Optimal Investment Choices Post Retirement in a Defined Contribution Pension Scheme.

By Russell Gerrard[a], Steven Haberman[b] and Elena Vigna[c]

Abstract

In defined contribution pension schemes, the financial risk is borne by the member. Financial risk occurs both during the accumulation phase (investment risk) and at retirement, when the annuity is bought (annuity risk). The annuity risk faced by the member can be reduced through the "income drawdown option": the retiree is allowed to choose when to convert the final capital into pension in a certain period of time after retirement. In some countries there is a limiting age when annuitization becomes compulsory (in UK the extreme age is 75). In the interim, the member can withdraw periodic amounts of money to provide for daily life, within certain limits imposed by the scheme's rules (or by law).

In this paper, we investigate the income drawdown option and define a stochastic optimal control problem, looking for optimal investment strategies to be adopted after retirement, when allowing for periodic fixed withdrawals from the fund. The risk attitude of the member is also considered, by means of a parameter that determines the risk aversion coefficient.

Numerical examples are also presented, to investigate other issues – relevant to the pensioner – when the optimal investment allocation is adopted. This issues are, for example, risk of outliving the assets before annuitization occurs (risk of ruin), average time of ruin, probability of reaching a certain pension target (that can be linked for example to a DB formula and is certainly greater than the pension the member could buy straight at retirement), final outcome that can be reached (distribution of annuity that can be bought at limit age), and how the risk attitude of the member affects the key performance measures mentioned above.

Keywords: annuity risk, income drawdown option, stochastic optimal control.

[a] Faculty of Actuarial Science and Statistics, Cass Business School, City University, London. Email: r.j.gerrard@city.ac.uk.

[b] Faculty of Actuarial Science and Statistics, Cass Business School, City University, London. Email: s.haberman@city.ac.uk

[a] Department of Applied Mathematics and Statistics, University of Turin. Email: elena.vigna@unito.it.

Optimal Investment Choices Post Retirement in a Defined Contribution Pension Scheme.

INTRODUCTION

The income-drawdown option in defined contribution (DC) pension schemes allows the member who retires not to convert the accumulated capital into annuity immediately at retirement but to defer the purchase of the annuity at a certain point of time after retirement. During this period the member can withdraw periodically a certain amount of money from the fund within prescribed limits. The period of time is also limited: usually freedom is given for a fixed number of years after retirement and at a certain age the annuity must be bought.

In the UK, where the option was introduced in 1995, the periodic income drawn is bounded between 35% and 100% of the amount that the member would receive if she bought a level annuity at retirement. At 75 the annuity must be bought with the remaining fund.

Comparing the drawdown option with the purchase of an annuity at retirement, we observe two important points: the member is given complete investment freedom (instead of locking the fund into bond-based assets, as is usual with annuities) and a bequest desire can be satisfied should the member die before buying the annuity (because in case of death the fund remains part of the individual's estate).

A number of authors have dealt with the problem of managing the financial resources of a pensioner after retirement, also due to the fact that whole life annuities are felt by policyholders as "poor value for money" (M. Orszag) and have investigated other alternatives given to a retiree at retirement. Among others, there are papers by Albrecht and Maurer (2002), Blake, Cairns and Dowd (2001), Kapur and Orszag (1999), Khorasanee (1996), Lunnon (2002), Milevsky (1998, 2001), Milevsky and Robinson (2000), Mitchell (2001), Wadsworth, Findlater and Boardman (2001).

In this paper we investigate what should be the optimal investment allocation of the fund after retirement and until the purchase of the annuity, given that the pensioner wishes to achieve a certain target when she/he buys the annuity.

THE MODEL

In our model we consider the position of an individual who chooses the drawdown option at retirement, i.e. withdraws a certain income until either the remaining fund allows her/him to buy a certain (and relatively high) level annuity or she/he achieves the age at which the purchase of the annuity is compulsory.

The fund is invested in 2 assets, a riskless and a risky asset.

The equation that describes the growth of the fund in discrete time is the following:

$$X_{t+h} = (X_t - b_0 h)[y_t e^{\lambda_t h} + (1 - y_t)e^{rh}]$$
$$X_0 = x_0$$

with:

X_t fund at time t (with X_0 being the fund at retirement)
b_0 income drawn from the fund in one unit time
y_t proportion of the fund invested in the risky asset
λ_t instantaneous force of interest of the risky asset
r instantaneous force of interest of the riskless asset, assumed to be constant over time

We assume that the return on the risky asset is lognormally distributed:

$\lambda_t h \sim N((\lambda-\sigma^2/2)h, \sigma^2 h)$

Following Merton (1969), we obtain the following stochastic differential equation that describes the evolution of the fund in the continuous time:

$$\begin{aligned} dX(t) &= \{X(t)[y(t)(\lambda - r) + r] - b_0\}dt + X(t)y(t)\sigma dW(t) \\ X(0) &= x_0 \end{aligned} \qquad (1)$$

with W(t) standard Brownian motion.

We justify our use of a 2 asset model by recalling one of the results of Merton (1971) that "whenever log-normality of asset prices is assumed, we can work, without loss of generality, with just two assets, one 'risk-free' and the other risky with its price log-normally distributed".

Formally, we have a controlled stochastic differential equation:

$$\begin{aligned} dX(t) &= b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t) \\ X(0) &= x_0 \end{aligned} \qquad (2)$$

where X(t) is the state process and u(t) (in our case y(t)) is the control variable and where:

b: $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$
σ: $\mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$

We observe that the functions b( ) and σ( ) as defined above satisfy the conditions for the existence and uniqueness of the solution of the SDE (1).

We consider the class of Markov controls:

$y(t, \omega) = y(t, X_t(\omega))$

i.e. controls whose value at time t is known, if the state of the system at time t is known.

We introduce the following loss (or disutility) function:

$$L(t, X(t), y(t)) = L(t, X(t)) = (F(t)-X(t))^2 + \alpha\ (F(t) - X(t)) \qquad (3)$$

It can be shown (see Haberman and Vigna, 2002) that α is a parameter that measures the risk attitude of the individual: the higher its value, the lower the risk aversion of the individual.

The coefficient F(t) is a target that the individual wishes to achieve: deviations from this target are penalised so that a “cost”, measured by the loss function, is paid when the fund is below the target. Actually, the real target pursued is not F(t) but $F(t) + \alpha/2$, and since α measures the risk aversion of the person, it is clear that the lower the risk aversion the higher the target pursued and vice versa.

The targets are time dependent because as time passes the individual becomes older and the future life expectancy decreases: hence, the value of the annuity that would be purchased at the interruption of the income drawdown option decreases, ceteris paribus. We also observe that as time passes the fund on the one hand decreases due to the periodic income drawn, and on the other hand changes in value (and hopefully increases) due to the investment return of the 2 assets in which it is invested.

In a later section we will produce results for two different specification of the targets: constant targets and exponential targets.

We now define the open set $G \subset \mathbf{R}_+ \times \mathbf{R}$, where the couples (t, X(t)) are allowed to range:

$$G = \{(0, T) \times (-\infty, +\infty)\}, \qquad (4)$$

where T is the time when purchase of the annuity becomes compulsory.

A more realistic application would set finite bounds to the process X(t).

In fact, retiree members of a DC scheme take the income drawdown option in the hope of doing better than buying an annuity at retirement. Therefore, it makes sense for them to have the wish of being able to buy a better annuity at a certain point of time after retirement than the annuity they would have purchased had they bought it at retirement. The option is thus taken with the final aim of buying a reasonably high pension and if the size of the fund allows the purchase of the high pension before the compulsory age the individual should stop investing the fund and lock it into an annuity. Therefore the existence of a finite maximum bound for the fund process would be realistic.

Even more desirable than the existence of a maximum bound would be the existence of a minimum finite bound. A minimum limit would be intended to protect the retiree from outliving his/her assets and not being able to buy a minimum level pension at time T. Therefore, a minimum limit equal to at least 0 would be appropriate, as many other similar applications of HJB equation show (see, among others, the examples contained in Björk, 1998).

However, adding finite bounds to the state process means adding boundary conditions to the problem and this makes it very difficult to solve analytically, noting that a quadratic disutility function is here used (in the applications mentioned the problem was solved using a power utility function)[1]. For this reason, we have left the state process unbounded, sacrificing an element of realism in order to have a solution in closed form.

We are thus assuming that the individual will use the income drawdown option until the maximum age allowed (time T), regardless of the size of the fund[2].

The first exit time of (t, X(t)) from the open set G is T.

We introduce now the "bequest function" K: $\mathbf{R}_{+}\times\mathbf{R}:\rightarrow\mathbf{R}$:

$$K(t, x) = \varepsilon\, e^{-\rho t}\, [(F(t)-X(t))^2 + \alpha\, (F(t) - X(t))] \tag{5}$$

Therefore, K( ) has the same form of the loss function, multiplied by a constant ε, that gives a weight to the cost experienced at time t.

THE STOCHASTIC OPTIMAL CONTROL PROBLEM

We are now ready to define the stochastic optimal control problem we wish to solve.
The objective is to minimise the expected losses that can be experienced from retirement until the interruption of the income drawdown option, therefore the aim is to minimise the following expected value:

$$E_{0,x_0}\left[\int_0^T e^{-\rho s} L(s, X(s), y)ds + K(T, X(T))\right] \tag{6}$$

where ρ is the (subjective) intertemporal discount factor and where the expectation is done at time t=0, when the state of the system is x_0.

To solve this stochastic control problem, we define the "performance" criterion:

---

[1] Other examples of application of HJB equation in the context of pension fund dynamics with particular cases of quadratic loss function can be found in Boulier et al (1995), Boulier et al (1996) and Cairns (2000).
[2] Actually, this is probably what a rich pensioner, unwilling to convert the capital into annuity and willing to manage his/her money until the maximum age allowed by law, would do. Thus, the absence of limits to the wealth process can be considered not so unrealistic for some classes of individuals.

$$J(t,x,y) = E_{t,x}\left[\int_t^T e^{-\rho s} L(s, X(s), y)ds + K(T, X(T))\right] \quad (7)$$

where expectation is conditional on the state x at time t.

The value function is defined as:

$$H(t,x) := \inf_y J(t, x, y) = J(t, x, y^*) \quad \forall (t, x) \in G \quad (8)$$

where y*(t, x) is the optimal control (if it exists).

We now want to determine the optimal control y*(t, x).

Let us consider, for any $v \in \mathbf{R}$ and any function $f \in C^2(\mathbf{R} \times \mathbf{R})$ the infinitesimal operator:

$$A^v f(t,x) := \frac{\partial}{\partial t} f(t,x) + b(t,x,v)\frac{\partial}{\partial x} f(t,x) + \frac{1}{2}\sigma^2(t,x,v)\frac{\partial^2}{\partial x^2} f(t,x) \quad (9)$$

where the functions b( ) and σ( ) are the drift and diffusion terms of the process X(t) defined by (2).

In our case $A^v f$ becomes:

$$A^v f(t,x) := \frac{\partial f}{\partial t} + \{x[v(\lambda - r) + r] - b_0\}\frac{\partial f}{\partial x} + \frac{1}{2}x^2 v^2 \sigma^2 \frac{\partial^2 f}{\partial x^2} \quad (10)$$

Applying the HJB equation (see for example Oksendal, 1998) we get:

$$\begin{cases} \inf_{v \in \Re}[e^{-\rho t} L(t, X(t), v) + A^v H(t,x)] = 0 \ldots\ldots \forall (t,x) \in G \\ H(t,x) = K(t,x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \forall (t,x) \in \partial G \end{cases} \quad (11)$$

Applying (11) we obtain:

$$\inf_{y \in \Re}\left\{e^{-\rho t}[(F(t) - x)^2 + \alpha(F(t) - x)] + \frac{\partial H}{\partial t} + \{x[y(\lambda - r) + r] - b_0\}\frac{\partial H}{\partial x} + \frac{1}{2}x^2 y^2 \sigma^2 \frac{\partial^2 H}{\partial x^2}\right\} = 0$$

(12)

with the boundary condition:

$$H(T, x) = K(T, x), \tag{13}$$

To have an easier notation, let us define:

$$\Phi(y,t,x) := \left\{ e^{-\rho t}[(F(t)-x)^2 + \alpha(F(t)-x)] + \frac{\partial H}{\partial t} + \{x[y(\lambda - r) + r] - b_0\}\frac{\partial H}{\partial x} + \frac{1}{2}x^2 y^2 \sigma^2 \frac{\partial^2 H}{\partial x^2} \right\} \tag{14}$$

Equation (12) becomes:

$$\inf_y \Phi(y, t, x) = 0 \qquad \rightarrow \qquad \Phi(y^*, t, x) = 0 \tag{15}$$

The first and second order conditions are:

$$\Phi'_y(y^*, t, x) = 0 \tag{16}$$
$$\Phi''_{yy}(y^*, t, x) > 0 \tag{17}$$

therefore:

$$x(\lambda - r)\frac{\partial H}{\partial x} + x^2 y^* \sigma^2 \frac{\partial^2 H}{\partial x^2} = 0$$

so that:

$$y^* = \frac{r-\lambda}{x\sigma^2}\frac{H'_x}{H''_{xx}} \tag{18}$$

The sufficient condition is satisfied if and only if:

$$x^2\sigma^2 \frac{\partial^2 H}{\partial x^2} > 0, \text{ that holds if and only if: } \qquad \frac{\partial^2 H}{\partial x^2} > 0 \tag{19}$$

We will show later that this condition is actually satisfied, so that the solution is a minimum.

By substituting (18) into (15) we obtain:

$$0 = e^{-\rho t}[(F(t)-X(t))^2 + \alpha(F(t)-X(t))] + \frac{\partial H}{\partial t} + (rX(t) - b_0)\frac{\partial H}{\partial x} - \frac{1}{2}\left(\frac{r-\lambda}{\sigma}\right)^2 \frac{H'^2_x}{H''_{xx}} \tag{20}$$

We try a solution of the form:

$$H(t, X(t)) = e^{-\rho t} [A(t)X(t)^2 + B(t)X(t) + C(t)] \tag{21}$$

The boundary condition (13) becomes:

$$\varepsilon\, e^{-\rho T} [(F(T)-X(T))^2 + \alpha\, (F(T) – X(T))] = e^{-\rho T} [A(T)X(T)^2 + B(T)X(T) + C(T)]$$

$$\begin{cases} A(T) = \varepsilon \\ B(T) = -\varepsilon(2F(T) + \alpha) \\ C(T) = \varepsilon(F(T)^2 + \alpha F(T)) \end{cases} \tag{22}$$

The partial derivatives of H are:

$$\begin{cases} H'_t = -\rho e^{-\rho t}[A(t)x^2 + B(t)x + C(t)] + e^{-\rho t}[A'(t)x^2 + B'(t)x + C'(t)] \\ H'_x = e^{-\rho t}[2A(t)x + B(t)] \\ H''_{xx} = 2e^{-\rho t}A(t) \end{cases} \tag{23}$$

From (18) we derive the optimal investment strategy at time t:

$$y^*(t, x) = \frac{r - \lambda}{\sigma^2}\left(1 + \frac{B(t)}{2A(t)x}\right) \tag{24}$$

Substituting the partial derivatives of H in (20) we have:

$$\begin{aligned} 0 = &\{1 – \rho\, A(t) + A'(t) – \beta^2 A(t) + 2rA(t)\}\, X(t)^2 + \\ &+ \{B'(t) – 2\, F(t) – \alpha – \rho\, B(t) + rB(t) – 2b_0\, A(t) – \beta^2 B(t)\}\, X(t) + \\ &+ \{F(t)^2 + \alpha F(t) - \rho C(t) + C'(t) – b_0 B(t) – \beta^2\, \frac{B(t)^2}{4A(t)}\} \end{aligned} \tag{25}$$

by defining: $\beta = \dfrac{r - \lambda}{\sigma}$.

Since (25) must hold $\forall$ (t, X(t)), we obtain the following system of ordinary differential equations:

$$\begin{cases} A'(t)=[\rho+\beta^2-2r]A(t)-1=aA(t)-1 \\ B'(t)=[\rho+\beta^2-r]B(t)+(2F(t)+\alpha)+2b_0A(t)=(a+r)B(t)+2F(t)+2b_0A(t)+\alpha \\ C'(t)=\rho C(t)-F(t)^2-\alpha F(t)+b_0B(t)+\beta^2B(t)^2(4A(t))^{-1} \end{cases} \quad (26)$$

by defining a:= $[\rho + \beta^2 - 2r]$ and with the boundary conditions (22).

SOLUTION OF THE PROBLEM

We have solved the problem with two definitions for the targets.

Case 1: targets constant over time.
We assume that $F(t) = F\ \forall t$.

The solution of (26) is:

$$\begin{cases} A(t)=(\varepsilon-\frac{1}{a})e^{-a(T-t)}+\frac{1}{a} \\ B(t)=-\varepsilon(2F+\alpha)e^{-(a+r)(T-t)}-\frac{2F+\alpha+2b_0/a}{a+r}(1-e^{-(a+r)(T-t)})-\frac{2b_0(a\varepsilon-1)}{ar}(e^{-a(T-t)}-e^{-(a+r)(T-t)}) \\ C(t)=\varepsilon(F^2+\alpha F)e^{-a(T-t)}-\rho^{-1}(F^2+\alpha F)(e^{-\rho(T-t)}-1)-e^{-\rho(T-t)}\left[b_0\int_t^T e^{\rho(T-s)}B(s)ds+\beta^2\int_t^T e^{\rho(T-s)}\frac{B(s)^2}{4A(s)}ds\right] \end{cases}$$

(27)

The condition $H''_{xx} > 0$ is also satisfied. In fact:

$$H''_{xx} = 2e^{-\rho t}A(t) = 2e^{-\rho t}\left(\varepsilon e^{-a(T-t)} + a^{-1}(1-e^{-a(T-t)})\right)$$

If a > 0, then $\left(\varepsilon e^{-a(T-t)} + a^{-1}(1-e^{-a(T-t)})\right) > 0$, obviously.

If a < 0, then $\left(\varepsilon e^{-a(T-t)} + a^{-1}(1-e^{-a(T-t)})\right) > 0$, because $a^{-1} < 0$ and also $1 - e^{-a(T-t)} < 0$.

Case 2: exponential targets.
If we consider targets that vary over time, a reasonable choice for the target can be the price of a certain level annuity. Considering that mortality can occur at any time t after retirement, we assume that the annuity will be paid continuously for $\Omega - t$ years (with Ω being expected remaining time to death from retirement) with certainty.

Therefore, the actual value at the rate of return r of the riskless asset of the annuity which pays an amount b_1 per unit time[3] is:

$$F(t) = b_1 \int_0^{\Omega - t} e^{-rs} ds = (b_1/r)(1 - e^{-r(\Omega - t)}) = \gamma\,(1 - e^{-r(\Omega - t)}) \tag{28}$$

by defining $\gamma := b_1/r$.

The solution of (26) is now:

$$\begin{cases} A(t) = (\varepsilon - \frac{1}{a}) e^{-a(T-t)} + \frac{1}{a} \\ B(t) = e^{-(a+r)(T-t)} (2\varepsilon\gamma e^{-r(\Omega - T)} - 2\varepsilon\gamma - \varepsilon\alpha) - \frac{2\gamma + \alpha + 2b_0 a^{-1}}{(a+r)} (1 - e^{-(a+r)(T-t)}) - \frac{2b_0(a\varepsilon - 1)}{ar} (e^{-a(T-t)} - e^{-(a+r)(T-t)}) \\ + \frac{2\gamma}{a} e^{r(t-\Omega)} (1 - e^{-a(T-t)}) \\ C(t) = \varepsilon(F(T)^2 + \alpha F(T)) e^{-a(T-t)} - \rho^{-1}(F(T)^2 + \alpha F(T))(e^{-\rho(T-t)} - 1) - e^{-\rho(T-t)} \left[b_0 \int_t^T e^{\rho(T-s)} B(s) ds + \beta^2 \int_t^T e^{\rho(T-s)} \frac{B(s)^2}{4A(s)} ds \right] \end{cases} \tag{29}$$

We observe that also the sufficient condition for the minimum is satisfied, as A(t) does not change when the targets change.

INFINITE TIME HORIZON

As an extreme situation, we have analysed the case where the time horizon is equal to infinity. Therefore the objective is to minimise the following expectation:

$$E_{0,x_0}\left[\int_0^\infty e^{-\rho s} L(s, X(s), y) ds \right] \tag{30}$$

with:

$$G = \{(0,\infty) \times (-\infty, +\infty)\} \tag{31}$$

Targets were chosen fixed over time (F(t) = F for any t) and the bequest function has been chosen equal to 0 (K(t, X(t)) = 0 for any t).

This problem is much easier to solve, because the boundary condition (“transversality condition”) is automatically satisfied and in the trial solution we can separate time and

[3] A reasonable choice for b_1 would be $b_1 > b_0$.

wealth, which we could not do with a finite time horizon and a quadratic (dis)utility function (see Merton, 1971).

The trial solution for H(t, x) is now:

$$H(t, x) = e^{-\rho t}\,[Ax^2 + Bx + C] = e^{-\rho t}\,G(x) \qquad (32)$$

with boundary condition:

$$\lim_{t\to+\infty} H(t, X(t)) = 0 \qquad (33)$$

which is satisfied.

By calculating the partial derivatives H'_t, H'_x and H''_{xx} and replacing them into equation (15) we obtain:

$$\begin{aligned} 0 &= \{1 - \rho A - \beta^2 A + 2rA\}\, X(t)^2 + \\ &+ \{-2F - \alpha - \rho B + rB - 2b_0 A - \beta^2 B\}\, X(t) + \\ &+ \{F^2 + \alpha F - \rho C - b_0 B - \beta^2 B^2/(4A)\} \end{aligned} \qquad (34)$$

and setting equal to 0 the coefficients of X_t^2 , X_t and the known term, we obtain:

$$A = (\rho + \beta^2 - 2r)^{-1} \qquad (35)$$

$$B = \frac{2F + \alpha + 2b_0 A}{r - \rho - \beta^2} \qquad (36)$$

$$C = [F^2 + \alpha F - b_0 B - \beta^2 B^2/(4A)]/\rho \qquad (37)$$

These results correspond to letting $T \to \infty$ in (25).

The optimal investment strategy is:

$$y^*(t, x) = \frac{r - \lambda}{\sigma^2}\left(1 + \frac{B}{2Ax}\right) \qquad (38)$$

which is independent of time.

The sufficient condition for y*(t, x) to be a minimum is A > 0, which does not automatically hold. In fact:

$$A > 0 \qquad \text{iff} \qquad 2r < \rho + \beta^2. \qquad (39)$$

We observe that in all the considered cases the amount invested in the risky asset $y^*X(t)$ is linear in the size $X(t)$ of the fund:

$$y^*X(t) = g(t)\,X(t) + h(t) \tag{40}$$

with g and h functions at most of time, and this is consistent with one of the results of Merton (1971).

SIMULATIONS

We have carried out some simulations to see what is the behaviour of the optimal investment strategy and its appropriateness in terms of:

a) the risk of out living the assets: risk of ruin;
b) average time of ruin, when ruin occurs;
c) the probability of reaching the target (e.g. the desired level of annuity) at any time between retirement and time T;
d) final outcome of the income drawdown option, analysed by looking at the distribution of the annuity that can be bought at time T;
e) how the risk attitude of the individual can affect optimal choices and final results.

ASSUMPTIONS

The assumptions we have made are the following:

- the member retires at the age of 60
- the compulsory age for annuitization is 75
- the expected age of death is 85, therefore Ω = 25 (chosen so that the target at age 75 approximately equals expected present value of a whole life annuity of b_1 per annum)
- the initial fund is X(0)=100
- the assumptions on investment returns parameters are the following: r = 5%; λ = 10%; σ = 20%; ρ =5%;
- the weight given to the loss at the time horizon T is $\varepsilon = 1$;
- the mortality table used in computing b_0 is the Italian projected mortality table (RG48)
- the targets considered are exponential with $b_1 = 3/2\ b_0$
- the values of α considered are 0, 50 and 100, in order to consider different risk profiles.

In discretizing the process we have chosen the time interval h equal to 1 week: this simplification leads to 780 time points in which the pensioner has to decide about the investment strategy and therefore the aim is to find the values of y*(t) for t=0,1,…,779.

We have carried out 1000 Monte Carlo simulations for each value of α. In each simulation we have simulated the Brownian motion (with the discretization chosen) and therefore the behaviour over time (15 years) of the risky asset. In each scenario of market returns, the optimal value y*(t) has been calculated (for t = 0,1,…,779) and then adopted in the growth of the fund.

SIMULATION RESULTS

The results from the simulations provide the following information:

- the optimal investment strategy is analysed by looking at some percentiles (5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th}), mean and standard deviation of the distribution of $y^*(t)$;
- the risk of outliving the assets (ruin probability) is analysed by looking at the probability of outliving the assets (frequency over the 1000 simulations of the event $X(t) < 0$ for some $t < 780$) and the average time of ruin (when ruin occurs) is also calculated;
- the probability of reaching the target is analysed by looking at the probability that the target is hit (frequency over the 1000 simulations of the event $X(t) > F(t)$ for some $t < 780$);
- the final outcome of the income draw-down option is considered by looking at the distribution of the annuity which can be bought at age 75 with the remaining fund (percentiles, mean and standard deviation);
- the effect of risk aversion is considered by comparing results relative to different values of α.

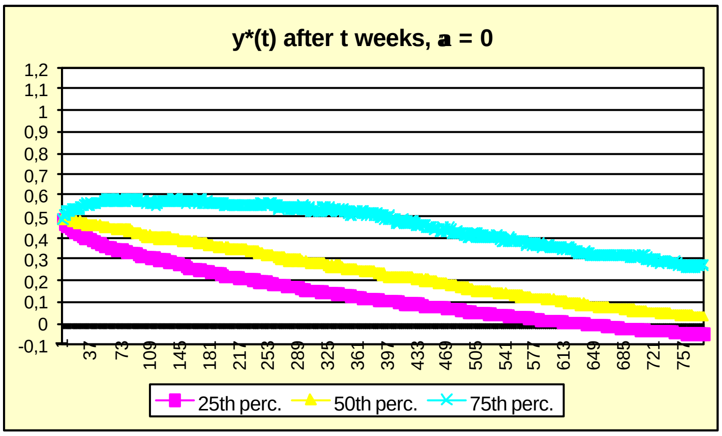
OPTIMAL INVESTMENT STRATEGY

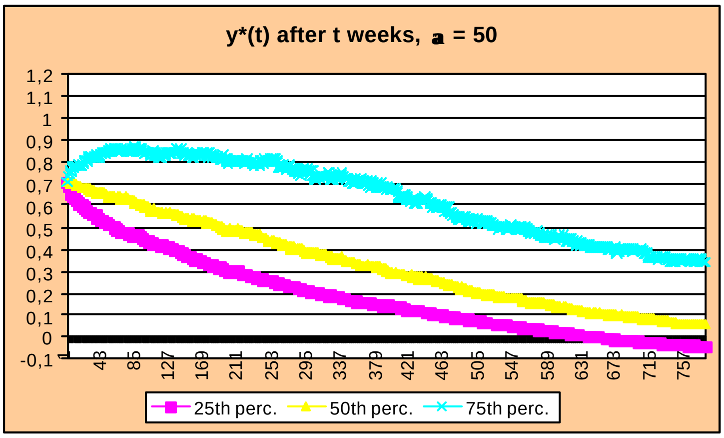
The following graphs show the 25^{th}, 50^{th} and 75^{th} percentiles of the distribution of the optimal investment strategy $y^*(t)$ ($t=0,1,\dots,779$) for the different degrees of risk aversion (the higher the value of α, the lower the risk aversion). In other words, the graphs capture the behaviour over time of 50% of the trajectories of $y^*(t)$ obtained by applying the optimal strategy.

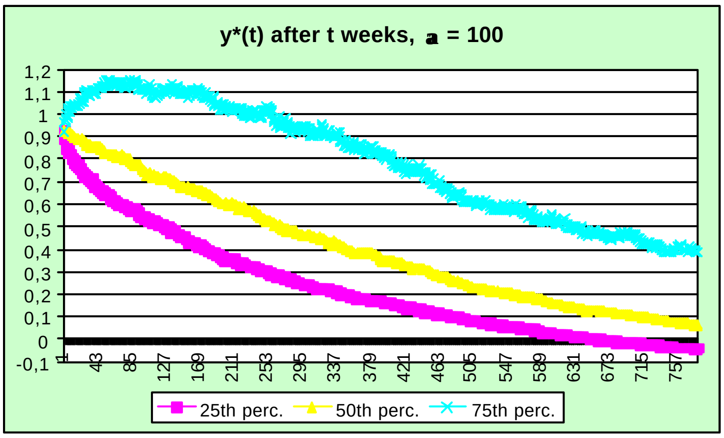
We notice that $y^*(t)$ has a decreasing trend over time, which means that the optimal investment in risky asset seems to decrease as time T approaches.

We observe that the values of $y^*(t)$ are higher with higher values of α, which is intuitive: the lower the risk aversion the higher the proportion of portfolio invested in the risky asset. This can be also proved mathematically, by looking at the expression for $y^*(t)$ (equations (24) and (29)): the function $B(t)$ is decreasing in α, and $y^*(t)$ is decreasing in $B(t)$, therefore $y^*(t)$ is increasing in α.

GRAPH 1: BEHAVIOUR OF $y^*(t)$ OVER TIME







PROBABILITY OF RUIN, OF HITTING THE TARGET AND FINAL ANNUITY

The table that follows reports, for the three different risk profiles, the ruin probability, which is the probability of outliving the assets (calculated as the frequency in the simulations that at a certain time t before T the pensioner runs out of money), the average time of ruin when ruin occurs, the probability of hitting the target and some percentiles of the distribution of the annuity that can be bought at time T with the remaining fund (noting that, with the assumptions made, the targeted annuity is 11,34).

	a = 0	a = 50	a = 100
probability of ruin	5,20%	9,10%	12,80%
probability final annuity < target annuity	63,10%	28,40%	19,30%
probability final annuity < initial potential annuity	14,00%	11,20%	10,40%
probability of hitting the target sometime before 75	40,70%	75,10%	85,30%
average ruin time | ruin occurs: weeks/years	459 / 8.8	410 / 7.9	357 / 6.9
final annuity: 5th perc.	3,809	2,966	2,253
final annuity: 25th perc.	8,993	10,757	12,598
final annuity: 50th perc.	10,894	13,606	16,318
final annuity: 75th perc.	11,676	14,822	17,980
final annuity: 95th perc.	12,457	15,873	19,303
final annuity: mean	9,824	11,756	13,936
final annuity: standard deviation	5,357	6,684	8,118

It is a remarkable fact that the probability of failing the target at retirement dramatically decreases when the risk aversion decreases (i.e. when riskier strategies are adopted, observing that $y^*(t)$ is an increasing function of α): from 63% with α=0 to 28% when α=50, down to 19% when α=100. At the same time, we notice that the probability of outliving their own assets does increase when the risk aversion decreases, but not so remarkably: from 5% to 9% up to 13% with the lowest risk aversion.

Furthermore, in our model income drawdown is taken until compulsory annuitization, regardless of the size of the fund reached in the interim and therefore it may happen that sometime before 75 it becomes possible for the member to buy the desired annuity. This happens in 40% of the cases when α=0, in the 75% of the cases when α=50, and in 85% of the cases when α=100. In the real world, member would probably stop the income drawdown plan and buy the annuity with the remaining fund (unless there are some bequest reasons): considering also those people who could buy the desired annuity before 75, the probability of failing the target reduces to 60%, to 25% and to 15%, in the cases α=0, α=50, and α=100 respectively.

Looking at the distribution of the final annuity bought at retirement (graph 2 that follows), we notice that the distribution moves towards the right when risk aversion decreases. Also the mean and the standard deviation of the final annuity increase when α increases. Higher mean and higher standard deviation of the distribution of the final annuity leads to a much longer right tail of the distribution, and to a slightly longer left tail of the distribution, and this results in a good chance of being better off by adopting riskier strategies than by adopting more cautious strategies. A more detailed analysis of the percentiles of the distribution (not shown here) shows that the percentiles from the 8th onwards increase when α increases, whereas the first 7 percentiles decrease when α increases. The three histograms of graph 2 have the same scale, in order to facilitate comparisons between the different values of α. We notice that negative values of the annuity, although reported for the sake of completeness, are not realistic, as in this case the retiree would simply run out of money and stop the process, without continuing by investing negative money (i.e. borrowing).

A detailed comparison between the different aggressiveness of the strategies may be done by considering the percentage of individuals who would be better off (i.e. would receive a higher annuity at retirement) by increasing the target, i.e. increasing the value of α. We found that 932 pensioners would receive a higher annuity if they choose α=50, instead of α=0, 933 if they choose α=100, instead of α=0, 929 if they choose α=100, instead of α=50. This means that 93% of pensioners would end up with a higher income at age 75 by adopting more aggressive strategies during the income drawdown option.

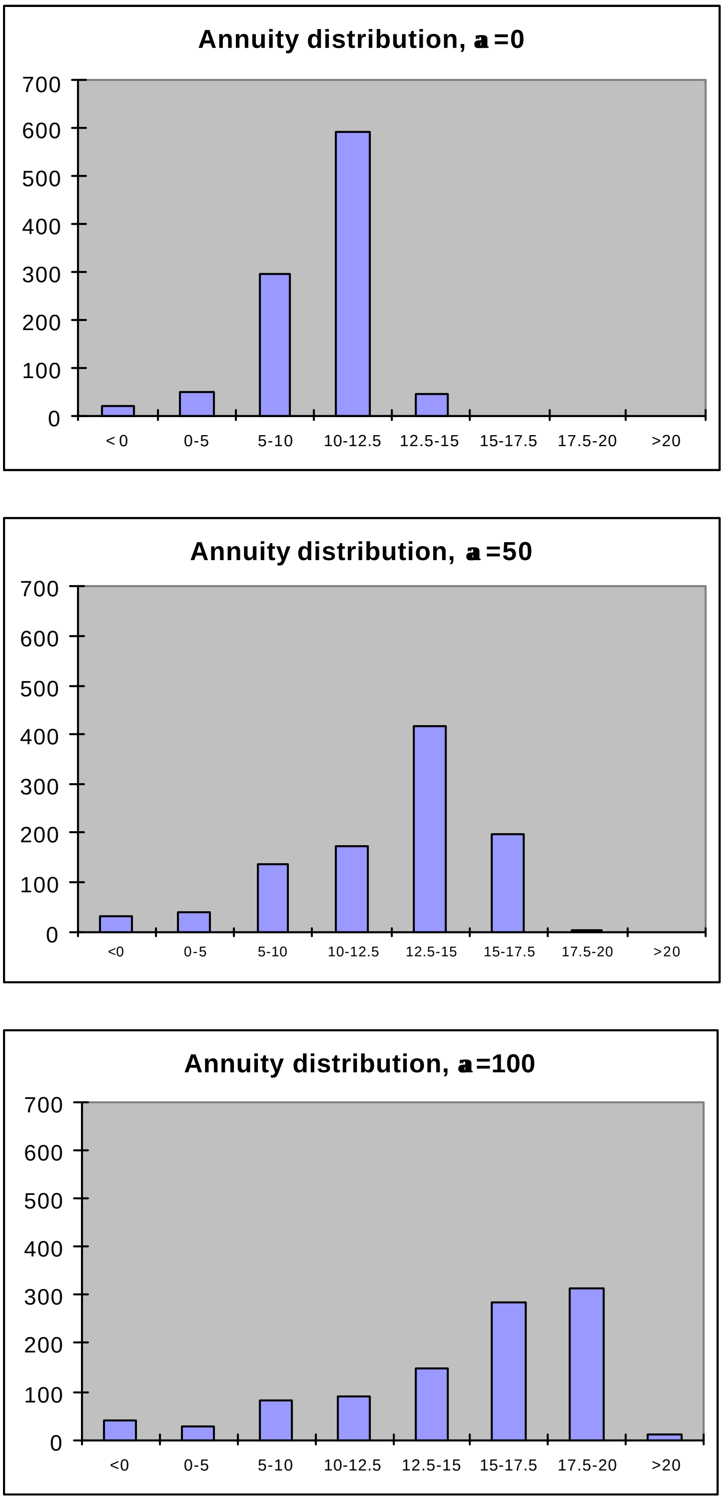
Another interesting result relative to the aggressiveness of the strategy adopted may be obtained by looking at the best strategy (in terms of highest annuity at retirement) in each of the 1000 scenarios of market returns. In graph 3, two histograms report what would be the "optimal α" (of the 3 choices investigated) to be adopted when comparing the final annuity in each of the 1000 simulations carried out.

In the first histogram, when ruin occurs, the individual is allowed to continue investing in the market (short selling the risky asset and investing in the riskless asset), so that in many cases the fund ends up being positive again. In contrast, in the second histogram, ruin is not allowed: if the fund becomes negative, the investor stops investing money and becomes bankrupt. In this histogram, the label "ruin" collects all individuals who invest the fund in a period of such an unfavourable scenario of market returns that ruin occurs with the three values of α. The number of individuals who outlive their assets, no matter what investment strategies they adopt, is 52, i.e. 0.5% of the total number of individuals (i.e. market scenarios) considered. The surprising result is that the value of α=100 is dominant in both graphs, with about 900 pensioners being better off by adopting riskier strategies (in particular, 921 when further investment is allowed in the case of ruin and 857 when further investment is not allowed).

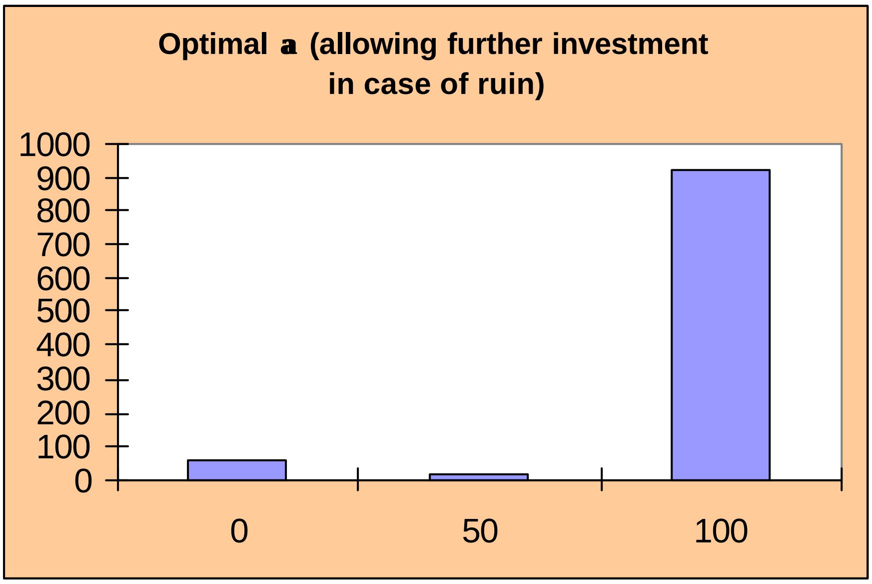
The price that one has to pay when investing in riskier strategies (higher probability of failing the target) seems to be largely compensated by the extra return on riskier asset, which leads to a much higher chance of being better off when the final annuity is bought.

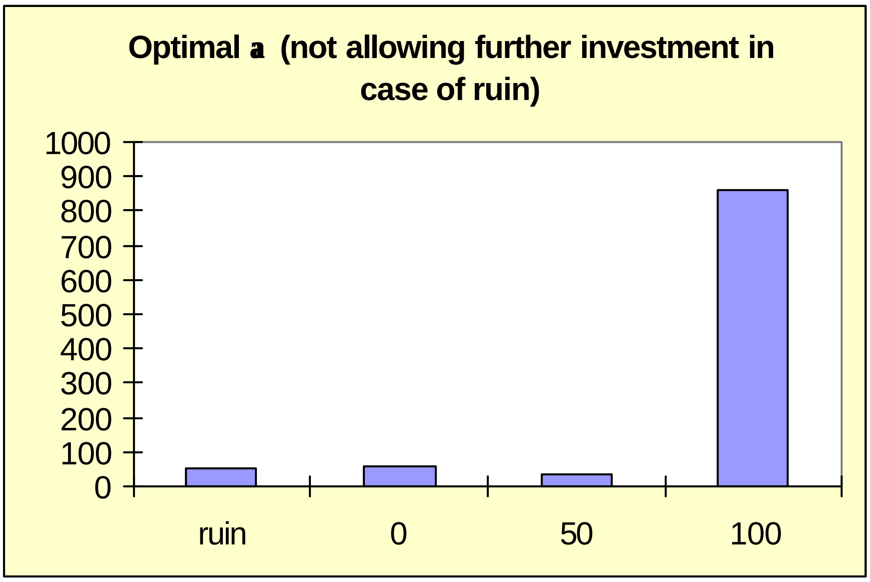
However, we notice that the average time of ruin, given that ruin occurs, decreases when increasing the aggressiveness of the strategy: from 9 years with α=0 to 8 years with α=50 down to 7 years with α=100. This means that, with riskier strategies, ruin occurs earlier on average than with more cautious strategies.

GRAPH 2: DISTRIBUTION OF THE FINAL ANNUITY



GRAPH 3: OPTIMAL a RESPECT TO FINAL ANNUITY





REFERENCES

Albrecht, P., Maurer, R. (2002). *Self-Annuitization, Consumption Shortfall in Retirement and Asset Allocation: The Annuity Benchmark*, Journal of Pension Economics and Finance, vol **1**, 269-288.

Björk, T. (1998). *Arbitrage Theory in Continuous Time*, Oxford University Press.

Blake, D., Cairns, A.J.G., Dowd, K. (2002). *Pensionmetrics 2: Stochastic Pension Plan Design during the Distribution Phase*, mimeo.

Boulier, J-F., Trussant, E., Florens, D. (1996). *A dynamic model for pension funds management*, Proceedings of the 5th AFIR International Colloquium **1**, 361-384.

Boulier, J-F., Michel, S., Wisnia, V. (1996). *Optimising investment and contribution policies of a defined benefit pension fund*, Proceedings of the 6th AFIR International Colloquium **1**, 593-607.

Cairns, A. (2000). *Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time*, ASTIN Bulletin, Vol. **30**, No. 1, 19-55.

Findlater A. *The annuity – a proposal*, The Actuary, May 1999.

Gerrard, R., Haberman, S., Vigna, E. (2003), *The income drawdown option: quadratic loss*, mimeo.

Haberman, S., Vigna, E. (2002). *Optimal Investment Strategies and Risk Measures in Defined Contribution Pension Schemes*, Insurance: Mathematics and Economics 31, 35-69.

Kapur S., Orszag J.M. *A Portfolio Approach to Investment and Annuitization During Retirement*, mimeo.

Khorasanee M.Z. *Annuity Choices for Pensioners*, Journal of Actuarial Practice, vol. 4, No. 2, 1996.

Lunnon, M. (2002). *Annuitization and alternatives*, Trans 27th ICA.

Merton, R. (1969). *Lifetime Portfolio Selection under Uncertainty. The Continuous Time Case*, Review of Economics and Statistics **51**, 247-257.

Merton, R. (1971). *Optimum Consumption and Portfolio Rules in a Continuous Time Model*, Journal of Economic Theory **3**, 373-413.

Milevsky M. A. *Optimal Asset Allocation Towards The End of the Life Cycle: To Annuitize or Not to Annuitize?*, The Journal of Risk and Insurance, vol. **65**, No. 3, 1998.

Milevsky M. A. *Optimal Annuitization Policies: Analysis of the Options*, North American Actuarial Journal, vol. **5**, No. 1, 2001.

Milevsky M. A., Young, V. R. (2002) *Optimal Asset Allocation and The Real Option to Delay Annuitization: It's Not Now-or-Never*, mimeo.

Milevsky M. A., Robinson C. *Self- Annuitization and Ruin in Retirement*, North American Actuarial Journal, vol. **4**, No 4, 2000.

Mitchell, O., S. (2001). *Developments in Decumulation: The Role of Annuity Products in Financing Retirement*, The Pension Institute, Discussion Paper PI-0110.

Mitchell, O., S., Poterba, J. M., Warshawsky, M. J., Brown, J. R. (1999), *New Evidence on the Money's Worth of Individual Annuities*, American Economic Review, 89: 1299-1318.

Øksendal, B. (1998). *Stochastic Differential Equations, 5th Edition*, Springer Verlag, Berlin.

Wadsworth, M., Findlater, A., Boardman, T. (2001). *Reinventing annuities*, Staple Inn Actuarial Society, London, January 2001.