PRICING OF GUARANTEED INDEX-LINKED PRODUCTS BASED ON LOOKBACK OPTIONS

Jochen Ruß
Abteilung Unternehmensplanung
University of Ulm
89069 Ulm
Germany

Tel.: +49 731 50 23592 /-23556
Fax: +49 731 50 23585
email: russ@mathematik.uni-ulm.de

Abstract

This paper introduces a new class of guaranteed index-linked products, where the payoff function depends on extreme values of the underlying index in certain time intervals. Within a Black/Scholes model environment, we derive closed-form pricing formulas for securities with such payoff functions using well-known results about lookback options. We then use our pricing formulas to calculate the fair rate of index participation (i.e. the percentage rate by which the investor participates in index gains) using actual market data. We analyse in detail how the rate of index participation depends on the product design and on the given market conditions.

1 Introduction

Guaranteed index-linked products are becoming more and more popular in Germany. This holds for both, pure investment contracts as well as insurance products like guaranteed equity-linked life insurance policies. The common feature of all these products is that the investor is guaranteed a certain deterministic prefixed amount at maturity. Additionally, the contract is linked (via the so-called rate of index participation) to an underlying equity-index. If the index performes well over the term of the contract, the payoff may exceed the guaranteed sum. In a series of papers (cf., e.g., [No 96], [No/Sc 97], [No/Ru 97a], [No/Ru 97b], [No/Ru 98], [Ru/Sc 98]) focussing especially on insurance products, our research team has investigated such contracts in great detail. We have, in particular, analysed the pricing of such products and the calculation of policy reserves according to German legislation for different payoff patterns.

The present paper now introduces a new class of products where the payoff function depends on extreme values (maximums and/or minimums) of the underlying index in certain time intervals. The pricing of such products is closely related to the pricing of lookback options. Closed-form solutions for the value of such options in a Black/Scholes model environment were developed in 1979 in [Go/So/Ga 79] (cf. also [Co/Vi 91]). We use these results to derive closed-form solutions for the value A_0 at time 0 of several guaranteed products based on lookback options within the model framework introduced in [No 96]. We derive our formulas for the general case of a term of $T \in I\!N$ years and annual premium payment at the beginning of the first $b \in I\!N$, $b \leq T$ years. We then calculate the fair rate of index participation and give empirical results for the case $b = 5$, $T = 12$.[1] Furthermore, we analyse in detail how the rate of index participation depends on the product design and the market conditions.

2 Product Design

In this Section, we introduce several index-linked guaranteed products with a term of T years and an annual premium payment at the beginning of the first b years ($b \leq T$, b, $T \in I\!N$).[2] We only look at the investment part of the product meaning that we assume that all involved costs (including acquisition costs, management

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[1]These are the minimum requirements that have to be fulfilled by insurance policies to be privileged with respect to taxes in Germany. Note, that in the case $b = 1$ our formulas can be used for the pricing of single premium contracts.

[2]In what follows, $t = 0$ denotes the start of the contract. Therefore, our contract matures at $t = T$, the end of year T. The premiums are paid at $t = 0, \ldots, b - 1$. Letting $b = 1$, we get a single premium product.

fees and in the case of an insurance policy, the risk premium associated with the part of the sum payable at death that exceeds the value of the investment[3]) have been subtracted from certain gross premiums leading to b equal net premiums NP.[4]

These b net premiums are used to buy a security with payoff A_T at time T. Here, A_T depends on an (equity) index but is guaranteed to exceed a certain deterministic minimum payoff G. Usually, G is generated by earning some guaranteed rate of interest $i_g \geq 0$ on the net premiums, i.e. $G = NP \sum_{i=1}^{b} (1 + i_g)^{T-i+1}$.

We will introduce some payoff functions depending on extreme values of the underlying index in certain time intervals. In order to demonstrate the dependence of the rate of index participation[5] on certain properties of the product, we introduce a rather high number of six different products. We will show that differences between two products that seem to be small[6] may have an extreme impact on the rate of index participation. Hence, from a marketing point of view, it might be worthwile to create a product that allows for a high rate of index participation.

We denote by S_t the value of the underlying index at time t. In particular, for $j \in \mathbb{N}$, S_j is the value of the index at the end of year j. For $t_1 \leq t_2$, we furthermore define $M_{t_1}^{t_2} = \max_{t_1 \leq \tau \leq t_2} S_\tau$ and $m_{t_1}^{t_2} = \min_{t_1 \leq \tau \leq t_2} S_\tau$.

The payoff functions at time T of our six products are denoted by A_T^k, $k = 1, \ldots, 6$. We let x_k, $k = 1, \ldots, 6$ denote the corresponding rate of index participation[7] and let $G = NP \sum_{i=0}^{b-1} (1 + i_g)^{T-i}$. The six payoff functions are:

$$A_T^1 = G + NP \sum_{i=1}^{b} \frac{M_{i-1}^T - S_{i-1}}{S_{i-1}} x_1$$

$$A_T^2 = NP \sum_{i=1}^{b} \prod_{j=i}^{T} \left(1 + \max \left[\frac{M_{j-1}^j - S_{j-1}}{S_{j-1}} x_2, i_g \right] \right)$$

$$A_T^3 = NP \sum_{i=1}^{b} \left(1 + \sum_{j=i}^{T} \max \left[\frac{M_{j-1}^j - S_{j-1}}{S_{j-1}} x_3, i_g \right] \right)$$

[3]That this part of the death benefit can be treated independently of the investment part of the product is shown in [No/Ru 98].

[4]For a detailled discussion of these topics see, e.g., [No/Sc 97].

[5]which is also called index participation rate or participation rate

[6]or might not even be seen by an investor who is not too familiar with such products

[7]Note that the participation rate of a lookback contract is obviously smaller than of an analogous non-lookback-contract (i.e. a contract, where in the payoff-pattern $(S_T - S_i)^+$ and $(S_j - S_{j-1})^+$ are used instead of $(M_i^T - S_i)$ and $(M_{j-1}^j - S_{j-1})$, respectively), cf. equation (5). Nevertheless, the products introduced here are very innovative. Furthermore, it should be easy to explain to an investor that in order to always get the maximum value of the index, he has to accept a smaller rate of index participation.

$$A_T^4 = G + NP \sum_{i=1}^{b} \sum_{j=i}^{T} \frac{M_{j-1}^j - S_{j-1}}{S_{j-1}} x_4$$

$$A_T^5 = G + NP \sum_{i=1}^{b} \frac{M_{i-1}^T - m_{i-1}^T}{S_{i-1}} x_5$$

$$A_T^6 = NP \sum_{i=1}^{b} \prod_{j=i}^{T} \left(1 + \max\left[\frac{M_{j-1}^j - S_{j-1}}{S_{j-1}} x_6, i_g \right] \right.$$

$$\left. - \max\left[\frac{M_{j-1}^j - S_{j-1}}{S_{j-1}} x_6 - i_c, 0 \right] \right)$$

In product 1, we determine for each net premium, how much the maximum index value between the time of premium payment and maturity T exceeds the index value at the time of premium payment. At maturity, x_1 times the sum of these returns (applied to the net premiums that have already been paid) is paid additionally to the guaranteed sum G.

In product 2, we determine in every year, how much the maximum index value in this year exceeds the index value at the beginning of the year.[8] This return, multiplied by x_2, or i_g, whichever is higher, is then earned on the net premiums that have already been paid and on the previous gains. Hence, gains that are earned in a specific year are reinvested which leads to a compound interest effect. Products with these features are often called cliquet products with compound interest.

Product 3 is similar to product 2, but here, the gains that are earned are not reinvested but rather added up and paid at maturity (cliquet without compound interest). Note, that this is the only product where the guaranteed sum is not G but rather $\tilde{G} = NP \sum_{i=1}^{b} \left(1 + (T - i + 1)i_g \right) \leq G$.[9]

In product 4, like in products 2 and 3, the return of the maximum index value of each year compared to the index value at the beginning of the year is determined. Here, however, we do not compare this return to i_g but rather add x_4 times this return (applied to the net premiums already paid) to the guaranteed sum G. This is an additive variant of product 3. Here, we have a cliquet product without compound interest where the guaranteed sum is G.

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[8]This kind of locking in gains during the term of the policy is called cliquet version.

[9]Obviously, $G = \tilde{G}$ if and only if $i_g = 0$.

The payoff of product 5 is calculated similar to product one, but here, for every net premium, we determine the difference between the maximum and minimum index value between the time of the premium payment and maturity.

Product 6 is a collar version of product 2. The return that is earned in a specific year is calculated as in product 2. If that value, however, exceeds a cap rate $i_c > i_g$, then i_c is earned in that year.

The ideas used in our products can be combined to create a quite arbitrary series of further products.[10] Our six products have been constructed such that it is possible to derive within the model for the economy introduced in Section 3 closed form solutions for the price A_0^k at time 0 of a security that pays off A_T^k at time T. At the end of Section 5, we mention some products for which this is not possible.

3 Our Model for the economy

All our calculations are based on the following assumptions:

- The index follows a geometric Brownian motion.

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma dW_t, \tag{1}$$

where W_t denotes a Wiener process on a probability space (Ω, Σ, P) with a filtration $F = \{F_t\}$. We assume W_t to be adapted with respect to F.[11] Note that $\mu(t)$ is time dependent, whereas $\sigma > 0$ is constant.

Given $S_0 > 0$, the solution of (1) is given by (cf. [Ka/Sh 88])

$$S_t = S_0 e^{\int_0^t \mu(s) - \frac{\sigma^2}{2} ds + \sigma W_t}. \tag{2}$$

In particular, for $0 \le t_1 < t_2$, it follows that $\log \frac{S_{t_2}}{S_{t_1}}$ has a Gaussian distribution with expectation $\int_{t_1}^{t_2} \mu(s) - \frac{\sigma^2}{2} ds$ and variance $\sigma^2(t_2 - t_1)$.

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[10]In [Ru/Sc 98], we analyse a great variety of 42 different non-lookback products including several averaging products. In principal, for each of these products, we can construct two analogous lookback products, one of the type $\frac{M_{t_1}^{t_2} - S_{t_1}}{S_{t_1}}$ and one of the type $\frac{M_{t_1}^{t_2} - m_{t_1}^{t_2}}{S_{t_1}}$.

[11]The filtration is assumed to fulfill the so-called ordinary conditions. This allows for the case where F is the P-completed version of the filtration generated by W_t.

- The short rate process $r(t)$ is deterministic and fits the current riskless term structure of interest rates, i.e.

$$\int_{t_1}^{t_2} r(t)dt = (t_2 - t_1)f_{t_1,t_2},$$

where f_{t_1,t_2} denotes the continuous, annualised, observed forward rate for the period of time $0 \leq t_1 < t_2$.

According to [Ha/Kr 79] and [Ha/Pl 81], the value A_t^k at time t is given by

$$A_t^k = E_Q \left[e^{-\int_t^T r(s)ds} A_T^k \mid_t \right], \quad 0 \leq t \leq T, \tag{3}$$

where $E_Q[\cdot \mid_t] := E_Q[\cdot \mid F_t]$ denotes the conditional expected value under the information available at time t according to an equivalent martingale measure Q.[12] As a consequence of this change of measure, in (1) and (2), $\mu(t)$ is substituted by $r(t)$. This enables us to explicitly calculate the expected value in (3) for all our products.

For all the following calculations, we assume the underlying index to be the DAX30, a performance index, and use market data as of July 18, 1997. Hence, we use a volatility of $\sigma = 15.38\%$ and the following term structure of interest rates (in %):

t	1	2	3	4	5	6	7	8	9	10	11	12
$f_{0,t}$	3.4	3.75	4.13	4.49	5.14	5.4	5.63	5.81	5.97	5.81	6.06	6.15

4 Pricing Formulas for our Products

In [Co/Vi 91] and [Go/So/Ga 79], closed form solutions for the value of different lookback options are derived. This includes pricing formulas for the value at time t, $t_1 \leq t \leq t_2$ of standard lookback calls and standard lookback puts (these are securities with payoff functions $[S_{t_2} - m_{t_1}^{t_2}]$ and $[M_{t_1}^{t_2} - S_{t_2}]$, respectively) as well as for so-called options on extrema (securities with payoff functions $\max[M_{t_1}^{t_2} - K, 0]$ and $\max[K - m_{t_1}^{t_2}, 0]$, for some $K > 0$).[13]

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[12]The (unique) existence of such a measure Q is essentially equivalent to the assumption of a complete, arbitrage-free market. The name martingale measure corresponds to the fact, that under Q, the discounted price process $S_t^* = e^{-\int_0^t r(s)ds} S_t$ is a martingale. For a detailled discussion, cf., e.g., [Du 96].

[13]These two different options on extrema are often called "call on the maximum" and "put on the minimum", respectively.

To derive pricing formulas for the products introduced in Section 2, we need similar formulas for the case $t < t_1 \leq t_2$. These can be derived from those for $t_1 \leq t \leq t_2$ by a simple idea. We will explain this idea for the first two products and then give the corresponding pricing formulas for the other products without proof.[14]

To calculate the fair rate of index participation, we only need formulas for A_0^k.[15] The basic idea that enables us to derive the desired pricing formulas is the simple equation

$$E\left[E[.\ |_j]\ |_0\right] = E[.\ |_0],\ j \geq 0. \tag{4}$$

All our products are designed such that all included options are performance options. Hence, when we apply (4), in the case $j > 0$, all F_j-measurable terms cancel and there only remain F_0-measurable terms, that are not stochastic.

Letting $G_0 = e^{-Tf_{0,T}}G$, we get for product 1:[16]

$$A_T^1 = G + NP \sum_{i=0}^{b-1} \frac{M_i^T - S_i}{S_i} x = G + xNP \sum_{i=0}^{b-1} \frac{1}{S_i}\left((M_i^T - S_T) + (S_T - S_i)\right)$$

$$\Rightarrow A_0^1 = E_Q\left[e^{-Tf_{0,T}}A_T^1\ |_0\right]$$

$$= xNP \sum_{i=0}^{b-1} E_Q\left[e^{-if_{0,i}}\frac{1}{S_i}E_Q\left[e^{-(T-i)f_{i,T}}\left((M_i^T - S_T) + S_T - S_i\right)\ |_i\right]\ |_0\right] + G_0$$

Here, $M_i^T - S_T$ is the payoff function of a standard lookback put. Given the information available at time i, the pricing formula given in [Co/Vi 91] can be applied yielding

$$A_0^1 = xNP \sum_{i=0}^{b-1} E_Q\left[e^{-if_{0,i}}\frac{1}{S_i}S_i\left[-N(-d_1^i) + e^{-(T-i)f_{i,T}}N(-d_2^i)\right.\right.$$

$$
\begin{aligned}
&+\; e^{-(T-i)f_{i,T}}\frac{\sigma^2}{2f_{i,T}}\left(N\left(d_1^i-\frac{2f_{i,T}}{\sigma}\sqrt{T-i}\right)+e^{(T-i)f_{i,T}}N(d_1^i)\right)\\
&+\; 1-e^{-(T-i)f_{i,T}}\Bigg]\ |_0\Bigg]+G_0,
\end{aligned}
$$

where $d_1^i=\frac{(f_{i,T}+\frac{\sigma^2}{2})(T-i)}{\sigma\sqrt{T-i}}$, $d_2^i=d_1^i-\sigma\sqrt{T-i}$ and $N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{s^2}{2}}ds$.

Hence, after slight reshuffling, we get

$$
\begin{aligned}
A_0^1 \;=\;& Dx\sum_{i=0}^{b-1}\left\{e^{(T-i)f_{i,T}}N(d_1^i)-N(d_2^i)\right.\\
&+\; \frac{\sigma^2}{2f_{i,T}}\left(N\left(d_1^i-\frac{2f_{i,T}}{\sigma}\sqrt{T-i}\right)+e^{(T-i)f_{i,T}}N(d_1^i)\right)\Bigg\}+G_0,
\end{aligned}
$$

where $D=e^{Tf_{0,T}}NP$.

For product 2, we get

$$
\begin{aligned}
A_T^2 \;=\;& NP\sum_{i=1}^{b}\prod_{j=i}^{T}\left(1+\max\left[\frac{M_{j-1}^j-S_{j-1}}{S_{j-1}}x,i_g\right]\right)\\
=\;& NP\sum_{i=1}^{b}\prod_{j=i}^{T}\left(1+i_g+\frac{x}{S_{j-1}}\max\left[M_{j-1}^j-S_{j-1}\left(1+\frac{i_g}{x}\right),0\right]\right)
\end{aligned}
$$

Here, $C_{j-1}:=\max\left[M_{j-1}^j-S_{j-1}(1+\frac{i_g}{x}),0\right]$ is the payoff function of a call on the maximum, maturing at $t=j$, starting at $t=j-1$ with strike $S_{j-1}(1+\frac{i_g}{x})$. Applying (3), we get

$$
A_0^2=E_Q\left[e^{-Tf_{0,T}}A_T\ |_0\right]
$$

$$
=E_Q\left[D\sum_{i=1}^{b}\prod_{j=i}^{T}e^{f_{j-1,j}}E_Q\left[e^{-f_{j-1,j}}\left(1+i_g+\frac{x}{S_{j-1}}C_{j-1}\right)\ |_{j-1}\right]\ |_0\right].
$$

The inner expected value can again be calculated using results from [Co/Vi 91]. As above, the value of C_{j-1} is a multiple of S_{j-1}. After cancelling, there are no stochastic terms left, and hence we get

$$
\begin{aligned}
A_0^2 &= D \sum_{i=1}^{b} \prod_{j=i}^{T} \left\{ 1 + i_g + x \left(e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(i_g)\right) - \frac{x+i_g}{x} N\left(\tilde{d}_2^{j-1}(i_g)\right) \right. \right. \\
&\quad + \left. \left. \frac{\sigma^2}{2f_{j-1,j}} \left[-\left(\frac{x}{x+i_g}\right)^{\frac{-2f_{j-1,j}}{\sigma^2}} N\left(\hat{d}_1^{j-1}(i_g)\right) + e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(i_g)\right) \right] \right) \right\},
\end{aligned}
$$

where $\tilde{d}_1^{j-1}(i_g) = \frac{\ln\frac{x}{x+i_g} + f_{j-1,j} + \frac{\sigma^2}{2}}{\sigma}$, $\tilde{d}_2^{j-1}(i_g) = \tilde{d}_1^{j-1}(i_g) - \sigma$ and $\hat{d}_1^{j-1}(i_g) = \tilde{d}_1^{j-1}(i_g) - \frac{2f_{j-1,j}}{\sigma}$.

Similarly, we get for the other products:

$$
\begin{aligned}
A_0^3 &= D \sum_{i=1}^{b} \left\{ 1 + \sum_{j=i}^{T} \left(i_g + x \left(e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(i_g)\right) - \frac{x+i_g}{x} N\left(\tilde{d}_2^{j-1}(i_g)\right) \right. \right. \right. \\
&\quad + \left. \left. \left. \frac{\sigma^2}{2f_{j-1,j}} \left[-\left(\frac{x}{x+i_g}\right)^{\frac{-2f_{j-1,j}}{\sigma^2}} N\left(\hat{d}_1^{j-1}(i_g)\right) + e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(i_g)\right) \right] \right) \right) \right\}
\end{aligned}
$$

$$
\begin{aligned}
A_0^4 &= Dx \sum_{i=1}^{b} \sum_{j=i}^{T} \left\{ e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(0)\right) - N\left(\tilde{d}_2^{j-1}(0)\right) \right. \\
&\quad + \left. \frac{\sigma^2}{2f_{j-1,j}} \left[-N\left(\hat{d}_1^{j-1}(0)\right) + e^{f_{j-1,j}} N\left(\tilde{d}_1^{j-1}(0)\right) \right] \right\} + G_0
\end{aligned}
$$

$$
\begin{aligned}
A_0^5 &= G_0 + Dx \sum_{i=0}^{b-1} \left\{ -e^{(T-i)f_{i,T}} \left(-1 + 2N(d_1^i)\right) + 1 - 2N(d_2^i) \right. \\
&\quad + \left. \frac{\sigma^2}{2f_{i,T}} \left(1 - 2N\left(d_1^i - \frac{2f_{i,T}}{\sigma}\sqrt{T-i}\right) + e^{(T-i)f_{i,T}} \left(-1 + 2N(d_1^i)\right) \right) \right\}.
\end{aligned}
$$

$$
\begin{aligned}
A_0^6 \;=\; & D\sum_{i=1}^{b}\Bigg\{ \prod_{j=i}^{T}\Bigg(1+i_g+e^{f_{j-1,j}}x\left[N\left(\bar{d}_1^{j-1}(i_g)\right)-N\left(\bar{d}_1^{j-1}(i_c)\right)\right] \\
& -\;(x+i_g)N\left(\bar{d}_2^{j-1}(i_g)\right)+(x+i_c)N\left(\bar{d}_2^{j-1}(i_c)\right) \\
& +\;x\frac{\sigma^2}{2f_{j-1,j}}\left[\left(\frac{x}{x+i_c}\right)^{\frac{-2f_{j-1,j}}{\sigma^2}}N\left(\dot{d}_1^{j-1}(i_c)\right)\right. \\
& -\;\left(\frac{x}{x+i_g}\right)^{\frac{-2f_{j-1,j}}{\sigma^2}}N\left(\dot{d}_1^{j-1}(i_g)\right) \\
& +\;e^{f_{j-1,j}}\left(N\left(\dot{d}_1^{j-1}(i_g)\right)-N\left(\dot{d}_1^{j-1}(i_c)\right)\right)\Bigg]\Bigg)\Bigg\}
\end{aligned}
$$

If the same participation rate x and the same guaranteed rate of interest i_g is used in all products, then obviously, $\forall i_g \geq 0$, $\forall x > 0$: $A_T^2 \geq A_T^3$, $A_T^2 \geq A_T^6$, $A_T^4 \geq A_T^3$, $A_T^5 \geq A_T^1$ and hence, $\forall i_g \geq 0$, $\forall x > 0$: $A_0^2(x, i_g) \geq A_0^3(x, i_g)$, $A_0^2(x, i_g) \geq A_0^6(x, i_g)$, $A_0^4(x, i_g) \geq A_0^3(x, i_g)$, $A_0^5(x, i_g) \geq A_0^1(x, i_g)$ (cf. figures 1-6). Furthermore, the A_0^k are increasing in x_k, i_g and σ and A_0^6 is decreasing in i_c.

The following figures show the A_0^k as functions of x and i_g. In product 6, we let $i_c = 10\%$. In all products, we let $b = 5$ and $T = 12$ years.
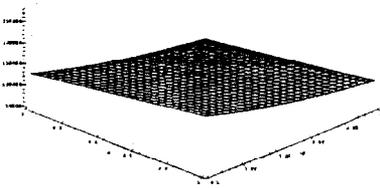
Figure 1: $A_0^1(x, i_g)$
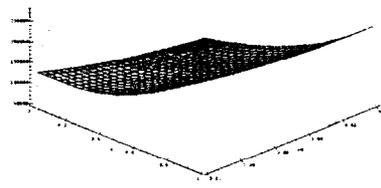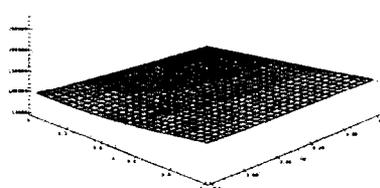

Figure 2: $A_0^2(x, i_g)$
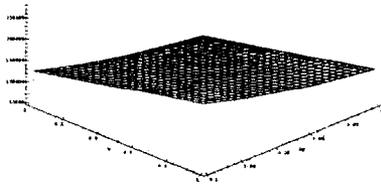

Figure 3: $A_0^3(x, i_g)$
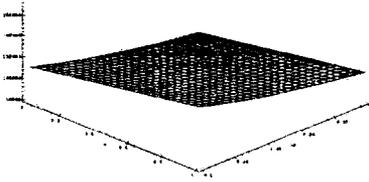

Figure 4: $A_0^4(x, i_g)$

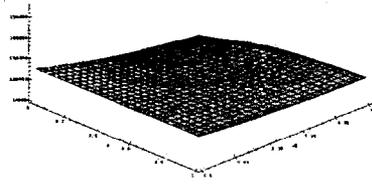Figure 5: $A_0^5(x, i_g)$ Figure 6: $A_0^6(x, i_g)$

5 The Rate of Index Participation

Given the term structure of interest rates, the index volatility, the guaranteed rate of interest and the cap rate (if applicable), the fair rate of index participation $x = x_k$ is calculated as positive solution of the (implicit) equation

$$A_0^k = A_0^k(x) = NP \sum_{i=0}^{b-1} e^{-i f_{0,i}} = \text{present value of the net premiums.} \qquad (5)$$

if such a solution exists.[17]

According to our remark at the end of Section 4, we obviously get for given $i_g \geq 0$, if x is calculated from (5): $x_2 \leq x_3$, $x_2 \leq x_6$, $x_4 \leq x_3$, $x_5 \leq x_1$ (cf. figures 7 - 12).

The following table shows for $b = 5$, $T = 12$, different values of i_g und i_c and different market scenarios the solutions of (5) (in %). Here, Δr denotes a parallel shift of the term structure of interest rates by $\Delta r\%$, and $\Delta \sigma$ a change of volatility by $\Delta \sigma\%$. Furthermore, x_k denotes the solution of (5) for product k, $k = 1, \ldots, 5$ and $x_6(i_c)$ the solution of (5) for product 6 with a cap rate of $i_c\%$.

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[17] Obviously, for product k, $k = 1, \ldots, 5$ there exists a constant i_k^*, such that there exists such a solution if and only if $i_g < i_k^*$. The solution is then unique. In case of product 6, there exists a constant i_5^*, such that there exists no positive solution of (5), if $i_g \geq i_6^*$. If, however, $i_g < i_6^*$, then there exists some $i_c^*(i_g)$, such that there exists a positive solution of (5) if and only if $i_c > i_c^*(i_g)$. The solution is then unique. Furthermore, $x_6 \to \infty$, if $i_c \searrow i_c^*(i_g)$. The constants i_k^*, $k = 1, \ldots, 6$ only depend on the given term structure of interest rates and the way of calculating the guaranteed sum from i_g, but not on other features of the payoff function or on σ. Hence, the i_k^* for $k \neq 3$ are all the same.

i_g	Δr	$\Delta\sigma$	x_1	x_2	x_3	x_4	x_5	$x_6(10)$	$x_6(15)$	$x_6(20)$
0	0	0	73,73	39,55	54,43	54,43	65,28	52,97	41,53	39,94
0	0	-2	78,44	44,06	60,62	60,62	70,77	57,62	45,92	44,39
0	0	2	69,21	35,82	49,29	49,29	60,27	48,98	37,88	36,26
0	-1	0	68,62	34,79	45,48	45,48	58,77	40,45	35,57	34,91
0	-1	-2	73,73	38,96	50,93	50,93	64,68	44,66	39,67	39,06
0	-1	2	63,85	31,37	41,02	41,02	53,75	36,96	32,20	31,52
0	1	0	77,74	43,94	63,71	63,71	70,63	74,91	48,09	44,88
0	1	-2	82,00	48,69	70,59	70,59	75,75	79,93	52,64	49,51
0	1	2	73,53	39,95	57,94	57,94	65,80	70,37	44,23	41,00
2	0	0	56,33	38,51	53,70	41,58	49,88	50,72	40,26	38,85
2	0	-2	59,93	42,97	59,86	46,32	54,07	55,33	44,61	43,25
2	0	2	52,88	34,83	48,60	37,66	46,05	46,79	36,65	35,21
2	-1	0	48,40	33,44	44,47	32,08	41,46	38,30	34,65	33,53
2	-1	-2	52,01	37,52	49,86	35,92	45,47	42,41	38,09	37,59
2	-1	2	45,04	30,10	40,07	28,93	37,92	34,89	30,78	30,21
2	1	0	62,71	43,13	63,17	51,39	56,97	72,16	46,96	43,98
2	1	-2	66,15	47,86	70,03	56,94	61,10	77,20	51,50	48,59
2	1	2	59,32	39,17	57,41	46,73	53,08	67,65	43,11	40,11
4	0	0	35,47	34,88	51,29	26,18	31,41	43,40	35,94	35,04
4	0	-2	37,74	39,08	57,30	29,17	34,05	47,74	40,07	39,22
4	0	2	33,30	31,42	46,33	23,72	29,00	39,76	32,55	31,62
4	-1	0	24,17	28,43	41,07	16,02	20,70	30,95	28,67	28,46
4	-1	-2	25,91	32,09	46,18	17,94	22,71	34,61	32,30	32,10
4	-1	2	22,49	25,47	36,90	14,45	18,94	27,96	25,74	25,50
4	1	0	44,69	40,34	61,40	36,63	40,61	63,46	43,20	40,92
4	1	-2	47,15	44,92	68,19	40,58	43,55	68,39	47,64	45,41
4	1	2	42,27	36,52	55,72	33,31	37,83	59,13	39,47	37,17

The following figures show for $b = 5$ and $T = 12$ the rate of index participation as a function of the guaranteed rate of interest and the cap rate (if applicable).
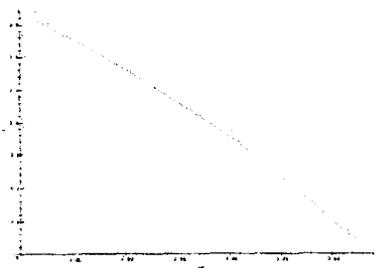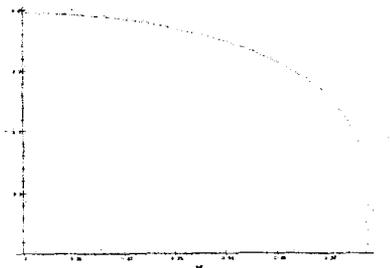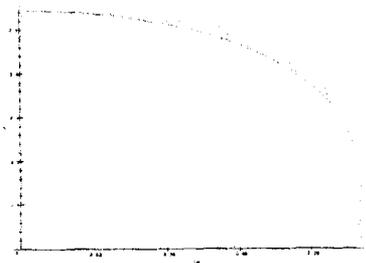
Figure 7: $x_1(i_g)$

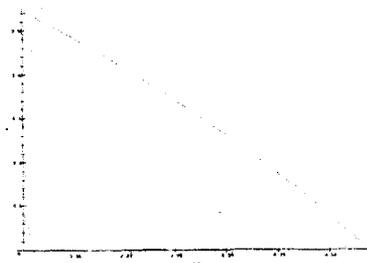Figure 8: $x_2(i_g)$

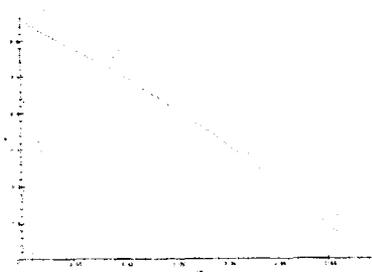Figure 9: $x_3(i_g)$

Figure 10: $x_4(i_g)$
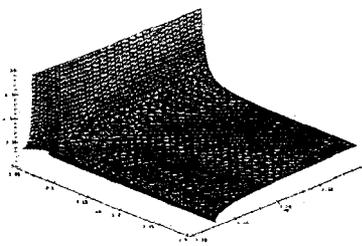

Figure 11: $x_5(i_g)$

Figure 12: $x_6(i_g, i_c)$

The following figures show (also for $b = 5$ and $T = 12$) the sensitivity of the rate of index participation with respect to parallel shifts of the term structure of interest rates and changes in the index volatility. Here, we fix $i_g = 2\%$ and $i_c = 10\%$.

In all the figures, the rate of index participation is decreasing in σ and increasing in Δr (cf. also Table 1).
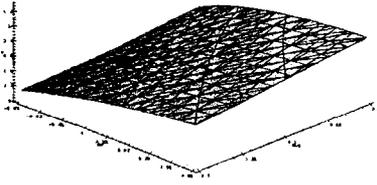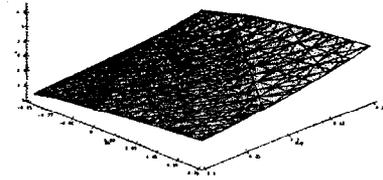
Figure 13: $x_1(\Delta r, \sigma)$

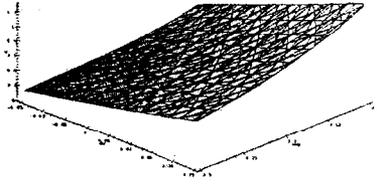Figure 14: $x_2(\Delta r, \sigma)$

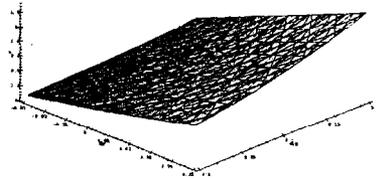Figure 15: $x_3(\Delta r, \sigma)$
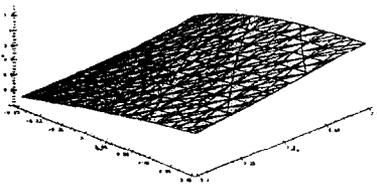

Figure 16: $x_4(\Delta r, \sigma)$
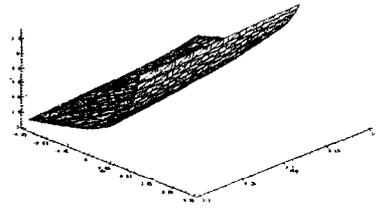

Figure 17: $x_5(\Delta r, \sigma)$

Figure 18: $x_6(\Delta r, \sigma)$

Our analysis shows, that the rate of index participation depends heavily on the product design. Products where the index-dependent payoff is added to the guaranteed sum (like our products 1, 4 and 5) show a very strong sensitivity of the rate of index participation with respect to changes in the guaranteed rate of interest. Non-additiv products, like our products 2, 3 and 6, however, show a rather moderate sensitivity with respect to i_g.

Depending on the market scenario, introducing a cliquet can either increase or decrease the rate of index participation.

Obviously, the rate of index participation of a product with compound interest is always lower than that of a similar product without compound interest. Imposing a cap rate always increases the participation rate. Substituting $M_{t_1}^{t_2} - S_{t_1}$ by $M_{t_1}^{t_2} - m_{t_1}^{t_2}$

(cf. products 1 and 5) obviously always reduces the rate of index participation.

A problem of all the discussed lookback products is that the fair participation rate is rather low. A possible way of increasing the partipation rate is using averaging products. In [No/Ru 97a] and [No/Ru 97b] we analysed three different averaging products and it turned out, as expected, that the index participation rate of averaging products is by far higher than of analoguos non-averaging products. From a marketing point of view, it might be interesting to combine lookback with averaging ideas (one could, e.g. link the payoff to the arithmetic average of each year's maximum index value). For such products, no explicit pricing formulas are available. Hence, numerical methods like Monte Carlo simulation are required, cf. [No/Ru 97b].

6 Summary and Outlook

In this paper, we introduced a new class of index-linked contracts based on lookback options. We derived closed form pricing formulas for these contracts within a model for the economy introduced in [No 96]. We furthermore analysed in detail the dependence of the rate of index participation on the product design as well as on different market conditions.

Some of our contracts are not very attractive from a marketing point of view, as the index participation rate is rather low. Nevertheless, our analysis helps understanding the impact of different product features on the participation rate and shows how it is possible to construct products being of interest to potential investors.

At present, many insurance companies in Germany consider offering guaranteed index-linked policies. In order to be innovative and contrast with the policies that are already being sold, it might be worthwile for those companies to consider such new products.

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