WHY THE CAPITAL ASSET PRICING MODEL FAILS IN A MULTI-CURRENCY WORLD

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ABSTRACT

The Capital Asset Pricing Model (CAPM) assumes that all investors measure risk and return in the same currency. When different currencies exist this assumption breaks down. A very simple model is investigated in this paper, where investors in two countries, A and B, can each invest in domestic cash and in domestic shares and foreign shares. It is immediately clear that the risk-free assets for the two groups of investor are different, as are their desired market portfolios.

It is then shown that, if the markets in the two countries are of different size, an equilibrium can only be obtained if the prices of shares change in some way. The model is investigated under the assumption that the total market value of shares remains unchanged, and numerical results are obtained.

A different assumption, that the market price of risk in the two countries remains unchanged, is mentioned but not investigated.

The advantage of foreign currency hedging, in the simplest case, is demonstrated.

The existence of a "throw-away asset" is postulated, and it is shown that the minimum variance portfolio, among all portfolios consisting of risky assets, is never efficient. A "free lunch" can be obtained by not investing all one's assets in risky securities, but by throwing away a fraction of them.

The multiple currency problem is compared, in general terms, with the situation where some investors measure in money terms, and others in real terms, or where investors have different liabilities, and different durations of liability, and it is concluded that many of the results obtained from the CAPM are of doubtful validity in this wider setting.

KEYWORDS

Multiple Currencies; Equilibrium Models; Amount of Wealth; Adjusting Prices; Inadequacy of Capital Asset Pricing Model (CAPM); Hedging; The 'Throw-away' Asset

1. INTRODUCTION

1.1 In the paper I presented to the first AFIR Colloquium in Paris in 1990 (Wilkie, 1990) I observed that the Capital Asset Pricing Model (CAPM) was not a feasible equilibrium model in a world in which different investors used different currencies. I made the point then in general terms. In this paper I discuss some numerical results, which show how the weight of money can be taken into account.

1.2 I also make two other points that have come out of my investigations: hedging foreign currencies is beneficial in reducing variance (a well-known idea) (Section 6). And in the portfolio selection model there is always a particular risk-free asset, which I call the "throw-away asset", so that the usual pictures of the mean-variance frontier with no risk-free asset are wrong (Section 7). The minimum variance portfolio is never efficient. I have not seen this point made before, but I am not familiar with all the literature.

2. THE DOMESTIC MODEL

2.1 There are two countries, A and B, which I sometimes call America and Britain, or US and UK. They have different currencies, currency A and currency B, which I sometimes call dollars and pounds. In each country the investment possibilities include a market portfolio of risky assets, which I call shares, and a risk-free asset, which I call cash. The markets in each country meet the usual criteria for the CAPM, in that investors use mean-variance efficiency, have homogeneous beliefs and a single time horizon (which I call a year), and there are no taxes, no transaction costs, etc. It is a very simple world, but more complex and more realistic than the usual CAPM world.

2.2 Initially there is exchange control between the countries, and investors can only invest in the cash and shares in their own countries. I shall put numerical values on everything to illustrate the argument. For algebraic convenience I assume that the returns on shares are lognormally distributed. This is of no significance in the mean-variance optimisation process.

2.3 I consider first country A. Country B will turn out to be almost identical, but the mirror image. The risk-free *return* is r, that is, \$1 invested now provides \$$r$ at the end of one year. I shall put $r = 1.05$ in the numerical examples, a *rate of return* of 5 per cent. The return on shares is lognormally distributed, that is, \$1 invested now provides \$$R_1$ at the end of one year, where $\log R_1$ is normally distributed with mean, $\mu_1 = 0.1$, and standard deviation, $\sigma_1 = 0.2$. The expected return on shares $E_1 = E[R_1] = \exp(\mu_1 + \sigma_1^2/2) = 1.1275$, and the variance of the return on shares $V_1 = \text{Var}[R_1] = E_1^2.(\exp(\sigma_1^2) - 1) = 0.051881$, so that the standard deviation of the return, $S_1 = \sqrt{V_1} = 0.2278$.

2.4 The efficient frontier in A is the straight line in the E-σ plane joining the point representing the risk-free asset, i.e. cash, $(0, r)$ and the point representing the market portfolio, i.e. shares (S_1, E_1). Investors who choose the market portfolio of risky assets can obtain an extra expected return of $7.75\% = 100(1.1275 - 1.05)$ for a standard deviation of 22.78%, so the *market price of risk*, the slope of the line joining cash and shares is 0.3402 in this case; this is the additional expected return per unit of standard deviation; or, better, the extra percentage rate of return per 1% standard deviation.

2.5 Country B is, at this stage, identical to A. All the numbers are the same, except that they are measured in pounds. Investors have the same attitude to risk.

3. The International Model

3.1 Now imagine that exchange control is lifted, so that investors in each country can also invest in the assets available in the other, foreign cash and foreign shares. The currencies are subject to a floating exchange rate, which we assume in this model is not affected by portfolio investors (in the real world it very likely is so affected). We assume that the exchange rate is one for one at the beginning of the year (this assumption is not important), and that its change during the year is lognormally distributed with zero drift and standard deviation $\sigma_c = 0.1$.

3.2 Consider the investor in A. He can still invest in US cash, with risk-free return r, and in US shares with expected return E_1 and standard deviation S_1. But he can now also invest in pound (sterling) cash and British shares, subject, however, as far as he is concerned, to exchange rate risks. The return on British cash is r_c, with $E_c = E[r_c] = r.\exp(\sigma_c^2/2) = 1.0553$, and variance $V_c = E_c^2.(\exp(\sigma_c^2) - 1) = 0.011192$ and standard deviation $S_c = \sqrt{V_c} = 0.1058$.

3.3 The American investor calculates his return on British shares in dollars, so the pound return is subject to exchange rate risk. We denote it as R_2, and calculate the expected value as $E_2 = E[R_2] = \exp(\mu_1 + (\sigma_1^2 + \sigma_c^2)/2) = 1.1331$, with variance $V_2 = \text{Var}[R_2] = E_2^2.(\exp(\sigma_1^2 + \sigma_c^2) - 1) = 0.065833$, and standard deviation, $S_2 = \sqrt{V_2} = 0.2566$. For simplicity I assume that returns on shares in A and B are uncorrelated, and that neither (in domestic currency) is correlated with the exchange rate. This may be artificial but it simplifies the algebra.

3.4 We can readily see that, for the American investor, US cash is still risk-free, and US shares are risky. British cash gives a slightly better expected return, with a rather higher variance, but the independence of returns means that there might be an advantage in diversification. British cash gives a trivially higher return than US cash, for a considerable extra variance, but it is positively correlated with British shares (through the exchange rate). It turns out that, if we allow negative holdings, the optimum portfolio contains a negative holding of British cash, but that if we restrict investors to non-negative holdings, British cash does not enter into the optimum portfolio of the US investor. For the time being, therefore, we shall not permit holdings of foreign cash; we reintroduce them again in Section 6.

3.5 Some simple algebra (see Appendix A) shows that the optimum portfolio for the US investor now consists of US shares and British shares in fixed proportions, forming a new "market portfolio", together with a holding of US cash that reflects his particular attitude to risk. If we put $g_1 = V_1/(E_1 - r)$ and $g_2 = V_2/(E_2 - r)$, then x_1 and x_2, the proportions of the market portfolio in domestic shares and foreign shares respectively, are given by $x_1 = g_2/(g_1 + g_2)$ and $x_2 = g_1/(g_1 + g_2)$. The numerical values in my example are $x_1 = 0.54185$ and $x_2 = 0.45815$. Thus the market portfolio for the American investor contains 54.185% US shares and 45.815% UK shares.

3.6 The market portfolio in each country now has expected return 1.1301, variance 0.029051 and standard deviation 0.1704. The expected return is increased slightly, and the variance is reduced considerably. Diversification pays off. The market price of risk is now 0.4699.

3.7 Now consider the British investor. His situation is the image of the American one, except that for him domestic shares are British and foreign shares are American. His market portfolio consists of 54.185% UK shares and 45.815% US shares. The market portfolios for the two investors are different. So also are the risk-free securities, US cash for the dollar investor, UK cash for the pound investor. There are two different capital market lines, one in each currency, and there is no unique CAPM.

3.8 While the algebra and arithmetic so far is without flaw, we can readily see that the proportions of investment in foreign shares are not equilibrium proportions unless the two countries are the same size, in the sense that the values of the pool of risky assets in each country are the same. If the original total holding of risky assets in America is the same size as the total holding of risky assets in Britain, then 45.815% of the one will equal 45.815% of the other, and it will be possible for the investors to exchange holdings and achieve a new equilibrium. But if the two countries are not the same size, such an exchange is not possible and a different equilibrium must be sought.

4. A NEW EQUILIBRIUM MODEL

4.1 Now let us assume that the original weight of shares and investment in country A is w times that of country B. To be precise: the value of shares in country A is W_A and the value of shares in country B is W_B and $W_A = w.W_B$. For numerical examples I shall initially assume that $w = 5$, and treat W_B as the unit. Now 45.815% of W_A is $2.29075W_B$, which is a great deal bigger than W_B, so if American investors try to invest 45.815% of their (risky) wealth in Britain, they will not be able to do it. If we are find a new equilibrium something must change.

4.2 I assume, for the purposes of this example, that the risk-free rate, like the exchange rate, is exogenously given, and is not changed by the activities of investors. I also assume that the returns per "share" are determined independently of the stock market, being the proceeds obtained by some sort of industrial or commercial activity. They are not affected by investment activity. All that we can change are the present prices of shares. I have assumed so far a unit price for shares, or at least I have described the returns as being per unit invested, but I now allow the share prices in countries A and B to change to p_A and p_B respectively.

4.3 Changing the prices changes the returns and the variances of returns. To be precise: in general, if the price changes from 1 to p, and the new expected values, variances and standard deviations are denoted by primes, then $E' = E/p$, $V' = V/p^2$ and $S' = S/p$. What we called g becomes $g' = V'/(E' - r) = V/p(E - p.r)$.

4.4 In order to reach an equilibrium we need to make UK shares less attractive to US investors, so that they wish to buy fewer of them, and make US shares more attractive to UK investors, so that they wish to buy more of them. A necessary constraint is that the

market values of shares exchanged are equal, so that the UK shares bought by American investors have the same value as the US shares bought by British investors, i.e. $x'_{2A}.W'_A = x'_{2B}.W'_B$, where primes represent the new equilibrium.

4.5 The equality of exchange gives us one constraint, but we have two prices we can alter, so I impose a second constraint, that the total market value of risky assets is unchanged, so that $W'_A + W'_B = W_A + W_B$. I assume that prices change before the exchange takes place, so that $W'_A = p_A.W_A$ and $W'_B = p_B.W_B$. Hence $p_B = 1 + w.(1 - p_A)$.

4.6 I now continue with my numerical example. Assume that country A is five times the size of country B, so that $w = 5$. We want to find a price p_A so that $x'_{2A}.W'_A = x'_{2B}.W'_B$ after we have gone through the required calculations. We can immediately see that $p_A = 1$ is too high, and we soon find that $p_A = 0.98$ is too low. Successive bisection gives the solution as $p_A = 0.9884$ with $p_B = 1.0578$. (In fact I have calculated the answers to many more decimal places; I quote only rounded answers.)

4.7 Thus we can achieve a new equilibrium with share prices in America falling by 1.16%, and share prices in Britain rising by 5.78%. For the US investor the expected return on US shares rises from 1.1275 to 1.1407, and the standard deviation rises from 0.2278 to 0.2304. If he invests in UK shares his expected return falls from 1.1331 to 1.0710, with a reduced standard deviation of 0.2425 (compared with that in Section 3 of 0.2566). His optimum risky portfolio, his market portfolio, consists of 82.69% in US shares and 17.31% in UK shares. This portfolio gives an expected return of 1.1287 with standard deviation 0.1951 and a market price of risk of 0.4031.

4.8 The British investor also faces new numerical values, but these are no longer the image of the American ones. For him, the expected return on UK shares is 1.0657, with standard deviation 0.2153, and on US shares he gets an expected return of 1.1464 with standard deviation 0.2596. His market portfolio consists of 19.13% UK shares and 80.87% US shares. This gives him an expected return of 1.1310, with standard deviation 0.2139 and a market price of risk 0.3786.

4.9 The wealth (in risky assets) in America is now 98.84% of 5 units, or 4.9420 units, and the wealth in Britain is now 1.0580 units. The investors can now exchange shares, the American investors buying 17.31% of 4.9420 = 0.8556, and the British investors buying 80.87% of 1.0580, which has the same value.

4.10 We can carry out the same calculations with different values of w as shown in Table 1. Two interesting features emerge. As w increases, the equilibrium price of shares in B increases, but at a reducing rate, whereas the price of shares in A first falls, and then rises again. If the countries are about the same size, but not equal, then B has a noticeable effect on A, but if B is very small relative to A it has a small effect. In all cases A has a noticeable effect on B.

4.11 Secondly, as the weight, w, increases beyond 19 the proportion of shares of country A held by investors in B goes beyond 100%, with a negative holding in domestic shares. Shares in B are still attractive to investors in A, to provide diversification, but the return on shares in B falls first to equal and then to go below the risk-free return, and shares in A become overwhelmingly attractive.

Table 1. Results for different values of w, the ratio of wealth in countries A and B.

Weight of wealth in country A relative to B	New price of shares in A	New price of shares in B	Proportion of foreign shares held by investors in:	
w	p_A	p_B	A	B
1	1	1	45.815%	45.815%
2	0.9856	1.0289	31.92%	61.15%
3	0.9853	1.0442	24.81%	70.23%
4	0.9868	1.0527	20.37%	76.39%
5	0.9884	1.0580	17.31%	80.87%
6	0.9897	1.0615	15.07%	84.28%
7	0.9908	1.0641	13.34%	86.97%
8	0.9917	1.0660	11.98%	89.15%
9	0.9925	1.0675	10.87%	90.96%
10	0.9931	1.0687	9.95%	92.47%
...				
19	0.9961	1.0737	5.66%	99.82%
20	0.9963	1.0740	5.40%	100.28%

5. AN ALTERNATIVE APPROACH TO EQUILIBRIUM

5.1 In the development in Sections 3 and 4 I assumed that the overall market value of shares remained constant. This was an arbitrary assumption, and not necessarily realistic. An alternative approach would be to assume that the market price of risk remained unchanged.

5.2 In the case where the two countries are of equal size we can (by empirical calculation with a spreadsheet) readily obtain new share prices where the market price of risk is the same as it was before exchange controls were lifted. If p_A and p_B both change to 1.0210 then the market portfolios in both countries become the same with 53.52% in domestic shares, 46.48% in foreign shares, with expected return 1.1068, standard deviation 0.1670 and market price of risk 0.3402, as it was in paragraph 2.4.

5.3 However, if we try to impose this constraint when the two countries are not of equal size, we discover that we cannot find values of p_A and p_B that will meet all three desired conditions: the quantity of shares exchanged is equal, and the market prices of risk in countries A and B are both unchanged. We need to relax some other constraint, for example allowing the risk free rate to change. I have not pursued this further at this stage.

6. The Benefits of Hedging

6.1 I mentioned in paragraph 3.4 that the optimum risky portfolios would contain negative holdings of foreign cash, if such holdings were permitted. I excluded them at that stage, and I have not investigated fully the results if foreign cash is permitted. However, in the simple case where the countries are of equal size, the optimum unconstrained portfolio consists of 79.44% in domestic shares, 77.57% in foreign shares, with a negative holding of 57.01% in foreign cash. The expected return on this portfolio is 1.1729, with standard deviation 0.2557 and a market price of risk of 0.4806. Thus hedging is beneficial, though the hedged risky portfolio provides a much higher expected return with a much higher standard deviation than that found in Section 3, where there was no foreign cash.

6.2 In fact hedging may more often be done nowadays by purchasing currency futures, rather than by borrowing foreign cash. The details are different, but the principles are the same.

7. The Throw-away Asset

7.1 In the course of the calculations shown above I at one stage made the mistake of putting the risk-free return equal to zero, implying that money invested in the risk-free asset is, in effect, thrown away. It provides zero return with certainty. Note that I do not mean a zero *rate of return*, but a *rate of return* of -100%.

7.2 It is easier to demonstrate the consequences by using different values of the returns on two risky securities than those used above. I assume two securities, A_1 and A_2, with expected returns $E_1 = 1.1$ and $E_2 = 1.3$, standard deviations $S_1 = 0.2$ and $S_2 = 0.3$ and zero correlation.

7.3 Since $r = 0$, we get $g = V/E$ so $g_1 = 0.04/1.1 = 0.36364$ and $g_2 = 0.09/1.3 = 0.069231$. Thus the market portfolio consists of proportions $x_1 = g_2/(g_1 + g_2) = 65.56\%$ of A_1 and $x_2 = g_1/(g_1 + g_2) = 34.44\%$ of A_2, with an expected return of 1.1689 and a standard deviation of 0.1669. Any positive mixture of the market portfolio and the risk-free asset is possible.

7.4 If the risk-free asset is excluded then the efficient frontier in E-σ space is a hyperbola, and a little algebra shows that the minimum variance portfolio consists of 69.23% of A_1 and 30.77% of A_2, with an expected return of 1.1615 and a standard deviation of 0.1664. However, if we take a mixture consisting of 0.63% of the throw-away asset, and 99.37% of the market portfolio we discovered above, then we can construct a portfolio with expected return 1.1615, the same as the minimum variance portfolio, and standard deviation 0.1659, smaller than that of the minimum variance portfolio. This portfolio is therefore more efficient.

7.5 In the E-σ space, we put the throw-away asset at the origin, (0, 0), and draw a tangent from that point to the usual hyperbola that shows the frontier of minimum variance portfolios for each value of the expected return. Assuming that this tangent touches the hyperbola above its "nose", then the true efficient frontier consists of this tangent as far as where it touches the hyperbola, and the usual efficient frontier thereafter. The minimum

variance portfolio is, therefore, never efficient. One cannot expect to be able to go short in the throw-away asset, i.e. borrow with no obligation to repay, so the tangent cannot be continued above the point of tangency.

7.6 One can reasonably suggest that in many cases a better asset than the throw-away asset is available. If one measures in units of a nominal currency, then one can usually hold money with no interest and no loss. This may not be possible if one wishes to measure "in real terms".

7.7 I do not imagine that this result is new, but most text books on the subject do not mention it, so it may be new to some readers. For example neither Ingersoll (1987) nor Huang & Litzenberger (1988), both quite thorough books, appears to mention the point. Nor does either discuss a multi-currency world. Nor even, so far as I can see, does the originator of all mean-variance optimisation theory, Markowitz (1987).

7.8 What does one do with the money invested in the throw-away asset. Well, one suggestion might be to buy a good lunch for oneself and as many of one's friends as it will pay for. Thus the phrase that "there is no such thing as free lunch" is also controverted!

8. CONCLUSIONS

8.1 I have shown that, in a simplified world where there are investors who work in two different currencies, the naive Capital Asset Pricing Model does not hold. Investors measure risk in different terms (dollars and pounds in my example), so they have different concepts of what constitutes a risk-free asset. The expected returns and variances (and also covariances — I assumed that these were zero just to simplify the algebra and arithmetic; they could have been included) are different for the different investors so that they have different portfolios on their efficient frontier, different market portfolios and different "capital market lines". An equilibrium cannot be found without taking into account the weight of money held by investors in the two currencies.

8.2 The same argument can be applied in one country where some investors think "in money terms" and others "in real terms". In the United Kingdom there is a significant market in government stocks whose interest and capital payments are linked to the value of the Retail Prices Index, so investors can construct portfolios measured "in real terms" with a risk-free asset available. For such investors, conventional bonds are "risky". For investors who measure in money terms, index-linked stocks are "risky". Any equilibrium must depend on the weight of money in each camp.

8.3 Some intermediary investors have liabilities fixed in money terms, others have liabilities fixed in real terms. Others may be ambivalent. Life offices who issue fixed money annuities fall into the first camp, pension funds with contractually index-linked annuities fall into the second. But do life offices that issue with profits life assurances, which contain fixed money guarantees, aim to maximise the money return to policyholders or the real return? Do pension funds that provide final salary pensions, and aim to increase them in line with the cost of living, but nevertheless have some fixed money guarantees, do their calculations in money terms or in real terms? The balance is complex.

8.4 The same can apply to investors whose liabilities occur at different times. The classical CAPM assumes a single time horizon for all investors, although this can be extended to multiple time horizons, if the assumption is made that investors maximise their returns over successive years (and possibly that the returns are homogeneous in time). But if different intermediate investors have liabilities that differ in their duration profiles, then their attitudes to risk may depend on the period over which they are looking. In particular, an intermediary such as a pension fund may have predominantly real liabilities spanning many years, and its attitude to risk might therefore depend on the real outcomes over such a time period. Arguments drawn from a naive single period CAPM, measured in dollar terms (one notices that the risk-free asset in most text books is always US dollars, regardless of the currency of the investor!), and with unspecified liabilities, should be treated with great suspicion.

8.5 In particular the concept that there is one "market price of risk" and that all investments can be assessed in terms of this one market price is untenable.

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APPENDIX

A.1 Assume a risk-free security S_0, with certain return r, and two risky securities, A_1 and A_2 with expected returns E_1 and E_2, variances V_1 and V_2 and no correlation between the returns.

A.2 Construct a portfolio by putting x_0 in the risk-free security and x_1 and x_2 in A_1 and A_2, where $x_0 + x_1 + x_2 = 1$. The expected return on the portfolio is:
$$E_P = x_0.r + x_1.E_1 + x_2.E_2$$
and the variance of the portfolio is:
$$V_P = x_1^2.V_1 + x_2^2.V_2.$$

A.3 We now wish to find a minimum variance portfolio for a particular expected return, say E. We do this by forming the Lagrangian:
$$W = x_1^2.V_1 + x_2^2.V_2 + \lambda.(x_0.r + x_1.E_1 + x_2.E_2 - E) + \mu.(x_0 + x_1 + x_2 - 1),$$

then differentiating with respect to x_0, x_1, x_2, λ and μ and setting all the partial derivatives to zero, to give:

$$\lambda.r + \mu = 0$$
$$2x_1.V_1 + \lambda.E_1 + \mu = 0$$
$$2x_2.V_2 + \lambda.E_2 + \mu = 0$$
$$x_0.r + x_1.E_1 + x_2.E_2 - E = 0$$
$$x_0 + x_1 + x_2 - 1 = 0$$

Putting $\mu = -\lambda.r$ and substituting we get:

$$\lambda = -2x_1.V_1/(E_1 - r)$$
$$\lambda = -2x_2.V_2/(E_2 - r)$$

Equating these and simplifying gives:

$$x_1.V_1/(E_1 - r) = x_2.V_2/(E_2 - r)$$

We then put $g_1 = V_1/(E_1 - r)$ and $g_2 = V_2/(E_2 - r)$, to give:

$$x_1.g_1 = x_2.g_2$$

But at the market portfolio, $x_0 = 0$, so $x_1 + x_2 = 1$, whence

$$x_1 = g_2/(g_1 + g_2)$$
$$x_2 = g_1/(g_1 + g_2)$$

as required in paragraph 3.5.