \tag{19}$$

with $R_f = r + \pi - \xi'\lambda$ is the nominal risk free rate and $\lambda = -\rho(\phi - \xi)$ is the vector of nominal market prices of risk. Any payoff at time T can be valued using these pricing kernels. For example, a nominal payoff X_T has time 0 value

$$X_0 = \mathsf{E}[M_T X_T] \tag{20}$$

Now consider a Defined Benefit pension with full indexation. In our very stylized model, the payoff of this pension can be written as $L \cdot \Pi_T$, where L is the present value of all the rights built up so far. The value of this claim is

$$L_0 = L\mathsf{E}[M_T \Pi_T] = L\mathsf{E}[M_T^*] \tag{21}$$

which is equal to the price of an index linked bond with face value L paying off at time T. This is of course not surprising as this ILB is the perfect hedge instrument for this particular claim.

Things become a little more interesting when indexation is limited. Again, as a very stylized example consider a pension where indexation is limited to a maximum of 5% per year (continuously compounded) on average.[5] The payoff of this pension is $L\min\{\Pi_T, \exp(0.05T)\}$, which can be written as

$$X_T = L\Pi_T - \max\{\Pi_T - \exp(0.05T), 0)\} \tag{22}$$

The conditional indexation is therefore a call option on inflation, written by the pension fund participant. Numerical valuation of this option is fairly straightforward by simulating paths for the pricing kernel and the option.

In practice, the check for indexation is every year. This will lead to a path dependent option, with payoff

$$X_T = L \prod_{t=1}^{T} \min\{\Pi_t / \Pi_{t-1}, 1.05\} \tag{23}$$

This option must be valued by numerical methods.

5 Conclusion

In this paper we considered the optimal investment policy of pension funds. Using a continuous time long term investments framework, we show that the optimal portfolio consists of two parts, a speculative part and a hedge part, that covers the long term

[5]In practice, the check for indexation is every year. This will lead to a path dependent option that we shall discuss later.

interest rate and price risks. We show that the speculative part is a fairly standard stock-bond portfolio with roughly a 50-50 mix for stocks and medium term nominal bonds, and a 65-35 mix for stocks and long term index linked bonds. The hedge part depends on whether index linked bonds are available. With nominal bonds only, the model suggests to invest in medium term, around 4 year, nominal bonds (for a 20 year investment horizon). The hedge effectiveness of this portfolio is low, however, because of the substantial unhedgeable inflation risk that these bonds carry. A long term index linked bond is much better as it provides the pension fund with a perfect hedge instrument. If long term index linked bonds are unavailable, medium term ILB's are a good substitute with an almost perfect hedge, but a leveraged position is required to obtain the right exposure to real interest rate fluctuations.

The overall optimal portfolio depends on the risk aversion assumed for the fund. A very conservative fund that aims to replicate Defined Benefit guarantees should invest almost exclusively in long term index linked bonds. Without ILB's, medium term nominal bonds are the best alternative, but the hedge effectiveness of this policy is very limited with a lot of unhedged inflation and interest rate exposure. A not so risk averse fund should invest around 60% in stocks and take on substantial leverage. Effectively then the fund runs a speculative investment portfolio on behalf of its participants.

Appendix

In this appendix we show how to derive the optimal portfolio of stocks, nominal bonds and index linked bonds, with and without cash positions. The basis is the dynamics of optimal wealth, derived by Brennan and Xia (2002)

$$d\ln F_t = [..]dt + \left[-\frac{1}{\gamma}\phi' - \left(1-\frac{1}{\gamma}\right)\sigma_{tT}\right]dZ_t \equiv c'dZ \tag{24}$$

where $\sigma_{tT} = (0, \sigma_r B(T-t), 0, 0)'$. This dynamics has to be equated with the actual (feasible) wealth dynamics at $t = 0$, the time of planning the wealth.

The menu of assets consists of risky assets, with portfolio weights x and price dynamics

$$dP/P = [..]dt + \sigma dZ \tag{25}$$

Therefore, the real wealth dynamics are given by

$$dX^*/X^* = [..]dt + (x'\sigma - \xi')dZ \tag{26}$$

where $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_u)'$ are the loadings of the price level on the factors. The optimal portfolio is given by minimizing the norm of the difference between optimal and feasible wealth dynamics

$$\min_x ||(x'\sigma - \xi' - c')dZ|| \tag{27}$$

with $||a'dZ|| = a'\rho a$. If there is a nominally risk free asset, x is unconstrained and the first order condition is

$$\sigma\rho(\sigma' x - \xi - c) = 0 \tag{28}$$

and hence the optimal portfolio rule follows by substituting c as

$$\begin{aligned} x^{opt} &= (\sigma\rho\sigma')^{-1}\left[-\frac{1}{\gamma}\sigma\rho\phi - \left(1 - \frac{1}{\gamma}\right)\sigma\rho\sigma'_{0T}\right] + (\sigma\rho\sigma')^{-1}\sigma\rho\xi' \\ &= (\sigma\rho\sigma')^{-1}\left[\frac{1}{\gamma}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)\sigma\rho(b+\xi)'\right] \end{aligned} \tag{29}$$

where $b = (0, -B(T)\sigma_r, 0, 0)'$.

Without a risk free asset, we have to impose the constraint $\iota' x = 1$. The first order conditions for optimality are

$$\sigma\rho(\sigma' x - \xi - c) - \mu\iota = 0 \tag{30}$$

where μ is the Lagrange multiplier for the constraint $\iota' x = 1$. Solving for μ gives

$$\mu = \frac{1 - \iota' x_{opt}}{\iota'(\sigma\rho\sigma')^{-1}\iota} \tag{31}$$

The optimal portfolio with constraints then is

$$x^{optc} = x^{opt} + (1 - \iota' x^{opt})x^{min}, \quad x^{min} = \frac{(\sigma\rho\sigma')^{-1}\iota}{\iota'(\sigma\rho\sigma')^{-1}\iota} \tag{32}$$

where x^{min} is the minimum variance portfolio of the risky assets.

References

Basak, Suleyman and Alexander Shapiro (2001), Value-at-Risk based management: Optimal policies and asset prices, Review of Financial Studies 14, 371-405.

Blake, David, Andrew Cairns and Kevin Dowd (2002), Pensionmetrics 2: Stochastic pension plan design during the distribution phase, working paper The Pensions Institute, Birkbeck College London.

Bodie, Zvi (2002), An analysis of investment advice to retirement plan participants, Pension Research Council Working Paper 2002-15, Wharton School, University of Pennsylvania.

Brennan, Michael J. and Yihong Xia (2002), Dynamic asset allocation under inflation, Journal of Finance 57, 1201-1238.

Buraschi, Andrea and Alexei Jiltsov (2002), Is inflation risk priced? working paper London Business School.

Campbell, John Y. and Luis M. Viceira (2002), Strategic Asset Allocation, Oxford University Press.

Exley, Jon (2001), Pension funds and the UK economy, working paper GEMS.

Lopes, Paula (2002), A dynamic model of nominal and inflation-indexed annuities, working paper Financial Markets Group, London School of Economics.

Merton, Robert C. (1969), Lifetime portfolio selection under uncertainty: The continuous time case, Review of Economics and Statistics 51, 247-257.

Sack, Brian en Robert Elsasser (2002), Treasury Inflation-Indexed Debt: A review of the U.S. experience, working paper Board of Governors of the Federal Reserve System.

Samuelson, Paul A. (1969), Lifetime portfolio selection by dynamic programming, Review of Economics and Statistics 51, 239-246.

Theebe, Marcel (2002), Housing Market Risks, Tinbergen Institute Research Series 283.

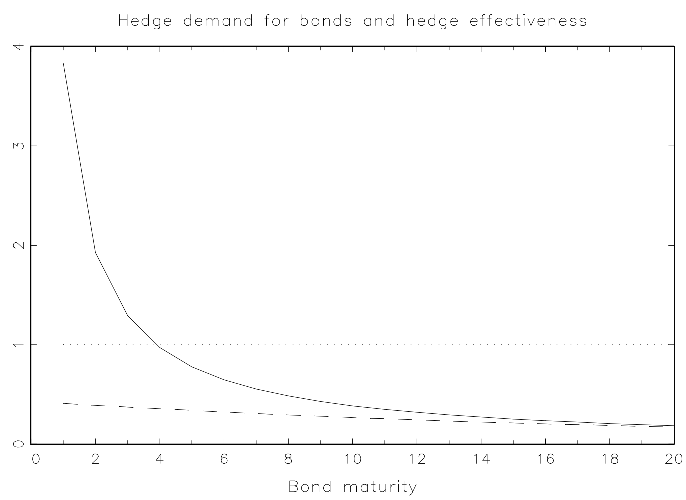

Figure 1: **Optimal bond weight in hedge portfolio**

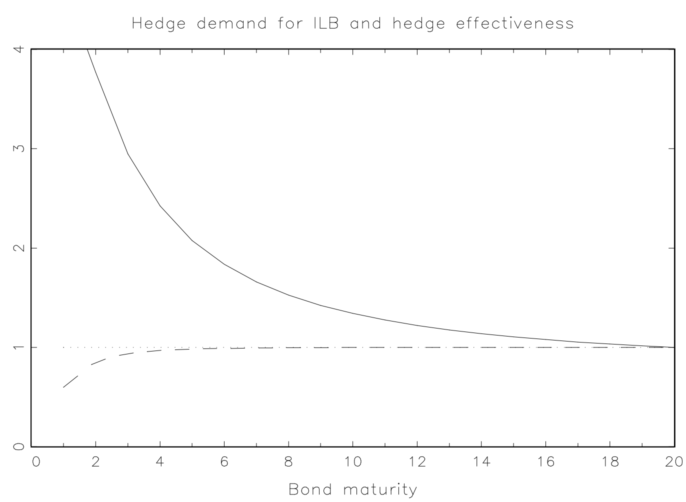

Figure 2: **Optimal index linked bond weight in hedge portfolio**