OPTIMAL SUBSTRUCTURE OF ASSET AND LIABILITY IN THE MULTI-FACTOR ECONOMY

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abstract

This paper investigates simplified frameworks of optimal capital structure and substructure issues in tax-shelter bankruptcy cost models where the multiple asset and liability instruments evolving according to the Gaussian type multi-factor stochastic processes are incorporated in the capital substructure. First, assuming for simplicity to exposition that liability instruments have a single maturity horizon, we derive global optimality conditions of the capital (sub-)structure, and elucidate that when such conditions are satisfied, maximization of the firm value or market value of debt admits a global optimal solution of capital (sub-)structure, whereas maximization of the equity value with given a debt ratio leads to an extremal point solution. Second, employing Clark's representation formula obtained directly from the recently-established stochastic methodolody, *Malliavin Calculus*, we study the risk management model of asset and liability substructure.

Keywords: Asset-liability model, term structure of interest rates, Malliavin Calculus

I. Introduction

In the corporate finance the optimal capital structure of the firm has been one of quite important issues; under what environment does optimum capital structure exist, and if it exists, then what natures does it has, and how is its comparative statics to relevant financial parameters. Innovative works of Modigliani and Miller(MM)(1958, 1963) elucidated that in the perfect market the firm value is invariant to the capital structure, and the introduction of a proportional corporate income tax provide an incentive for the firm to maximize their use of debt financing, since the deductibility of interest charges from taxable income enhances the after-tax operating income of the levered firm. Since MM's works, a lot of articles inspired by MM have appeared in the subjects of optimal capital structure. Krauss and Litzenberger(1973), Scott(1976) and Kim(1978) demonstrated that when in addition to taxes the leverage-related costs such as bankruptcy cost, agency cost and non-debt tax shields etc. are introduced, the firm has an optimal capital structure. This is due to the trade-off between the tax advantage of debt and such financial distress costs. However it should be noted that almost descriptions of these researches rely upon an assumption that asset and liability structure consists of only a single asset and debt for simplicity of exhibition, in order to prove the existence of optimal capital structure for various types of market imperfections like corporate, personal taxes and leverage-related costs. For example, the existence of optimal corporate debt capacity is proved in addition to the existence of optimal capital structure under an assumption of a single asset and debt in Kim(1978), and subsequnt papers of Kim(1982), Bradley, Jarrell, Kim(1984) deal with the issue of tax disadvantage of debt at the personal level, pointed out by Miller(1977), and provide some comparative statics and conduct empirical studies.

Independently to the lines of addressing the issues of optimal capital structures Merton(1974) has developed a pricing model of corporate debt, in which an equity value is expressed in closed form by regarding it as a call option of the asset value with debt payment being taken as the strike price. Merton's model and subsequent almost articles based upon his framework also assume a single debt and a single asset framework.

Unfortunately, capital structure theories developed so far have not established themselves as available models describing real financial policies and corporate activities. From the standpoint of providing insight into the more realistic firms' capital structure a single asset and debt assumption seems unsatisfactory, because almost firms finance the funds by issuing several corporate bonds or borrowing from financial firms, and invest those to several assets. Although there are a lot of financial and operating policies in the actual firm management, it is still a matter of considerable concern for the firm manager to diversify various species of risks that the firm bears in capital substructure, exploiting whole asset and debt instruments available in the finacial markets, as well as to determine the optimal debt ratio. Then the main practical interest of the financial manager seems to be (i) what kind of debt to issue in which country, (ii) what amount of the total debt to be financed and (iii) which asset to invest that fund. Along this line, Agmon, Ofer and Tamir(1981) has investigated the debt portfolio selection problems, especially to answer the above question (i), in which, however, the debt ratio as a bench mark of capital structure was exogenously given as a constraint of optimization problems and the substructure of asset side was not taken into consideration. Namely, the questions (ii) and (iii) were not addressed.

The first aim of ours are to extend capital structure models investigated so far like tax shelter-bankruptcy cost models, Kim(1978), Kim(1982), Bradley et al.(1984), Ikeda(1993) and so on, in a way to incorporate the multiple asset and debt instruments, invoking to the Merton's option-theory based approach, and to answer the three questions above at the

same time by solving capital structure issues as well as substructure ones with respect to the debt ratio and weight vectors of multiple asset and debt instruments. The geometrical viewpoint from this enlarged space consisting of weight vectors and capital structure(debt ratio) facilitates characterizing these two approaches of Kim and Agmon et al.; the argument of Kim(1978) about optimal capital structure studied corresponds to an optimality along the direction of debt ratio when the weight vector is restricted to an extremal point of simplex space the weight vectors span, and that of Agmon et al.(1981) to an optimality in the interior or boundary region within a simplex space when the debt ratio is exogenously given. Once entering into our option-theory based approach, these two kinds of optimalities are, *ceteris paribus*, reconfirmed as well by exploiting geometric nature(convexity/concavity) of objective functions (equity value, firm value and so on) originating from the embedded option.

From a closer look of equity and firm values formulae given later in closed form we are led to some interesting findings that maximizing the equity value has a tendency of changing capital substructure to a single asset and debt structure, and that maximizing either a market value of debt or a firm value has a tendency of diversifying capital substructure to appropriate multiple assets and debts. These remarkable differences of the features come from the above-mentioned geometric natures of option embedded in relevent contingent claims.

Second, we aim to provide a stochastic interest rates dynamics so as to be able to differentiate between fixed and floating rate instruments. The original Merton's model was described in non-stochastic interest rate (Black-Scholes(1973)) economy. Therefore, needless to say, Merton's model cannot deal with the floating rate assets or debts in itself. Recently Ikeda(1993) has investigated optimal capital structure issues under the stochastic interest rate assumption of Ornstein-Uhlenbeck, in a tax shelter-bankruptcy cost model, where optimal capital structure and debt capacity are proved for a single aggregate asset and fixed rate or floating rate debt. Thus, the recent developement of the research of modelling stochastic interest rate processes enables us to consistently evaluate interest rate sensitive securities under the stochastic interest rates, e.g., Heath-Jarrow-Morton(1992), Amin and Jarrow(1991) and so on. For a concrete application to pricing bonds, bond futures, interest rate futures and their options, see Nakamura(1991). In this article, we follow a multi-factor Gaussian type stochastic modelling of domestic, foreign interest rates, and exchange rates processes *a la* Heath-Jarrow-Morton(1992) and Amin and Jarrow(1991). Based upon a Martingale approach, we derive analytic expressions for the equity, firm value and market value of debt, in which the change of numéraire technique developed by Jarrow(1987), and El Karoui, N., R. Myneni, and R. Viswanathan(1992) is effectively applied. Taking an aggregate asset as a numéraire facilitates fairly the evaluation of contingent claims and gives us a deeper insight to the relationship of each risk of asset and liability substructures.

Third we explore some new possibilities of risk management models of asset and liability. Usually risk exposures of the relevent contingent claims are unclear until their analytical expressions are derived. Such a calculation of contingent claims including the multiple assets and debts is, however, seemingly quite difficult. Nevertheless, we show that there is a breakthrough technology, that is, the Clark's representation formula based upon the *Malliavin* calculus developed recently (see Ocone and Karatzas(1991) for financial applications, and Ikeda and Watanabe(1989) for mathematical foundations).

As for a numerical computation of analytical contingent claims expressions it seems to be difficult on account of the complicated multiple integrations included. A simple way out is that we develop approximating valuation methods utilizing the Edgeworth expansion (e.g., Jarrow and Rudd(1982) employs the method of such sort to pricing the standard

option, and Nakamura(1992) does to pricing arithmetic average options under the stochastic interest rates). Due to the page limit of proceedings the detail of computations as well as the numerical analyses including asset and liability optimization results is not stated in this article, see Nakamura(1996).

The present paper is organized as follows. In section II we introduce general settings of several assumptions and stochastic processes of every instrument treated in this article. In section III we provide a general framework of tax-shelter bankruptcy models and discuss the existence of optimal structure as well as substructure under certain conditions, elaborate on various implications of different criteria, examine the comparative statics associated with market frictions. Section IV explores some possibilities of risk management models of asset and liability by means of the *Malliavin* calculus. The last section is devoted to the summary and concluding remarks. The appendices provide the change-of-numéraire technique in more general form and contain some technical proofs omitted in the body of article.

II. General Setting of Dynamics

First of all, before proceeding into tackling with issues of capital substructure, we shall introduce a necessary setting of both the stochastic interest rates *a la* Heath, Jarrow and Morton (1992) and stochastic exchange rate *a la* Amin and Jarrow(1991). The terminology of our model, notation and some assumptions are presented as follows: [1]

Definitions

Money Market Accounts (domestic/foreign):

$$\begin{aligned} B_d(t) &= e^{\int_0^t r_d(u)du}, \\ B_f(t) &= e^{\int_0^t r_f(u)du}, \end{aligned} \tag{1}$$

Default-free Discount Bonds (domestic/foreign):

$$\begin{aligned} P_d(t,s) &= e^{-\int_t^s f_d(t,u)du}, \\ P_f(t,s) &= e^{-\int_t^s f_f(t,u)du}. \end{aligned} \tag{2}$$

Here $r_d(t)$ ($r_f(t)$) stands for the domestic (foreign) spot interest rate at the time t, and $f_d(t,s)$ ($f_f(t,s)$) the domestic (foreign) forward interest rate contracted at time t, prevailing at time s. Let $\mathbf{W} = \{W_t = (W_1(t), \cdots, W_K(t))^\top, \mathcal{F}_t; 0 \le t \le \tau_H\}$ be a K-dimensional independent Brownian motion on the probability space (Ω, F, P_0). $\{\mathcal{F}_t\}$ is the argumentation under the original probability measure P_0 of $\{\mathcal{F}^W\} = \sigma(W_k(s); 0 \le s \le t, 0 \le t \le \tau_H; k = 1, \cdots, K)$. Let P denote a risk-adjusted probability measure (equivalent to P_0) under which the reference asset is $B_d(t)$ and any value process divided by $B_d(t)$ is P-martingale. Let Q denote a probability measure transformed equivalently from P. Under Q any preferable asset staying alive during the horizon τ_H is chosen as a reference asset (numéraire) and then any value process divided by such a reference asset chosen becomes Q-martingale. [see Jarrow(1987), and El Karoui, N., R. Myneni, and R. Viswanathan(1992)].

The assumptions imposed explicitly are as follows:

[1] The equality of random variable is supposed to be taken in the almost sure sense, but to avoid excessive notation that will not be mentioned hereafter.

(A.1) All instruments trade continuously in frictionless markets over $t \geq 0$ except for maturity dates of debt instruments.

(A.2) The market is complete.

(A.3) There are no arbitrage opportunities in the market.

(A.4) Under the original probability P_0 an asset value $A(t)$ evolves according to the stochastic differential equation (SDE): [2]

$$\frac{dA(t)}{A(t)} = \mu_A(t)dt + \sum_{k=1}^{K} \gamma_{Ak}(t)dW_k(t). \tag{3}$$

(A.5) Under the P_0 an exchange rate $E(t)$ evolves according to the SDE: [3]

$$\frac{dE(t)}{E(t)} = \mu_e(t)dt + \sum_{k=1}^{K} \gamma_{ek}(t)dW_k(t). \tag{4}$$

(A.6) Under the P_0, a domestic and foreign forward interest rates follow stochastic processes($t \leq s \in [0, \tau_H]$), respectively,

$$df_d(t,s) = \alpha_d(t,s)dt + \sum_{k=1}^{K} \beta_{dk}(t,s)dW_k(t), \tag{5}$$

$$df_f(t,s) = \alpha_f(t,s)dt + \sum_{k=1}^{K} \beta_{fk}(t,s)dW_k(t). \tag{6}$$

Their discount bonds evolve according to

$$\frac{dP_d(t,s)}{P_d(t,s)} = \mu_d(t,s)dt + \sum_{k=1}^{K} \gamma_{dk}(t,s)dW_k(t), \tag{7}$$

$$\frac{dP_f(t,s)}{P_f(t,s)} = \mu_f(t,s)dt + \sum_{k=1}^{K} \gamma_{fk}(t,s)dW_k(t), \tag{8}$$

with the same K dimensional Brownian motions, where the volatility function, $\beta_k(t,s)$,

[2] The single asset value, here, may be regarded as an aggregated value of multiple asset instruments. Later on, multiple asset and debt case is studied as well.

[3] When the firm under consideration has multi-currency debts in the liability substructure, the dynamics of the corresponding exchange and interest rates must be introduced as an initial setting. Then a variety of debt instruments are supposed to be available as domestic as well as multi-foreign countries' fixed/floating rate debts. In the succeeding description of our capital structure model it is shown that these quantities affect the dynamic behavior of debt instruments only through their volatility functions of diffusion terms. Hence, for the simplicity of exposition we shall take a description style such that the firm issues some foreign debts in one foreign currency in addition to some domestic debts.

$(i = d, f)$ of the interest rate processes is, by the definition, related to the cumulated volatility function, $\gamma_{ik}(t,s)$, $(i = d, f)$ of each discount bond through the integral formula,

$$\gamma_{ik}(t,s) = -\int_t^s \beta_i(u,s)du, \tag{9}$$

and the drift function $\mu_i(t,s)$ is defined by

$$\mu_i(t,s) = r_i(t) - \int_t^s \alpha_i(u,s)du + \frac{1}{2}\sum_{k=1}^{K} \gamma_{ik}^2(t,s). \tag{10}$$

It should be noted that the above $\beta(t,s)$ and $\gamma(t,s)$ must obey some appropriate regularity conditions.

The SDE's of the domestic money market account and the foreign one are given, respectively, by

$$\begin{aligned} \frac{dB_d(t)}{B_d(t)} &= r_d(t)dt, \\ \frac{dB_f(t)}{B_f(t)} &= r_f(t)dt. \end{aligned} \tag{11}$$

For the later convenience, we present respectively the SDE's of a foreign discount bond denominated in domestic currency, $\hat{P}_f(t,s) \equiv E(t)P_f(t,s)$ and a foreign money market account denominated in domestic currency, $\hat{B}_f(t) \equiv E(t)B_f(t)$:

$$\begin{aligned} \frac{d\hat{P}_f(t,s)}{\hat{P}_f(t,s)} &= (\mu_e(t) + \mu_f(t,s) + \gamma\gamma_{ef}^{\top})dt + \sum_{k=1}^{K}(\gamma_{ek}(t,s) + \gamma_{fk}(t,s))dW_k(t), \\ \frac{d\hat{B}_f(t)}{\hat{B}_f(t)} &= (\mu_e(t) + r_f(t))dt + \sum_{k=1}^{K}\gamma_{ek}(t)dW_k(t). \end{aligned} \tag{12}$$

The variance-covariance matrix of diffusion terms are expressed in terms of instantaneous rate of returns $(dA(t)/A(t), dE(t)/E(t), dP_d(t,T)/P_d(t,T), dP_f(t,T)/P_f(t,T))$:

$$\begin{aligned} &COV_{ij}(t,T) \\ &= \begin{pmatrix} \sum_{k=1}^{K}\gamma_{Ak}^2(t) & \sum_{k=1}^{K}\gamma_{Ak}(t)\gamma_{ek}(t) & \sum_{k=1}^{K}\gamma_{Ak}(t)\gamma_{dk}(t,T) & \sum_{k=1}^{K}\gamma_{Ak}(t)\gamma_{fk}(t,T) \\ & \sum_{k=1}^{K}\gamma_{ek}^2(t) & \sum_{k=1}^{K}\gamma_{ek}(t)\gamma_{dk}(t,T) & \sum_{k=1}^{K}\gamma_{ek}(t)\gamma_{fk}(t,T) \\ & \star & \sum_{k=1}^{K}\gamma_{dk}^2(t,T) & \sum_{k=1}^{K}\gamma_{dk}(t,T)\gamma_{fk}(t,T) \\ & & & \sum_{k=1}^{K}\gamma_{fk}^2(t,T) \end{pmatrix} \\ &\equiv \begin{pmatrix} \sigma_A^2 & \rho_{Ae}\sigma_A\sigma_e & \rho_{Ad}\sigma_A\sigma_d & \rho_{Af}\sigma_A\sigma_f \\ & \sigma_e^2 & \rho_{ed}\sigma_e\sigma_d & \rho_{ef}\sigma_e\sigma_f \\ & \star & \sigma_d^2 & \rho_{df}\sigma_d\sigma_f \\ & & & \sigma_f^2 \end{pmatrix}, \end{aligned} \tag{13}$$

These entries could be estimated from the historical data and that allows us to evaluate relevant contingent claims dealt with in the succeeding subsections.

Throughout this article we assume that a firm issues zero coupon bonds with the same time-to-maturity as debt instruments, where all accumulated interest on zero coupon bonds is paid at the maturity, and its amount is dependent upon the debt covenant. Evidently

one-period model is a portion of the multi-period model, and therefore any findings in the one-period model are not always kept valid, when entering into the multi-period framework. But no-go theorems discovered at the one-period model unambiguously cannot be upset even if it moves on the multi-period one. Besides, although this assumption seems to simplify the realistic firms' operating policy excessively, that would be satisfactory to some extent when we suppose that the firm has a policy of rebalancing asset and debt portfolio by the subdivision time (which is to be identified with debt maturity T) of a prespecified much longer project horizon. Thus in that sense it would be worthwhile investigating the one-period models of capital (sub-)structure.

Armed with these notations we can deal with a variety of debt instruments with an identical maturity time T. For example, consider the following typical four debts; domestic domestic fixed rate debt $P_d(t,T)/P_d(0,T)$, floating rate debt $B_d(t)$, foreign foreign fixed rate debt $E(t)P_f(t,T)/(E(0)P_f(0,T))$ and floating rate debt $E(t)B_f(t)/E(0)$, in which all quantities are time t values for an initial principal of unity denominated in a certain currency. Note that it is easily extended to incorporate arbitrary n debt instruments denoted by a vector $\mathbf{D}(T) = (D_1, \cdots, D_n)$. As in Nakamura(1993), it is convenient to change the numéraire from a money market account to any asset instrument, which may be merely a specific single asset or an aggregated one. According to the general prescription developed in Appendix A the SDE of $\mathbf{D}^0 \equiv \mathbf{D}/A(t)$ is given by

$$\frac{dD_j^0(t)}{D_j^0(t)} = \sum_{k=1}^{K} \gamma_{jk}^0(t,T) dW_k^Q(t),$$

for $j = 1, \cdots, n$. In particular, the above four debt instruments have the coefficient vector of diffusion terms is given in terms of the volatility functions:

$$\gamma_{jk}^0(t,T) = \begin{pmatrix} \gamma_{dk} - \gamma_{Ak} \\ -\gamma_{Ak} \\ \gamma_{ek} + \gamma_{fk} - \gamma_{Ak} \\ \gamma_{ek} - \gamma_{Ak} \end{pmatrix}. \tag{14}$$

Note that $\gamma_{jk}^0(t,T)$ is thought of as a ***relative volatility*** function of each debt instrument against the uncertain asset value process. When random variables attached with 0 are used without any remark in succeeding sections, it means that they are divided by a numéraire of the asset value such as $P_d^0(t,T) = P(t,T)/A(t), A^0(t) = 1$, etc. The above SDE has a solution:

$$D_j^0(t) = D_j^0(0) \exp(\sum_{k=1}^{K} \int_0^T \gamma_{jk}^0(t,T) dW_k^Q(t) - \frac{\sigma_j^2}{2}), \tag{15}$$

with

$$\sigma_j^2 = \sum_{k=1}^{K} \int_0^T {\gamma_{jk}^0}^2(u,T) du. \tag{16}$$

Given a total debt amount, say L, [4] a weight vector of n debt instruments in the liability structure composes $(n-1)$-simplex, $\Delta^n \equiv \{(\theta_1, \cdots, \theta_n) | \sum_{s=1}^n \theta_s = 1; \theta_s \geq 0, s = 1, \cdots, n\}$.

[4] Our framework developed here is not furnished with some logic by which overall scale, $A(0)$, can be determined uniquely. Therefore it must be given exogenously. When normalizing $A(0) = 1$, the total debt amount is identical to the debt ratio in magnitude. As long as we follow such a normalization convension, it will not be necessary to distinguish between the total debt amount and debt ratio.

The relative volatility function, γ^0_{jk}, is assigned to j-th debt vertex of the simplex. Swapping i-th debt to j-th one means graphically that starting from the vertex of i-th debt, moving along the extremal ray connecting i-th and j-th vertices in the simplex, and terminating at the vertex of j-th debt. In Nakamura(1993) the equity value associated with such movement along the extremal ray connecting two debts was explicitly computed in analytic form, and it was pointed out that the intermediate equity value is identified with just a homotopy mapping of a single debt function to another debt one.

III. Optimal Capital Structure And Substructure

A. *Tax shelter-bankruptcy cost model*

In this subsection we shall describe a tax shelter-bankruptcy cost model, introducing some tax and bankruptcy cost functions in the similar way of Ikeda(1993). There may be a lot of possible friction forms, but in this paper we shall investigate as their naive candidates the following two ones; the first is of the form paying constant amount in any case, the second is of the form paying the amount proportional to the firm value subtracted by debt obligation value. To avoid the complexity we do not consider a personal tax and a non-debt tax shield, as treated in Miller(1977), Kim(1982), Bradley, Jarrel and Kim(1984). Let t_c be a generic notation of representing the tax rate (amount) for proportional(constant) paying case. Suppose that the tax function the firm has to pay at maturity date T is

$$T_x = \begin{cases} t_c 1(A(T) \geq D(T)) & \text{(constant)} \\ t_c (A(T) - D(T))^+ & \text{(proportion)}, \end{cases} \tag{17}$$

where $1(X)$ denotes an indicator; if $X \geq 0$, then it is X, otherwise zero, and $(X)^+$ stands for $\max(X, 0)$. The terminal debt value $D(T)$ is assumed to decompose into a sum of multiple debt elements $(D_1, \cdots, D_n)$ maturing at T with some weights, $(x_1, \cdots, x_n)$; $x_j \geq 0, \sum x_j = 1$. Let c be also a generic notation which represents a bankruptcy cost rate (amount) for proportional(constant) paying case. When the bankruptcy takes place, it is supposed that the firm is required to pay the cost of financial distress,

$$C_{bk}(T) = \begin{cases} c1(A(T) \leq D(T)) & \text{(constant)} \\ c(D(T) - A(T))^+ & \text{(proportion)}. \end{cases} \tag{18}$$

Using these notations, the terminal payoff of the equityholder is given by asset value after repaying the outstanding debt and paying corporate tax which is nonzero some amount if no bankruptcy takes place, or zero otherwise. That is,

$$\begin{aligned} \mathcal{S}(T) &= (A(T) - T_x(T) - D(T))1(A(T) \geq D(T)) \\ &= (A(T) - D(T))^+ - t_c \begin{cases} 1(A(T) \geq D(T)) & \text{(constant)} \\ (A(T) - D(T))^+ & \text{(proportion)} \end{cases}. \end{aligned} \tag{19}$$

The terminal payoff of the bondholder is of the form:

$$\begin{aligned} \mathcal{B}(T) &= D(T)1(A(T) \geq D(T)) + (A(T) - C_{bk})1(A(T) \leq D(T)) \\ &= -(A(T) - D(T))^+ + A(T) - c \begin{cases} 1(A(T) \leq D(T)) & \text{(constant)} \\ (D(T) - A(T))^+ & \text{(proportion)} \end{cases}. \end{aligned} \tag{20}$$

The first term is a debt obligation amount without bankruptcy, while the second term is collateral value secured by asset value after subtracting the cost in bankruptcy.

Together with those, the firm value is defined as the sum of these two contingent claims:

$$\begin{aligned}\mathcal{V}(T) &= \mathcal{S}(T) + \mathcal{B}(T) \\ &= A(T) - t_c \left\{ \begin{array}{ll} 1(A(T) \geq D(T)) & \text{(constant)} \\ (A(T) - D(T))^+ & \text{(proportion)} \end{array} \right\} - c \left\{ \begin{array}{ll} 1(A(T) \leq D(T)) & \text{(constant)} \\ (D(T) - A(T))^+ & \text{(proportion)} \end{array} \right\}. \end{aligned} \tag{21}$$

This form means that there are four expressions depending upon whether the tax is constant or proportional one in the equity function, and whether the bankruptcy cost is constant or proportional one in the debt present value. We shall hereafter classify those as follows; case (i) (constant, constant) of (tax, bankruptcy cost) , case (ii) (constant, proportion), case (iii) (proportion, constant), and case (iv) (proportion, proportion).

The present value $\mathcal{C}(0)$ of $(A(T)-D(T))^+/B_d(T)$ under the probability measure P is regarded as a call option of the underlying asset value with the strike price of terminal debt value, and the present value $\mathcal{P}_b$ of $1(A(T) < D(T))/B_d(T)$ is thought of as a risk-adjusted discounted bankruptcy probability that will be of representing a credit risk of the firm. Using the put-call parity, we obtain the present value formulae of the above contingent claims:

$$\begin{aligned} \mathcal{S}(0) &= A(0)\mathcal{C}(0) - t_c \left\{ \begin{array}{ll} 1 - \mathcal{P}_b & \text{(constant)} \\ \mathcal{C}(0) & \text{(proportion)}, \end{array} \right. \\ \mathcal{B}(0) &= A(0) - \mathcal{C}(0) - c \left\{ \begin{array}{ll} \mathcal{P}_b & \text{(constant)} \\ \mathcal{C}(0) - (A(0) - D(0)) & \text{(proportion)}, \end{array} \right. \\ \mathcal{V}(0) &= A(0) - t_c \left\{ \begin{array}{ll} 1 - \mathcal{P}_b & \text{(constant)} \\ \mathcal{C}(0) & \text{(proportion)} \end{array} \right\} - c \left\{ \begin{array}{ll} \mathcal{P}_b & \text{(constant)} \\ \mathcal{C}(0) - (A(0) - D(0)) & \text{(proportion)} \end{array} \right\}. \end{aligned} \tag{22}$$

B. *Optimal capital structure*

Either analytic or geometric nature of contingent claims,(22) is essentially of importance to elaborate on the optimal capital (sub-)structure issue in the subsequent sections. As in Kim(1978), Turnbull(1979), Ikeda(1993), examine the first order partial derivatives of contingent claims with respect to the debt ratio. For optimal capital structure,

$$\frac{\partial \mathcal{V}(0)}{\partial L} = \begin{cases} (t_c - c)\frac{\partial \mathcal{P}_b}{\partial L} & \text{(case (i))} \\ t_c \frac{\partial \mathcal{P}_b}{\partial L} - c\frac{\partial \mathcal{C}}{\partial L} - c\hat{D}(y) & \text{(case (ii))} \\ -t_c \frac{\partial \mathcal{C}}{\partial L} - c\frac{\partial \mathcal{P}_b}{\partial L} & \text{(case (iii))} \\ -(t_c + c)\frac{\partial \mathcal{C}}{\partial L} - c\hat{D}(y) & \text{(case (iv))}, \end{cases} \tag{23}$$

where $\hat{D}(x) \equiv \sum_{j=1}^n x_j D_j(0)$ ($D_j(0)$: face value of j-th debt, x_j: proportion of j-th debt such that $\sum_{j=1}^n x_j = 1$, $x_j \geq 0$ for $j = 1, \cdots, n$). We normalize the face value of each debt, $D_j(0)$ $(j = 1, \cdots, n)$, to be unity. [5] For the optimal debt capacity, examine the partial derivative,

[5] What we wish to determine is the face value of each debt (as well as investment amount of each asset). Let y_j denote the proprtion of j-th debt instrument whose price is normalized to be one. Let y'_j $(j = 1, \cdots, n)$ denote the unit number of j-th debt instrument whose price is $D_j(0)$ in one unit. The j-th debt amount is expressed as either Ly_j in the first convension or $y'_j D_j(0)$ in the second convension. In the first representation the budget constraint is $\sum_{j=1}^n y_j = 1$, while in the second representation it is $\sum_{j=1}^n y'_j D_j(0) = L$. Clearly the face value of each debt issued must be the same in both representaions on solving certain optimization problems, that means $Ly_j = y'_j D_j(0)$. This immediately provides a transformation rule such as $y'_j = Ly_j/D_j(0)$. Analogously for the asset prices the investment amount is the same in both representaion, so $x_i \cdot 1 = x'_i A_i(0)$ $(i = 1, \cdots, m)$ where total asset amount is normalized to be one, and x_i is the proportion of i-th asset with the asset price being normalized to be one, and x'_i is the number of i-th asset instrument

$$\frac{\partial \mathcal{B}(0)}{\partial L} = \begin{cases} -\frac{\partial \mathcal{C}}{\partial L} - c\frac{\partial \mathcal{P}_b}{\partial L} & \text{(constant)} \\ -(1+c)\frac{\partial \mathcal{C}}{\partial L} - c\hat{D}(y) & \text{(proportion).} \end{cases} \tag{24}$$

Given the weight vector of debts, optimal debt ratio and debt capacity are determined as zero point solutions of each of above two partial derivatives equating to zero. However, it is not the case, when unknown variables are $\mathbf{X} = (\mathbf{x}, L)$ and weight vector is constrained on a simplex region. Correspondingly, these two cases are involved in optimization issues formulated as (I) $\max_{\mathbf{X}} \mathcal{V}$ subject to $\sum_{j=1}^{n} x_j = 1, x_j \geq 0, L \geq 0$, (II) $\max_{\mathbf{X}} \mathcal{B}$ subject to the same constraints as (I). In order to discuss the optimality we must invoke to the Lagrange multiplier method. Let λ_j $(j = 1, \cdots n+1)$ and μ stand respectively for the Lagrange multipliers of $(n+1)$ inequality constraints and one equality constraint. Then the Lagrange function to be minimized is described as $\mathcal{L}_a = -\mathcal{V}$ (or $\mathcal{B}$) $-\sum_{j=1}^{n+1} \lambda_j X_j + \mu(\sum_{j=1}^{n} x_j - 1)$. To see the relationship between optimal capital structure and debt capacity for four market friction cases, we shall examine the first order conditions of optimality with respect to the debt ratio, i.e., $\partial \mathcal{L}_V / \partial L = 0$, $\partial \mathcal{L}_B / \partial L = 0$. As a result, except for case (i) it is by no means concluded without any additional condition in the multiple liability (and/or asset) substructure models that the optimal capital structure is less than the optimal debt capacity, as proved in Kim(1978) and so on. Let us see the reason subsequently by each case.

In the exceptional case (i) the first order optimality conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}_V}{\partial L} &= -(t_c - c)\frac{\partial \mathcal{P}_b}{\partial L} - \lambda_{n+1} = 0, \\ \frac{\partial \mathcal{L}_B}{\partial L} &= \frac{\partial \mathcal{C}^0(0)}{\partial L} + c\frac{\partial \mathcal{P}_b}{\partial L} - \lambda_{n+1} = 0. \end{aligned} \tag{25}$$

Using the fact that $\partial \mathcal{C}^0 / \partial L$ is negative, $\partial \mathcal{P}_b / \partial L$ is positive (as both proved in Appendix B), and any optimal solution of $(x_j, L; \lambda_j, \lambda_{n+1}, \mu)$ satisfies complementary conditions, $\lambda_j X_j = 0$ with $\lambda_j \geq 0$, $X_j \geq 0$, $\sum_{j=1}^{n} x_j - 1 = 0$, we find that the above first equation leads to the optimal capital structure, $L^* = 0$ $(\lambda_{n+1}^* > 0)$ only when $t_c < c$, and the above second one leads to the optimal debt capacity, $L^{**} > 0$ $(\lambda_{n+1}^{**} = 0)$. Thus desired order inequality $L^*(= 0) < L^{**}$ comes out somehow just in the case (i).

For the other three cases, as long as the liability (and/or asset) substructure is multiple, such a multiplicity of instruments as additional degrees of freedom to one-dimensional debt ratio would admit a break down of the order relationship between the capital structure and debt capacity, although that has been established as a matter of course in a single asset and liability framework. For instance, in case (iv) this phenomenon is illustrated in short: The relevent first order optimality conditions are of the form,

$$\begin{aligned} -\frac{\partial \mathcal{C}^0(0)}{\partial L}|_{X^*} &= \frac{1}{c + t_c}(c\hat{D}^0(x^*) - \lambda_{n+1}^*), \\ -\frac{\partial \mathcal{C}^0(0)}{\partial L}|_{X^{**}} &= \frac{1}{c+1}(c\hat{D}^0(x^{**}) - \lambda_{n+1}^{**}). \end{aligned} \tag{26}$$

For both of non-zero optimal capital structure and debt capacity to exist simultaneously, it

with asset price $A_i(0)$. Noticing the respective budget constraints, $\sum_{i=1}^{m} x_i = 1$ or $\sum_{i=1}^{m} x_i' A_i(0) = 1$, there is a transformation rule, $x_i' = x_i / A_i(0)$, too. As far as such transformation rules are taken into account, normalization convention such that the asset and debt prices are all unity does not lose the generality of subsequent arguments, so we shall hereafter follow such convention.

is sufficient that $\lambda^*_{n+1} = \lambda^{**}_{n+1} = 0$. Noting that $\partial\mathcal{C}^0(0)/\partial L$ and $\hat{D}^0(x)$ of both hand side in the above equations are varied according to the changes of the proportion of debt instruments, we recognize it sometimes occur that an order relationship $L^* < L^{**}$ at some debt configuration is made inverse at the other configuration. However, imposing additional appropriate conditions could make this sort of order relationship invariant under any change of configuration. As will be demonstrated in the next subsection, if the contingent claims of the market value of debt and the firm value are strictly concave (its Hessian is negative definite) in the extended space of X, then invariance of order $L^* < L^{**}$ is proved as follows: Consider the sign of marginal market value of debt carefully at the optimal capital structure, not but at the debt capacity. Note that the marginal market value of debt at optimal capital substructure in terms of the marginal option value is positive:

$$\frac{\partial\mathcal{B}(0)}{\partial L}|_{X^*} = -(c+1)\frac{\partial\mathcal{C}(0)}{\partial L}|_{X^*} - c\hat{D}|_{X^*} = \frac{c(1-t_c)}{c+t_c}\hat{D}(\mathbf{x}^*) > 0. \tag{27}$$

Since the market value of debt is strictly concave by assumption, the above fact readily suggests that there arises a debt capacity at some level over the optimal debt ratio when increasing the debt ratio. That is, inequality $L^* < L^{**}$ holds for any other debt instruments.

In the simplified case of (iv) (a single asset and liability substructure) the only marginal option value is allowed to vary along with L, and the other quantities are all fixed constants in Eq.(26). Taking into account a trivial inequality, $c/(c+1) < c/(c+t_c)$ and lower/upper bounds of marginal option value in Appendix, we surely re-establish the inequality $L^* < L^{**}$ of Kim(1978) and Ikeda(1993). Next in case (ii) the first order optimality conditions lead to

$$\begin{aligned} -\frac{\partial\mathcal{C}^0(0)}{\partial L}|_{X^*} &= \frac{1}{c}(-t_c\frac{\partial\mathcal{P}_b}{\partial L}|_{X^*} + c\hat{D}^0(x^*) - \lambda^*_{n+1}), \\ -\frac{\partial\mathcal{C}^0(0)}{\partial L}|_{X^{**}} &= \frac{1}{c+1}(c\hat{D}^0(x^{**}) - \lambda^{**}_{n+1}). \end{aligned} \tag{28}$$

These inequalities do not yield definite relationship of L^* and L^{**} on account of the multiplicity of debt, but when in single asset and debt case, it follows that $L^* < (>) L^{**}$ depending upon whether $-\partial\mathcal{P}_b/\partial L| < (>) c\hat{D}(0)/(t_c(c+1))$. Due to

$$\frac{\partial\mathcal{B}(0)}{\partial L}|_{X^*} = -\frac{t_c(c+1)}{c}\frac{\partial\mathcal{P}_b}{\partial L}|_{X^*} + \hat{D}|_{X^*},$$

the same criterion as the above is obtained. In the last case (iii) the first order optimality conditions are

$$\begin{aligned} -\frac{\partial\mathcal{C}^0(0)}{\partial L}|_{X^*} &= \frac{1}{t_c}(c\frac{\partial\mathcal{P}_b}{\partial L}|_{X^*} - \lambda^*_{n+1}), \\ -\frac{\partial\mathcal{C}^0(0)}{\partial L}|_{X^{**}} &= (c\frac{\partial\mathcal{P}_b}{\partial L}|_{X^{**}} - \lambda^{**}_{n+1}). \end{aligned} \tag{29}$$

These equations do not also yield any definite relationship of L^* and L^{**} in multiple debt case. However, from the same reason as case (iv) such that the marginal market value of debt at the optimal capital structure is positive,

$$\frac{\partial\mathcal{B}(0)}{\partial L}|_{X^*} = \frac{c(1-t_c)}{t_c}\frac{\partial\mathcal{P}_b}{\partial L}|_{X^*} > 0,$$

due to $\partial\mathcal{P}^b/\partial L > 0$ in the lemma of Appendix B, we get $L^* < L^{**}$, whenever the concavity condition of the market value of debt and the firm value is satisfied. Note that when such

an additional condition is not met, global optimality (uniqueness) of each solution, L^*, L^{**}, is no longer guaranteed, as will be demonstrated in next subsection. In the simplified case of (iii), that is, the case of single asset and liability substructure, a trivial inequality $1/t_c > 1$ immediately results in $L^* < L^{**}$ as well.

Now, as a preliminary stage of research it seems still quite instructive to study a single asset and liability substructure model. It is partly because the complexity of the computation of relevent contingent claims is reduced drastically, and moreover partly because optimization problems of relevent contingent claims are easily solved to come up with an optimal single asset and liability substructure. That is why in the remainder of this subsection we will confine ourselves to study a single asset and laibility substructure model, to derive closed form expressions of relevent contingent claims and to present their comparative statics.

Under an assumption of the multi risk factors economy the equity value is obtained as

$$\begin{aligned} \mathcal{S}(0) &= (1-t_c)A(0)\mathbf{E}^Q[(1-D^0(T))^+|\mathcal{F}_0] \\ &= (1-t_c)\{A(0)N_1(d) - LN_1(d-\sigma)\}, \end{aligned} \tag{30}$$

the market value of debt is

$$\begin{aligned} \mathcal{B}(0) &= A(0)\mathbf{E}^Q[D^0(T)\mathbf{1}(D^0(T) \le 1) + ((1+c) - cD^0(T))\mathbf{1}(D^0(T) > 1)|\mathcal{F}_0] \\ &= (1+c)A(0) - cL - (1+c)\{A(0)N_1(d) - LN_1(d-\sigma)\}, \end{aligned} \tag{31}$$

and the firm value is as the sum of both values,

$$\mathcal{V}(0) = (1+c)A(0) - cL - (c+t_c)\{A(0)N_1(d) - LN_1(d-\sigma)\}, \tag{32}$$

in the closed forms, where $N_1(\cdot)$ denotes a one-dimensional normal distribution function, and we used an expression of terminal debt value, $D^0(T) = D^0(0) \exp(\int_0^T \sum_k \gamma_k^0 dW_k(u) - \sigma^2/2)$ with an initial value $D^0(0) = L/A(0)$, and

$$\begin{aligned} d &= \frac{\ln(\frac{A(0)}{L}) + \frac{\sigma^2}{2}}{\sigma}, \\ \sigma^2 &= \mathbf{E}^Q[\{\sum_{k=1}^K \int_0^T (\gamma_{ik} - \gamma_{Ak})dW_k^Q(u)\}^2|\mathcal{F}_0] \\ &= \int_0^T (\gamma\gamma_{ii}^\top + \gamma\gamma_{AA}^\top - 2\gamma\gamma_{iA}^\top)du. \end{aligned} \tag{33}$$

Note that σ is regarded as a relative volatility of some debt instrument (subscript i in the above expression means i-th debt chosen from the available debt portfolio) relative to the aggregate asset one, not but as an absolute volatility of debt instrument alone, and we have $\lim_{L\to 0}\mathcal{V}(0) = (1+c)A(0) - (c+t_c)$.

The globally optimal (maximum) capital structure is attained at the face debt value L^* which is a solution of the transcendental equation:

$$\frac{\partial \mathcal{V}(0)}{\partial L} = -c + (c+t_c)N_1(d-\sigma) = 0. \tag{34}$$

Its maximum nature is due to the negative second derivative of the firm value with respect

to the debt ratio over whole range of debt ratio, i.e., $\partial^2\mathcal{V}(0)/\partial L^2 = -(c+t_c)n_1(d-\sigma)/(\sigma L) < 0$ ($n_1(\cdot)$ is a one-dimensional normal density function). Further, it will be verified in the next subsection that concavity of this sort is distribution-free nature. Noting the similar concavity nature of the debt value function, we find a face debt value L^{**} at which the globally optimal (maximum) debt value is attained as well. It is just a solution of

$$\frac{\partial\mathcal{B}(0)}{\partial L} = -c + (c+1)N_1(d-\sigma) = 0. \tag{35}$$

Equations $\partial\mathcal{V}(0)/\partial L = 0$, $\partial\mathcal{B}(0)/\partial L = 0$ yield, respectively, $N_1(d(L_V)-\sigma) = c/(c+t_c)$ and $N_1(d(L_B)-\sigma) = c/(c+1)$. As $c/(c+t_c) > c/(c+1)$, $N_1(\cdot)$ is an incresing function of argument, and $d(L)$ is a decreasing function of L, we can re-establish the relationship $L^* < L^{**}$ which has been already proved by kim(1978) and Ikeda(1993) and so on in the context of single asset and liability substructure models.

Let F stand for the left hand side of Eq.(34). The sensitivity of the optimal debt ratio with respect to σ (Eq.(33)) is represented by

$$\begin{aligned}\frac{dL^*}{d\sigma} &= -\frac{\partial F/\partial\sigma}{\partial F/\partial L^*} \\ &= -L^*\cdot d,\end{aligned} \tag{36}$$

employing the implicit function theorem. From this we notice that instantaneous changes of the optimal debt ratio to σ depend crucially upon the sign of d. Let $\bar{L}$ denote a critical debt ratio obtained as a solution of $d(L)=0$ which is equal to $A(0)\exp(\sigma^2/2)$. It follows from this that if $L^* < (\geq)\bar{L}$, then $\partial L^*/\partial\sigma < (\geq)0$. Since $\bar{L}$ is clearly over the natural upper bound, that is, an aggregated asset side value, $A(0)$, unless $\sigma = 0$, $\bar{L}$ will not be actually achieved. In other words, the firm manager taking solvent debt financing policy will not be willing to achieve $\bar{L}$. As to the sensitivity of the optimal debt ratio with respect to one of multi-risk factors, letting it be denoted by γ_{Lk}, the volatility function of a single debt to k-th risk factor ($k = 1,\cdots,K$), and assuming it to be of the form, $\gamma_{Lk}(u,T) = \gamma_{ik}\cdot(T-u)$ (that is, a constant volatility assumption in the interest rate processes), we have

$$\frac{dL^*}{d\gamma_{Lk}} = -\frac{L^*d}{\sigma}(\gamma_{Lk}-\gamma_{Ak})\cdot T. \tag{37}$$

This sign is evidently affected by $d\cdot(\gamma_{Lk}-\gamma_{Ak})$, and that requires messy further classification. We put these findings together into the proposition:

Proposition 1 Sensitivity of the Optimal Debt Ratio to the Relative Volatility of Single Debt

In the framework of single asset and liability substructure model with imperfect market frictions, when given an initial asset value $A(0)$, there exists an optimal debt ratio, L^*, and it is a decreasing (increasing) function of the relative volatility of single debt instrument to the asset one, σ, in the range of $L^* < (>)A(0)\exp(\sigma^2/2)$. Furthermore it is also a decreasing (increasing) function of an endogenous factor risk (k-th one) exposure parameter, γ_{Lk}, when $d\cdot(\gamma_{Lk}-\gamma_{Ak}) > (<)0$ under the assumption of constant volatility of uncertain interest rate processes.

C. *Optimal capital substructure*

In the proof of the global optimality of capital structure in a single asset and debt framework, the concavity of the firm value and the market value of debt was of considerable importance. Does this geometric nature still remain valid, when this framework is enlarged to accomodate itself to either multiple asset substructure or multiple debt one? The answer is No. More surprisingly this implies that the imperfect market assumption in the higher dimensional direct product space of weight vectors times debt ratio may cause the occurrence of several local optimal solutions of capital structure as well as its substructure for some parameter sets. Even if we manage to devise new other leverage-related costs in our framework alternatively, it will not necessarily leads to the unique existence of the optimal capital (sub-)structure. To this respect it makes remarkable contrast that in single asset and debt models the uniqueness of optimal capital structure is guaranteed in any case. For both the optimal capital structure and substructure to exist uniquely, it is expected that economic parameters such as leverage-related costs and volatility functions of instruments must satisfy certain conditions. We shall seek out those conditions, interpret their financial meaning and elaborate on their implications in the corporate finance.

First of all, we start again with an assumption that the capital substructure consists of an aggregate single asset and multiple debt instruments for simplicity of exposition. That assumption will be relaxed so as to incorporate the multiple asset instruments later. Let $X(t) \equiv 1 - L\sum_{i=1}^{n} x_j D_j^0(t)$, $(x \in \Delta^{n-1})$. Let $\mathcal{C}^0(0)$ be the present value of $X^+(T)$ under the probability measure Q, where the aggregate asset value is taken as a numéraire. Then relevent contingent claims are given in order: The equity value is expressed in terms of $\mathcal{C}^0(0)$ as

$$\mathcal{S}(0) = (1 - t_c)A(0)\mathcal{C}^0(0), \tag{38}$$

the market value of debt is

$$\begin{aligned} \mathcal{B}(0) &= -\frac{1+c}{1-t_c}\mathcal{S}(0) + (1+c)A(0) - cD(0) \\ &= -(1+c)A(0)\{\mathcal{C}^0(0) - \frac{c}{c+1}X(0) - \frac{1}{c+1}\} \end{aligned} \tag{39}$$

the firm value is

$$\begin{aligned} \mathcal{V}(0) &= -\frac{c+t_c}{1-t_c}\mathcal{S}(0) + (1+c)A(0) - cD(0) \\ &= -(c+t_c)A(0)\{\mathcal{C}^0(0) - \frac{c}{c+t_c}X(0) - \frac{1}{c+t_c}\}, \end{aligned} \tag{40}$$

using identities, $\mathbf{1}(X > 0) = 1 - \mathbf{1}(X \leq 0)$, and $X^+ = (-X)^+ + X$. As obviously seen from Eqs.(38),(39),(40), we have bounds of these contingent claims as

$$\begin{aligned} &(1-t_c)(A(0) - D(0))^+ \leq \mathcal{S}(0) \leq (1-t_c)A(0), \\ &\qquad -cD(0) \leq \mathcal{B}(0) \leq -(1+c)(A(0) - D(0))^+ + (1+c)A(0) - cD(0) \leq A(0), \\ &(1-t_c)A(0) - cD(0) \leq \mathcal{V}(0) \leq -(c+t_c)(A(0) - D(0))^+ + (1+c)A(0) - cD(0) \leq A(0). \end{aligned} \tag{41}$$

In general, the equity value embedded in the firm value is thought of as a call option whose strike price is the total principal of debt contracts, so it seems likely to inherit a geometric nature, i.e., convexity of call option's property. It is trivial to verify that the equity value is convex along the L-direction, but it is nontrivial whether the equity value is convex

or not in a simplex space of the weight vector. As with the latter, we can prove

Proposition 2 Convexity of Equity Value with Multiple Debts
Given a debt ratio, the equity value is a convex function with respect to the weight vector of debt instruments.
Proof: See the Appendix.

Assume that C^0 is C^2-class function with respect to the weight vector and debt ratio. From the above proposition it turns out that the Hessian of the equity value, $\partial^2 S/\partial x_i \partial x_j$, $(i, j = 1, \cdots, n)$, is positive semidefinite in the weight space without any explicit computation of partial derivatives. However, it is not so easy to check whether such geometric properties of convexity/concavity exist or not in enlarged space of both the weight vector and debt ratio. Let $\mathbf{X}$ denote $(\mathbf{x}, L)$. From Eqs.(38), (39) and (40) Hessians of the equity value, market value of debt and firm value are written by a generic notation as

$$\begin{aligned} H_a &= \eta_a A(0) \begin{pmatrix} \frac{\partial^2 C^0}{\partial \mathbf{x}^2} & \frac{\partial^2 C^0}{\partial \mathbf{x} \partial L} + \xi_a \mathbf{D}^0(0) \\ \frac{\partial^2 C^{0\top}}{\partial \mathbf{x} \partial L} + \xi_a \mathbf{D}^{0\top}(0) & \frac{\partial^2 C^0}{\partial L^2} \end{pmatrix} \\ &\equiv \eta_a A(0) \begin{pmatrix} \mathbf{h} & \mathbf{m} + \xi_a \mathbf{D}^0(0) \\ \mathbf{m}^\top + \xi_a \mathbf{D}^{0\top}(0) & l \end{pmatrix}, \end{aligned} \tag{42}$$

where index a runs from S(equity value),B(market value of debt), V(firm value), and $\eta = ((1 - t_c), -(c+1), -(c+t_c))$, $\xi = (0, c/(c+1), c/(c+t_c))$. Imagine a specific situation where the off-diagonal entries are all zero, $\mathbf{m} + \xi_a \mathbf{D}^0(0) = 0$. Then it is trivial that the Hessian is positive semidefinite, because $\mathbf{h}$ is positive semidefinite due to the proposition 2, and $l \geq 0$. Hence, as we wish to examine the positive (or negative) semidefiniteness of the Hessian of the equity value(market value of debt and firm value), it will not lose the generality of subsequent arguments to assume $\mathbf{m} + \xi_a \mathbf{D}^0(0) \neq \mathbf{0}$ hereafter.

Before working with the proof of positive definiteness of Hessians, we will provide a method suitable for exploring some implications of market frictions in the form isolated purely from the MM's leverage irrelevancy paradigm. To this end we may decompose the Hessian as $\mathbf{H} = \mathbf{H}_0 + \mathbf{F}$, where

$$\mathbf{H} = \begin{pmatrix} \mathbf{h} & \mathbf{m} \\ \mathbf{m}^\top & l \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} O & \xi \mathbf{D}^0(0) \\ \xi \mathbf{D}^{0\top}(0) & 0 \end{pmatrix}.$$

Taking notice of lemma 1, 2 of Appedix with regard to the regularity of Hessian and $\mathbf{H}^{-1} = (\mathbf{H}_0(\mathbf{I} + \mathbf{H}_0^{-1}\mathbf{F}))^{-1} = (\mathbf{I} + \mathbf{H}_0^{-1}\mathbf{F})^{-1}\mathbf{H}_0^{-1}$, we find that if the spectral radius, $\rho(\mathbf{H}_0^{-1}\mathbf{F})$, is less than 1, then it is expanded as a convergent infinite sum, $\sum_{i=0}^{\infty}(-\mathbf{H}_0^{-1}\mathbf{F})^i \mathbf{H}_0^{-1}$. Hence, when the market frictions are weak, that is, $\mathbf{F}$ is regarded as a perturbation to the MM's leverage irrelevancy world, and the above condition about the spectral radius is met, $\mathbf{H}^{-1}$ will be approximated to $\mathbf{H}_0^{-1} + \mathbf{H}_0^{-1}\mathbf{F}\mathbf{H}_0^{-1} + \cdots$, as in an analogy of the perturbation theory of quantum physics. This inverse would be helpful in examining the comparaive statics at every order of market frictions.

As concerns the positive or negative definiteness conditions of Hessians, the key inequality is one of lemma 4 in Appendix which is thought of as a second order polynomial of ξ, that is, $\xi^2 \mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{D}^0 + 2\xi(\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{m}) + \mathbf{m}^\top \mathbf{h}^{-1}\mathbf{m} - l (\leq 0)$. Here note that the quantities included in its coefficients, $l, \mathbf{m}, \mathbf{D}^0, \mathbf{h}^{-1}$, are all independent of ξ. Recalling that $\xi_S = 0, 0 < \xi_B = c/(c+1) <$

$1{,}0 < \xi_V = c/(c+t_c) < 1$, we understand that this condition can be transformed to a more concise form. Let Π denote a discriminant of that quadratic equation, which equals to

$$\Pi = (\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{m})^2 - (\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{D}^0)\cdot(\mathbf{m}^\top\mathbf{h}^{-1}\mathbf{m} - l). \tag{43}$$

Let two roots of that equation be denoted by $\xi_\pm$, which is $[-(\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{m}) \pm \sqrt{\Pi}]/(\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{D}^0)$ respectively, whenever $\Pi \geq 0$. Except for the equity value case, in which the key inequality reduces to a trivial one, $\mathbf{m}^\top\mathbf{h}^{-1}\mathbf{m} - l \leq 0$ due to $\xi_S = 0$, both the market value of debt and firm value cases require that the market friction parameter, $\xi_a (a = B, V)$ satisfying each of the key inequalities enters into the significant range of $(0,1)$. Namely, this conforms to the conditons, $\Pi \geq 0$, $0 < \xi_+$ and $\xi_+ < 1$. Putting these facts together, we get the following proposition about the positive or negative (semi-)definiteness conditions of Hessians which are indepedent of the market friction parameters.

Proposition 3 Positive or Negative Definiteness Conditions of Each Hessian

(i) For the Hessian of equity value, positive (semi-)definiteness condition is of the form,

$$l > (\geq)\mathbf{m}^\top\mathbf{h}^{-1}\mathbf{m}, \tag{44}$$

(ii) For the market value of debt and firm value, the negative (semi-)definiteness conditions of both cases coincide with

$$\begin{aligned} &l > (\geq)\mathbf{m}^\top\mathbf{h}^{-1}\mathbf{m} - \frac{(\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{m})^2}{(\mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{D}^0)}, \\ &\sqrt{\Pi} - \mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{D}^0 < \mathbf{D}^{0\top}\mathbf{h}^{-1}\mathbf{m} < -\sqrt{\Pi}(<0). \end{aligned} \tag{45}$$

There are two remarks. First, in case (i) if $\mathbf{h}$ is positive definite, then the right hand side is as a matter of course greater than zero for arbitrary $\mathbf{m}$. Second, the inequality $l \geq 0$ trivially proved implies that when the right hand side of the first inequality in (ii) is negative, that condition becomes trivial one.

Now we turn to deducing some definite results from the optimization of relevent contingent claims, employing the propositions above and some lemmas provided in Appendix. As demonstrated below, the results depend crucially upon which contingent claim is chosen as the objective. Therefore the subsequent analysis may be classified to two cases, (i) equity value, and (ii) market value of debt or firm value, as expected immediately from the proposition 3.

To begin with, consider the optimization of equity value. Maximization problem with the debt ratio being fixed leads to the proposition.

Proposition 4 Maximization of Equity Value with Multiple Debts

Given a total asset value, if the conditions, Eq.(44) is satisfied, then the maximization problem of the equity value has a trivial extremal point solution of the simplex space the weight vector spans and zero debt ratio. When a debt ratio is given in addition to the total asset value, this problem results in a trivial extremal point solution in the simplex region of weight vectors without requiring any other conditions like Eq.(44).

Proof:

Since $S(x, L)$ is a convex function of weight vector x and L, constrained to the simplex region (convex set) when the conditions, Eq.(44) is met, the above statement holds trivially. See relevant text of optimization. Due to the proposition 2 $S(x)$ is also a convex function of the weight vector x without requiring any other conditions. Hence the above statement results in a trivial extremal point solution of the weight vector. ||

In this proposition the extremal point solution means that a single debt substructure is optimal in the weight vector space. In regard to the latter part of proposition, maximization of the equity value yields an extremal point solution of the one-dimensional debt ratio space, $L^* = 0$, (i.e., no-leveraged policy is optimal) since $\mathcal{C}^0$ is monotonically decreasing function of L (which tends to zero as the debt ratio goes up to the infinity) on account of the lemma in Appendix. On the other hand, minimization of the equity value leads to an unrealistic result, $L^* = \infty$, due to the same reason as above. See the lower bound of Eq.(41) and lemma of marginal option value in Appendix. From these considerations we can state that unless total debt amount is bounded by any reasonable constraint from the financial point of view or other rational reasons, there exists no bounded solution of it. Thus when the firm manager chooses the equity value as an objective to be minimized or maximized in making decision, that policy leads us to a trivial or insignificant consequence.

For fixed L there is a further remark with respect to which debt instruments are chosen in maximizing the equity value, (i) when an aggregate single asset is already given, or (ii) when several multiple assets are already given.

In the case (i), the most volatile debt relative to the aggregate asset value is chosen at the optimal extremal point. [6] This is because a call option embedded in the equity value satisfies an inequality, $\max_x \mathbf{E}^Q[(1 - L\sum_{i=1}^n x_i D_i^0(T))^+|\mathcal{F}_0] \leq \max_x \sum_{i=1}^n x_i \mathbf{E}^Q[(1 - LD_i^0(T))^+|\mathcal{F}_0] \leq \max_i \mathbf{E}^Q[(1 - LD_i^0(T))^+|\mathcal{F}_0]$, where the last inequality comes from the fact that the extremal point solution is optimal. The right hand side of the above inequality looks like a put option of the relative price with the strike price being unity. Evidently the larger the relative volatility of debt is, the higher the premium of option is. Thus the premium of put is dominated by the most volatile single debt relative to the aggregate asset value, as apparently seen from Eq.(30) and the total relative risk, Eq.(33), (which may be rewritten as $(\sigma_A^2 + \sigma_L^2 - 2\rho_{AL}\sigma_A\sigma_L)^{1/2} \equiv \sigma_{tot}$, $\sigma_A^2 = \int_0^T \gamma\gamma_{AA}^\top du$, $\sigma_L^2 = \int_0^T \gamma\gamma_{LL}^\top du$, $\rho_{AL}\sigma_A\sigma_L = \int_0^T \gamma\gamma_{AL}^\top du$).

In the case (ii) it is not so easy to identify which single asset and debt is selected. The optimal combination of instruments is decided so as to maximize the relative volatility of Eq.(33), in other words, to maximize a *relative* distance (measured by a norm, $\sqrt{\mathbf{E}[f^2(\cdot)]}$ for some square integrable function $f(\cdot)$) of two stochastic integrals $\sum_k \int_0^T \gamma_{ik}(u)dW_k^P(u)$ (i:asset indices), $\sum_k \int_0^T \gamma_{jk}(u)dW_k^P(u)$ (j:debt indices), on the probability space. Hence the resulting optimal instruments do not have always the maximum (absolute) volatility among each of available asset and debt portfolios. Namely, the following statement, "*When the firm manager maximizes an equity value, it is optimal that he or she invests to the riskiest asset, financing with issuing the riskiest debt*", is not necessarily correct, contrary to our naive expectations. The reason is that the affection of off-diagonal entries of covariance matrix in the total relative risk cannot be disregarded in their optimization, and consequently there would exist a combination of asset and debt maximizing the relative distance defined above. Roughly speaking, from the completion of the square of total relative risk as $\sigma_{tot}^2(\sigma_L) = (\sigma_L - \rho_{AL}\sigma_A)^2 + \sigma_A^2(1 - \rho_{AL}^2) \geq \sigma_A^2(1 - \rho_{AL}^2)$, it seems likely that given a relatively small ρ_{AL}, as the increase of σ_A rises σ_{tot}^2, an asset with maximum (absolute) volatility will be chosen among

[6] The author acknoledges Professor Kusuoka for informing the author of this respect at the first meeting of Japan Financial Economics Association(1993).

available asset portfolio. For this asset chosen, when there is a portfolio of debt instruments with correlations, $\rho_{AL_j} > 0$ ($j = 1, \cdots$, number of debts) and volatilities, $\{\sigma_{Lj}\}$, all of which are larger than $2\rho_{AL_j}\sigma_A$ (say case 1), a debt instrument with maximum (absolute) volatility is chosen among such a debt portfolio. On the contrary, when all debts have the volatilities less than $2\rho_{Aj}\sigma_A$ and correlations, $\rho_{Aj} > 0$ for the other debts (case 2), a debt instrument with volatility being equal to $\max[\sigma_{tot}(\min\{\sigma_{L_j}\}), \sigma_{tot}(\max\{\sigma_{L_j}\})]$, is chosen. Thus the case 2 is just a counter-example of the above statement. Note that since the total relative risk at the extremal point is symmetric in exchanging an asset and a debt, above argument based upon the relative distance of volatility functions is valid in itself even for asset portfolio selection problems. As a specific, may be, exceptionally simple case when asset and debt instruments have no correlation with each other, the above statement holds, i.e., the single asset and debt with maximum (absolute) volatility in each portfolio are selected. This is just because σ_{tot}^2 reduces to a simple form, $\sigma_A^2 + \sigma_L^2$.

So far we have studied the equity value optimization. Next, let us proceed to the optimization of the market value of debt as well as the firm value. Here, we should notice that given a debt ratio exogenously, for arbitrary integer $s(s = 1, \cdots, p)$ $X(\sum_s \theta_s x^s) = 1 - \sum_{s,j} \theta_s x_j^s LD_j(t)/A(t) = \sum_s \theta_s(1 - \sum_j x_j^s LD_j(t)/A(t)) = \sum_s \theta_s X(x^s)$ and $X^+(\sum_s \theta_s x^s) \leq \sum_s \theta_s X^+(x^s)$ in the parameter space $\theta_s \in \Delta^{p-1}$. With those we can prove that a function $F(x;\alpha) \equiv \mathbf{E}^Q[X^+ - \alpha X|\mathcal{F}_0]$ satisfies an inequality,

$$\begin{aligned} F(\sum_s \theta_s x^s; \alpha) &\leq \mathbf{E}^Q[\sum_s \theta_s\{X^+(x^s) - \alpha X(x^s)\}|\mathcal{F}_0] \\ &= \sum_s \theta_s \mathbf{E}^Q[X^+(x^s) - \alpha X(x^s)|\mathcal{F}_0] \\ &= \sum_s \theta_s F(x^s; \alpha), \end{aligned}$$

that is, the function $F(x)$ is a convex function with respect to the weight vector. Since the firm value and market value of debt are of the form proportional to $-F(x; c/(c + t_c))$ and $-F(x; c/(c + 1))$, respectively, the above inequality proves that their values are concave functions in the weight vector space which is a subspace of entire space, $(\mathbf{x}, L)$. Imagine an asymmetric short straddle position in option contracts for $-F(\cdot)$. That heuristic example will help us get the intuitive understanding of their geometric natures. Thus, together with proposition 3, this observation leads to:

Proposition 5 Maximization of Firm Value (or Market Value of Debt) with Multiple Debts

Given a total asset value, if the conditions, Eq.(45) are satisfied, then the maximization problem of the firm value (or market value of debt) has a non-trivial global optimal solution of both the weight vector and debt ratio. When a debt ratio is given in addition to the total asset value, this problem results in a non-trivial global optimal solution of the weight vector without requiring any other conditions like Eq.(45).

We have two remarks associated with this proposition.

First, let us consider a problem of whether the global optimal solution of debt ratio belongs or not to the proper range of solvent debt financing, i.e., $L^*\hat{D}(0) \in [0, A(0)]$ ($\hat{D}(0) \equiv \sum_j^n x_j D_j(0)$). If we can prove an inequality like $\partial\mathcal{V}/\partial L|_0 \cdot \partial\mathcal{V}/\partial L|_{A(0)/\hat{D}(0)} \leq 0$ with respect to L when Eq.(45) is satisfied, then this verifies that the optimal capital structure, L^*, resides in a naturally solvent range, $(0, A(0)/\hat{D}(0)]$, due to the concavity property of $\mathcal{V}$. However,

unfortunately we can find a counter-example. In a single debt case, that inequality is $((c+t_c)N(d-\sigma)|_0 - cD(0)) \cdot ((c+t_c)N(d-\sigma)|_{A(0)} - cD(0))$, and a slightly close consideration elucidates that its sign is indefinite depending upon some parameters like c,t_c, L and so on. Therefore we cannot always say that the optimal debt ratio is reached below 100 % debt financing when gradually increasing the debt ratio from zero.

The second remark is as follows. As easily seen from the shape of cross section of the firm value function, its curvature along the debt ratio, i.e., l in Hessian(42), would be the largest around the debt ratio being nearly equal to the total asset value, $L \sim A(0)$, and it would increase as the total relative risk of instruments decreases through its successful diversification among the available asset and debt portfolio. Note that from the definition the l is by no means affected by the market friction parameters, whether they are large or not. Apparently, the essential condition of global optimality of capital (sub-)structure, Eq.(45), requires that l must exceed a certain variable lower bound which is thoroughly independent of the market imperfections. For l to be large enough to exceed such a lower bound, the total relative risk of instruments must be rendered small to a certain extent around the neighborhood of $L \sim A(0)$. In other words, if the diversification of risks involved in the asset and liability substructure is not sufficient, and Eq.(45) is not satisfied, then the global optimality of both the weight vectors and debt ratio will be spoiled, and consequently it allows for emergence of unexpected local optimum solutions. Thus the diversification of risks by optimal portfolio selection does not have only a well-accepted traditional meaning of reducing total risk originating from uncertain value changes of asset and debt instruments, but also, more importantly, another meaning of assuring us of a unique equilibrium capital (sub-)structure.

Now we proceed to the comparative statics of the optimal debt ratio and firm value, in particular, with respect to the bankruptcy cost and tax rate.

The behavior of optimal debt ratio within an appropriate range of (c, t_c) (nonzero and bounded one) is examined by computing dL^*/dc and dL^*/dt_c. When the condition of regularity of Hessian in the lemma of Appendix is satisfied, they are given by implicit function theorem as

$$\begin{aligned} \frac{dL^*}{dc} &= -\sum_{j=1}^{n+1} H^{-1}_{n+1,j} \frac{\partial^2 \mathcal{V}}{\partial X_j \partial c}, \\ \frac{dL^*}{dt_c} &= -\sum_{j=1}^{n+1} H^{-1}_{n+1,j} \frac{\partial^2 \mathcal{V}}{\partial X_j \partial t_c}, \end{aligned} \tag{46}$$

where $H_{n+1,j}$ denotes the $(n+1, j)$-th entry of the Hessian matrix of the firm value, H_V, with respect to the variables of weight n-vector and debt ratio, $\mathbf{X} = (\mathbf{x}, L)$. Here the subscript V was omited to avoid a cumbersome subscript notation. Using the first order optimality conditions at the optimum point $X^* \equiv (\mathbf{x}^*, L^*)$, we get

$$\begin{aligned} \frac{\partial \mathcal{C}^0}{\partial x_j}|_{X^*} &= -\frac{cL^* D_j(0)}{(c+t_c)A(0)} < 0, \\ \frac{\partial \mathcal{C}^0}{\partial L}|_{X^*} &= -\frac{c\sum_j x_j^* D_j(0)}{(c+t_c)A(0)} < 0, \end{aligned} \tag{47}$$

so that

$$\frac{\partial^2 \mathcal{V}}{\partial x_j \partial c}|_{X^*} = -\frac{t_c}{c+t_c} L^* D_j(0) < 0,$$
$$\frac{\partial^2 \mathcal{V}}{\partial x_j \partial t_c}|_{X^*} = \frac{c}{c+t_c} L^* D_j(0) > 0.$$

Together with the bounds of $\partial \mathcal{C}^0/\partial L$ in Appendix, we have

$$\frac{\partial^2 \mathcal{V}}{\partial L \partial c}|_{X^*} = -A(0)\frac{\partial \mathcal{C}^0}{\partial L}|_{X^*} - \sum_i x_i D_i(0) < 0,$$
$$\frac{\partial^2 \mathcal{V}}{\partial L \partial t_c}|_{X^*} = -A(0)\frac{\partial \mathcal{C}^0}{\partial L}|_{X^*} > 0.$$

Thus it follows that

$$\begin{aligned} &\frac{\partial^2 \mathcal{V}}{\partial X_j \partial c}|_{X^*} < 0, \\ &\frac{\partial^2 \mathcal{V}}{\partial X_j \partial t_c}|_{X^*} > 0. \end{aligned} \tag{48}$$

Since $\mathcal{V}(0)$ is concave with respect to $\mathbf{X}$ when negative definiteness condition of Eq.(45) is satisfied, we get $\partial \mathcal{V}(0)/\partial c < 0$. [7] According to the analogous consideration we get $dL^*/dt_c < 0$. From those inequalities we have a proposition,

Proposition 6 Sensitivity of the Optimal Debt Ratio to Market Frictions
If the negative definiteness condition of Eq.(45) is satisfied, then optimal debt ratio becomes a decreasing (increasing) function of the bankruptcy cost rate (tax rate).

The comparative statics of firm value with respect to those market friction parameters are simpler than that of optimal debt ratio, as given above. Using the bounds of the equity value, Eq.(41), and explicit differentiations,

$$\begin{aligned} &\frac{\partial \mathcal{V}(0)}{\partial c} = -\frac{\mathcal{S}(0)}{1-t_c} + A(0) - D(0) \leq 0, \\ &\frac{\partial \mathcal{V}(0)}{\partial t_c} = -A(0)\mathcal{C}^0(0) \leq 0, \end{aligned} \tag{49}$$

we are immediately led to the proposition,

Proposition 7 Sensitivity of the Optimal Firm Value to Market Frictions
Optimal firm value is decreasing function with respect to both of the bankruptcy cost rate and the tax rate.

[7] The Hessian H is diagonalized in terms of some orthogonal matrix P as $PDP^\top$ where D is the diagonal matrix whose diagonal entries are all negative. Then the inverse of Hessian in Eqs.(46) which is assumed to be regular is expressed as $PD^{-1}P^\top$. Therefore since the meanig of $\partial \mathcal{V}(0)/\partial c$ in Eqs.(46) is first rotating the negative vector,$\partial^2 \mathcal{V}/\partial \mathbf{X} \partial c$ by $P^\top$, second multiplying it by negative scale factor D^{-1}, third rotating it back inversely and finally multiplying it by -1 and extracting the last component of the reselting $(n+1)$-vector, we arrive at the desired inequality, $\partial \mathcal{V}(0)/\partial c < 0$.

IV. Risk Management Model of Asset And Liability

So far we have demonstrated that tax shelter-bankruptcy cost models give rise to the optimal capital (sub-)structures of asset and liability under the certain conditions. Then special emphasis was not put on the stochastic behaviors of the contingent claims themselves which were brought through uncertain changes of call option $\mathcal{C}$. However they play an important role in the risk management such as an immunization strategy of asset and liability models. In this subsection we shall first derive the stochastic processes of the equity value, market value of debt and firm value, and next discuss some aspects of asset and liability risk management based upon our multiple asset and liability substructure model.

Suppose that the asset and liability structure composes from stochastic m asset instruments and n debt ones. Let the stochastic variables with a superscript 1 indicate that they are devided by a numéraire of money market account. Consider a call option,

$$\begin{aligned}\mathcal{C}^1(t) &\equiv \mathbf{E}^P[\frac{B(t)}{B(T)}(A(T)-D(T))^+|\mathcal{F}_t] \\ &= \mathbf{E}^P[\frac{B(t)}{B(T)}(\sum_{i=1}^m x_i A_i(T) - L\sum_{j=1}^n y_j D_j(T))^+|\mathcal{F}_t],\end{aligned} \tag{50}$$

which is expressed formally in terms of solutions of the SDE's of asset and debt instruments as

$$\mathcal{C}^1(t) = \int_{\mathbf{R}} (\sum_{i=1}^m x_i A_i(t) e^{-\sigma_i^2/2+z_i} - L\sum_{j=1}^n y_j D_j(t) e^{-\sigma_{m+j}^2/2+z_{m+j}}) n_{m+n}(\mathbf{z};0,\sigma) d^{m+n}z \tag{51}$$

with the integration domain

$$\mathbf{R} = \{\mathbf{z}|\sum_{i=1}^m x_i A_i(t) e^{-\sigma_i^2/2+z_i} \geq L\sum_{j=1}^n y_j D_j(t) e^{-\sigma_{m+j}^2/2+z_{m+j}}\}.$$

This formula is too complicated to invoke to *Itô* calculus directly for the purpose of deriving its SDE. Alternatively, it is convenient to employ ***Malliavin*** calculus through Clark's representation formula (Ocone and Karatzas(1991) and references therein). Applying the Clark's representation formula to that call option, its value at time t (here $0 \leq t \leq T$, T: the common expiry date of debts) under risk-adjusted probability P is of the general form

$$\mathcal{C}^1(t) = \mathbf{E}^P[\mathcal{C}^1(t)] + \sum_{k=1}^K \int_0^t \mathbf{E}^P[D_{k,u}\mathcal{C}^1(t)|\mathcal{F}_u] dW_k^P(u), \tag{52}$$

where D_u is the ***Malliavin-Fréchet*** functional derivative on Wiener space. The expectation of such a functional derivative $(0 \leq u \leq t)$ is calculated as follows:

$$
\begin{aligned}
&\mathbf{E}^P[D_{k,u}\mathcal{C}^1(t)|\mathcal{F}_u] \\
&= -\mathbf{E}^P[(\sum_{i=1}^{m} x_i A_i^1(u)\gamma_{ik}(u,T)\exp(-\frac{\sigma_i^2}{2} + \sum_{k=1}^{K}\int_0^T \gamma_{ik}(v,T)dW_k^P(v)) \\
&\quad - L\sum_{j=1}^{n} y_j D_j(u)\gamma_{m+j,k}(u,T)\exp(-\frac{\sigma_{m+j}^2}{2} + \sum_{k=1}^{K}\int_0^T \gamma_{m+j,k}(v,T)dW_k^P(v))1(A^1(T) > D^1(T))|\mathcal{F}_u] \quad (53) \\
&\equiv \sum_{i=1}^{m} x_i A_i^1(u)\gamma_{ik}(u,T)J_i(u,T) - L\sum_{j=1}^{n} y_j D_j^1(u)\gamma_{m+j,k}(u,T)J_{m+j}(u,T) \\
&\equiv \beta_k(u,T)\mathcal{C}^1(u),
\end{aligned}
$$

where

$$
\begin{aligned}
J_\alpha(u,T) &\equiv e^{-\frac{\sigma_\alpha^2}{2}}\int_{\mathbf{R}} e^{z_\alpha} n_{m+n}(\mathbf{z};0,\sigma_{\alpha\beta})d^{m+n}z, \\
\mathbf{R} &= \{\mathbf{z}|A(T) > D(T)\}
\end{aligned}
$$

for $\alpha,\beta = 1,\cdots,m+n$, and $\sigma_{\alpha\beta}$ stands for the local covariation of the stochastic integral, $\int_0^T \gamma_{\alpha k}^0(v,T)dW_k^P(v)$. Taking total derivative of Eq.(52), we get

$$
d\mathcal{C}^1(t) = \sum_{k=1}^{K}\beta_k(t,T)\mathcal{C}^1(u)dW_k^P(t). \quad (54)
$$

Together with

$$
\begin{aligned}
dA^1(t) &= \sum_{k=1}^{K}\sum_{i=1}^{m} x_i A_i^1(u)\gamma_{ik}(t,T)dW_k^P(t), \\
dD^1(t) &= \sum_{k=1}^{K}\sum_{j=1}^{n} y_j D_j^1(u)\gamma_{m+j,k}(t,T)dW_k^P(t),
\end{aligned} \quad (55)
$$

we have their SDE's under the probability P, using Eqs.(38), (39) and (40), as

$$
\begin{aligned}
d\mathcal{S}(t) &= r_d(t)\mathcal{S}(t)dt + \sum_{k=1}^{K}(1-t_c)\beta_k(t,T)\mathcal{C}(t)dW_k^P(t), \\
d\mathcal{B}(t) &= r_d(t)\mathcal{B}(t)dt - (1+c)\sum_{k=1}^{K}\beta_k(t,T)\mathcal{C}(t)dW_k^P(t) + (1+c)dA^1(t) - cdD^1(t), \\
d\mathcal{V}(t) &= r_d(t)\mathcal{V}(t)dt - (c+t_c)\sum_{k=1}^{K}\beta_k(t,T)\mathcal{C}(t)dW_k^P(t) + (1+c)dA^1(t) - cdD^1(t).
\end{aligned} \quad (56)
$$

Let the coefficients of diffusion terms associated with k-th risk factor in the above equations be denoted by Σ_k^S, Σ_k^B and Σ_k^V, respectively. With this factor loadings and the drift vectors, μ_i^A, μ_j^L $(i = 1,\cdots,m; j = 1,\cdots,n)$ of asset and debt instruments under the original probability, we can provide a proto type of immunization strategy of asset and liability risk management model by maximizing an excess return of asset and liability:

$$
\max_{\mathbf{x},\mathbf{y}}\{\sum_{i=1}^{m} x_i\mu_i^A - \sum_{j=1}^{n} y_j\mu_j^L\} \quad (57)
$$

subject to

$$\Sigma_k^V(\mathbf{x},\mathbf{y}) \leq \bar{\Sigma}_k^V$$

for all factors, where the debt amount L is fixed and $\bar{\Sigma}_k^V$ is an allowable factor loading associated with the k-th risk. This optimization problem enables us to maximize the excess return of asset and liability, controlling uncertainties of the firm value. Replacing $\Sigma_k^S(\mathbf{x},\mathbf{y})$ (or $\Sigma_k^B(\mathbf{x},\mathbf{y})$) for $\Sigma_k^V(\mathbf{x},\mathbf{y})$ leads to the optimization problem dealing with uncertainties of the equity value(or market value of debt). Due to $\partial\mathcal{S}/\partial\tau = (1-t_c)\partial\mathcal{C}/\partial\tau$, $\partial\mathcal{B}/\partial\tau = -(1+c)\partial\mathcal{C}/\partial\tau$ and $\partial\mathcal{V}/\partial\tau = -(c+t_c)\partial\mathcal{C}/\partial\tau$ (τ: time to maturity), if the time decay of $\mathcal{C}$ occurs, i.e., $\partial\mathcal{C}/\partial\tau > 0$, [8] then the equity value decreases as the calendar time approaches the expiry date of debts, whereas both the market value of debt and the firm value increase on the contrary. Hence, as an alternative objective against the problem (57), maximization of the marginal firm value to the time-to-maturity, $\partial\mathcal{V}/\partial\tau$, would be viable, as often adopted in the literature of asset and liability management.

The detail of their numerical analyses is omitted here. It is stated in Nakamura(1996), in which a simple approximation method of relevent contingent claims whose first, second moments are used is provided. More general extention will be also treated in another subsequent paper, Nakamura(1997), in which the multi-period debt substructure, incorporating fixed rate, short-term floating rate (like a LIBOR) as well as long-term floating rate (like a constant maturity swap rate) debt instruments, is considered, and as a numerical valuation methodology recently highlighted the quasi-Monte Carlo method, using low descrepancy sequence such as the Sobol one, is employed. With this, it is also not difficult to evaluate the multi-variate integral such as Eq.(53) appearing in the above asset-liability model.

V. Summary and Concluding Remarks

In this paper we provided a model of determining optimal asset-liability substructures in the multi-factor risk economy. One of remarkable findings is that in the tax-shelter bankruptcy cost models, depending upon what criterion the firm manager employs in making decision there arise different aspects in the optimal asset and liability (sub)structures. If the firm manager behaves like equity value maximizer, then only one debt of the largest relative volatility to asset value is chosen among available debt portfolio. If the firm manager behaves like firm value maximizer that seems like the most plausible criterion, then there arise some non-trivial, optimal debt weights and the debt ratio when certain conditions are met. For some optimization problems we provided a numerical efficient approximation method and attempted to solve these problems to determine those simultaneously. Introducing a finer substructure for the asset, the optimal asset weights are also calculted. Thus, in the imperfect and friction economy issuing multiple debts takes advantage of issuing an appropriate single debt, because the former allows for a diversification of intrinsic risks originating from the asset and liability substructure by selecting an optimal combination of their instruments. In addition, by effective uses of the Clark formula based upon the Malliavin Calculus we developed some immunization strategies of asset and liability management corresponding to the preceeding ones, and also discussed related topics.

[8] Using an approximation formula of the call option within the precision of up to the second order moments of Edgeworth expansion, it is proved that $\partial\mathcal{C}/\partial\tau = L\sum_{j=1}^{n} y_j D_j(0) n_1(d-\sigma)\partial\sigma/\partial\tau$ which is positive because of $\partial\sigma/\partial\tau > 0$. But the sign of the exact formula of the call option, Eq.(51), is non-trivial.

When the firm includes debt instruments with various maturities in the liability substructure, our framework must be extended either to conform to the fashion of Kraus and Litzenberger(1973)'s dynamic programming like approach or to be able to invoke to the Monte Carlo simulation. Such a generalization will be studied more extensively in the subsequent paper, Nakamura(1997). Moreover, in this paper we did not conduct detailed empirical study of the optimal capital substructures. For the firms disclosing detailed accounting information of asset and liability substructure, our model would be applicable as the one-period model corresponding to their operating period. These subjects, together with a multi-period generalizaiton, are the furture researches.

Appendix A: Change of Numeraire from One Asset to Another

Assume that there are N assets and $K(< N)$ risk factors. Let $S_i(t)$ denote i-th price of any instrument constitueting the asset and liability substructure, which follows a stochastic differential equation (SDE):

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{i=1}^{K} \gamma_{ik} dW_k(t). \tag{A·1}$$

Under the risk-adjusted probability measure P in which a money market account $B_d(t)$ is chosen as a numéraire and $S_i(t)/B_d(t)(\equiv \bar{S}_i(t))$ is a P-martingale, that equation becomes

$$\frac{d\bar{S}_i(t)}{\bar{S}_i(t)} = r_d(t)dt + \sum_{i=1}^{K} \gamma_{ik} dW_k^P(t). \tag{A·2}$$

On the other hand, under a specific probability measure Q in which some instrument, say l-th one, is a new numéraire, the relative price, $S_i(t)/S_l(t)(\equiv S_i^0(t))$ is a Q-martingale. To check this, consider a Radon-Nikodym derivative defined over $\mathcal{F}_T$

$$\begin{aligned} \frac{dQ}{dP} &= \frac{S_l(T)}{S_l(0)B_d(T)} \\ &= \exp(\sum_{k=1}^{K} \int_0^T \gamma_{lk} dW_k^P(u) - \frac{1}{2}\hat{\gamma}_{ll}^2), \end{aligned} \tag{A·3}$$

where we denoted $\int_0^T \gamma\gamma_{ll}^\top dt$ by $\hat{\sigma}_{ll}^2$. By the Girsanov's theorem, the translation of a Wiener process under a new probability measure is $dW_k^Q = dW_k^P - \gamma_{lk}dt$. Then it follows from the *Itô*'s lemma that a relative price $S_i^0(t)$ evolves according to the SDE:

$$\begin{aligned} \frac{dS_i^0(t)}{S_i^0(t)} &= (\gamma\gamma_{ll}^\top - \gamma\gamma_{il}^\top) + \sum_{i=1}^{K}(\gamma_{ik} - \gamma_{lk})dW_k^P(t) \\ &= \sum_{i=1}^{K}(\gamma_{ik} - \gamma_{lk})dW_k^Q(t). \end{aligned} \tag{A·4}$$

The solution of that SDE is

$$S_i^0(T) = S_i^0(0)\exp(\sum_{i=1}^{K} \int_0^T (\gamma_{ik} - \gamma_{lk})dW_k^Q(u) - \frac{1}{2}\hat{\sigma}_{il}^2) \tag{A·5}$$

with

$$\delta_{ii}^2 \equiv \int_0^T (\gamma\gamma_{ii}^\top + \gamma\gamma_{ii}^\top - 2\gamma\gamma_{ii}^\top)dt. \qquad (A\cdot 6)$$

It immediately follows that $S_i^0(t)$ is a Q-martingale price process.

Appendix B: Proofs of Lemmas and Propositions

First, we provide the proof of convex (concave) property of contingent claim included in the equity value (market of debt and firm value).

Proof of Proposition 2: Suppose that a firm issues n sorts of debts with total debt amount(debt ratio when $A(0) = 1$) L in which each terminal payoff of debt instrument is given by $D_i(T)$ (initially it is assumed to be normalized as $D_i(0) = 1$), and the proportion of i-th debt is x_i ($\sum_{i=1}^n x_i = 1$). Then the equity value $\mathcal{S} : \mathbf{R}^n \to [0, +\infty)$ is expressed as $\mathcal{S}(x) = (1 - t_c)\mathbf{E}^P[1/B_d(T)(A(T) - L\sum_{i=1}^n x_i D_i(T))^+|\mathcal{F}_0]$. In general the convexity of some function $F(x)$ is defined by an inequality that $F(\sum_{s=1}^p \theta_s x_i^s) \leq \sum_{s=1}^p \theta_s F(x_i^s)$ for arbitrary p and all i, where $\theta_s \in \Delta^{p-1} \equiv \{(\theta_1, \cdots, \theta_p)|\sum_{s=1}^p \theta_s = 1; \theta_s \geq 0, s = 1, \cdots, p\}$. This is proved as follows; the left hand side $= (A(T) - L\sum_{s=1}^p \sum_{i=1}^n \theta_s x_i^s D_i(T))^+ = \{\sum_{s=1}^p \theta_s(A(T) - L\sum_{i=1}^n x_i^s D_i(T)\}^+ \leq \sum_{s=1}^p \theta_s(A(T) - L\sum_{i=1}^n x_i^s D_i(T))^+$. Taking expectation of both sides, we get the desired inequality. When the asset side also consists of multiple instruments like $\mathcal{S}(x) = (1 - t_c)\mathbf{E}^P[1/B_d(T)(\sum_{i=1}^n x_i A_i(T) - L\sum_{j=1}^n y_j D_j(T))^+]$, this proposition is valid, and it is proved in a similar way. Let $a_\alpha(T) = (A_i(T), -LD_j(T))$. Let $X_\alpha = (x_i, y_j)$. Then $(\sum_{\alpha=1}^{m+n} \sum_{s=1}^p a_\alpha(T)\theta_s X_\alpha^s)^+ \leq \sum_{s=1}^p \theta_s(a_\alpha(T)X_\alpha^s)^+$ for θ_s defined above. Taking expectation of both sides establishes the convexity of such a contingent claim. ||

Second, consider the general properties of Hessians in the generic notation:

$$\begin{aligned} H_a &= \eta_a A(0)\begin{pmatrix} \frac{\partial^2 \mathcal{C}^0}{\partial x^2} & \frac{\partial^2 \mathcal{C}^0}{\partial x \partial L} + \xi_a \mathbf{D}^0(0) \\ \frac{\partial^2 \mathcal{C}^{0\top}}{\partial x \partial L} + \xi_a \mathbf{D}^{0\top}(0) & \frac{\partial^2 \mathcal{C}^0}{\partial L^2} \end{pmatrix} \\ &\equiv \eta_a A(0)\begin{pmatrix} \mathbf{h} & \mathbf{m} + \xi_a \mathbf{D}^0(0) \\ \mathbf{m}^\top + \xi_a \mathbf{D}^{0\top}(0) & l \end{pmatrix}, \end{aligned} \qquad (B\cdot 1)$$

where index a runs from S(equity value),B(market value of debt), V(firm value), and $\eta = ((1 - t_c), -(c + 1), -(c + t_c))$, $\xi = (0, c/(c + 1), c/(c + t_c))$. Although the off-diagonal block of the above Hessian matrix is ambiguous in their signs or in magnitude due to the complexity of higher dimensional space, we shall examine in order the conditions assuring the regularity and positive (or negative) (simi-)definiteness of the Hessian.

Lemma 1
Let H_a^0 $(a = S, B, V)$ be denoted by

$$H_a^0 = \eta_a A(0)\begin{pmatrix} \mathbf{h} & \mathbf{m} + \xi_a \mathbf{D}^0(0) \\ \mathbf{m} + \xi_a \mathbf{D}^0(0) & 0 \end{pmatrix}.$$

If h *is positive definite, then H_a^0 is regular for any market friction parameters.*
Proof:
We show that when $H_a^0 \cdot (\mathbf{u}, v)^\top = 0$ holds for arbitrary n-vector, and scalar v, its solution is

$(0,0)$. Explicitly writing it down leads to the simultaneous equations, i.e., $\mathbf{h}\mathbf{u}+(\mathbf{m}+\xi_a\mathbf{D}^0(0))v = 0$ and $(\mathbf{m}+\xi_a\mathbf{D}^0(0))^\top\mathbf{u} = 0$. Suppose that there exists $\mathbf{u} \neq \mathbf{0}$. Multiplying the former equation by $\mathbf{u}$, and using the latter equation, we have $\mathbf{u}^\top\mathbf{h}\mathbf{u} + \mathbf{u}^\top(\mathbf{m}+\xi_a\mathbf{D}^0(0))v = \mathbf{u}^\top\mathbf{h}\mathbf{u} = 0$. By the assumption of positive definiteness of $\mathbf{h}$ this is a contradiction. Therefore $\mathbf{u} = \mathbf{0}$. Then the simultaneous equations becomes $(\mathbf{m}+\xi_a\mathbf{D}^0(0))v = \mathbf{0}$, which immediately yields $v = 0$ due to $(\mathbf{m}+\xi_a\mathbf{D}^0(0)) \neq \mathbf{0}$. Thus we get $(\mathbf{u}, v)^\top = 0$. ||

Let $||A||$ stands for the matrix norm, $\max\{||Ax||/||x|| | 0 \neq \mathbf{x} \in \mathbf{R}^n\}$ for matrix $A \in \mathbf{R}^{n\times n}$. Let $\rho(A)$ denote the spectral radius, defined by $\max_{i=1,\cdots,n}|\lambda_i|$, where λ_i is the i-th eigenvalue of matrix A, and $\rho(A) \leq ||A||$ holds. It is known that if $\rho(A) < 1$, then $(I-A)^{-1} = \sum_{i=0}^{\infty} A^i$. Using the above lemma, we have a proposition about the positive definiteness of H_a.

Lemma 2

Suppose that $\mathbf{h}$ *is positive definite. Let* $||H_a^{0-1}||$ *be equal to* α. *If* $l < 1/\alpha$ *holds, then the Hessian of Eq.(B·1) is regular for any market friction parameters, and its norm is less than or equal to* $\alpha/(1-\alpha l)$.

Proof:

As

$$H_a - H_a^0 = \eta_a A(0)\begin{pmatrix} O & O \\ O & l \end{pmatrix},$$

we have $||H_a - H_a^0|| = l$. The lemma 1 implies that $||H_a^{0-1}||$ is bounded, say its value equal to α in magnitude. Then $||I - H_a^{0-1}H_a|| \leq ||H_a^{0-1}|| \cdot ||H_a - H_a^0|| = \alpha l < 1$ by the assumption of this lemma. From this it follows that $(I-(I-H_a^{0-1}H_a))^{-1} = \sum_{i=0}^{\infty}(I-H_a^{0-1}H_a)^i$ and the right hand side infinite sum converges. This implies that there exists $(H_a^{0-1}H_a)^{-1}$, i.e., H_a^{-1}, which is just what we want to prove. Further it holds that $||H^{-1}|| = ||\sum_{i=0}^{\infty}(I-H_a^{0-1}H_a)^i \cdot H_a^{0-1}|| \leq \alpha\sum_{i=0}^{\infty}(\alpha l)^i = \frac{\alpha}{1-\alpha l}$. ||

Noting that the Hessian of Eq.(B·1) is decomposed to be 2×2 blocks, we can obtain its inverse with imposing some conditions.

Lemma 3 Inverse of Hessian of Equity, Debt and Firm values

If $\mathbf{h}$ *is regular, and* $\hat{l} \equiv l - \hat{\mathbf{m}}^\top\mathbf{h}^{-1}\hat{\mathbf{m}} \neq 0$ *(* $\hat{\mathbf{m}} \equiv \mathbf{m}+\xi_a\mathbf{D}(0)$ *), then the inverse of Hessian of Eq.(B·1) is given by*

$$H_a^{-1} = \frac{\hat{l}^{-1}}{\eta_a A(0)}\begin{pmatrix} \hat{l}\mathbf{h}^{-1} + \mathbf{h}^{-1}\hat{\mathbf{m}}\hat{\mathbf{m}}^\top\mathbf{h}^{-1} & -\hat{\mathbf{m}}^\top\mathbf{h}^{-1} \\ -\mathbf{h}^{-1}\hat{\mathbf{m}} & 1 \end{pmatrix}. \qquad (B\cdot 2)$$

Proof:

Multiplying H_a^{-1} by H_a, we can easily establish this lemma. ||

Third, we present the lemma about the positive or negative denifiteness conditions of Hessians.

Lemma 4 Positive or Negative Definiteness of Hessians of Equity, Debt and Firm values

The Hessian of Eq.(B·1) is positive (negative) semidefinite for the equity value (for the market value of debt and firm value), if and only if an inequality, $l - (\mathbf{m}+\xi\mathbf{D}^0)^\top\mathbf{h}^{-1}(\mathbf{m}+\xi\mathbf{D}^0) \geq 0$ *holds for the regular submatrix* $\mathbf{h}$ *and market friction parameters.*

Proof:
For an arbitrary $(n+1)$-vector $\mathbf{X}$ decomposed to n-vector and $(n+1)$-th element as $(\mathbf{x}, x_{n+1})$ the quadratic form, $Q(X) \equiv \mathbf{X}^\top \mathbf{H}\mathbf{X}$ becomes

$$\begin{aligned} Q(X) &= \mathbf{x}^\top \mathbf{h}\mathbf{x} + 2x_{n+1}(\mathbf{m} + \xi \mathbf{D}^0)^\top \cdot \mathbf{x} + l x_{n+1}^2 \\ &= \mathbf{y}^\top \mathbf{h}\mathbf{y} + b x_{n+1}^2, \end{aligned}$$

where n-vector $\mathbf{y} = \mathbf{x} + \mathbf{h}^{-1}(\mathbf{m} + \xi \mathbf{D}^0)x_{n+1}$ and $b = l - (\mathbf{m} + \xi \mathbf{D}^0)^\top \mathbf{h}^{-1}(\mathbf{m} + \xi \mathbf{D}^0)$. That is, $\mathbf{H}$ is expressed in terms of

$$\mathbf{R} = \begin{pmatrix} I_{n-1} & 0 \\ \mathbf{m}^\top \mathbf{h}^{-1} & 1 \end{pmatrix}, \quad \mathbf{H}_D = \begin{pmatrix} \mathbf{h} & 0 \\ 0^\top & \hat{l} \end{pmatrix},$$

by

$$\mathbf{H} = \eta_a A(0) \mathbf{R}\mathbf{H}_D \mathbf{R}^\top. \tag{B·3}$$

Since $\mathbf{h}$ is positive semidefinite, the necessary and sufficient condition of the positivity of $Q(X)$ is unambiguously $b \geq 0$. This, together with $\eta_S = 1 - t_c > 0, \eta_B = -(c+1) < 0, \eta_V = -(c + t_c) < 0$, completes the proof. ||

Finally, we provide a useful lemma in discussing the order relationship of optimal debt ratio and debt capacity in magnitude.

Lemma 5
The marginal equity value with respect to the debt ratio is negative. The marginal bankruptcy probability with respect to the debt ratio is positive as well.
Proof:
Let $\mathcal{C}^0(0) = \mathbf{E}^Q[(1 - L\mathcal{O})^+ | \mathcal{F}_0]$, where $\mathcal{O} = \sum x_i D_i^0(0) \exp(Y_i - \sigma_i^2/2)$.

$$\frac{\partial \mathcal{C}^0(0)}{\partial L} = -\int_{\mathcal{O} < \frac{1}{L}} \mathcal{O} d\mathcal{Q}(y) + \lim_{\varepsilon \to 0} \{ -\int_{\frac{1}{L+\varepsilon} < \mathcal{O} < \frac{1}{L}} (1 - L\mathcal{O}) d\mathcal{Q}(y) \} / \varepsilon.$$

The first term of the right hand(RHS) side is clearly negative. As for the second term of RHS the integral domain has a finite limit and the integrand is negative within such domain. From those the negativity of RHS is established. Since tax rate is less than one, together with $\mathcal{S}(0) = (1 - t_c)A(0)\mathcal{C}^0(0)$, this completes the proof. When the asset side has a multiple structure like $\sum_{i=1}^m x_i A_i^1(T)$, the line of proof is the same. Let $\mathcal{C}^1(0) = \mathbf{E}^P[(A^1(T) - L\hat{D}^1(T))^+ | \mathcal{F}_0]$, where $\hat{D}^1(T) = \sum_{j=1}^n y_j D_j^1(T)$, and the superscript 1 attached to the random variables indicate that the numéraire is the money market account. Then

$$\frac{\partial \mathcal{C}^1(0)}{\partial L} = -\int_{\hat{D}(T) < A(T)/L} \hat{D}(T) d\mathcal{P}(x, y) + \lim_{\varepsilon \to 0} \{ -\int_{\frac{A(T)}{L+\varepsilon} < \mathcal{O} < \frac{A(T)}{L}} (A(T) - L\hat{D}(T)) d\mathcal{P}(x, y) \} / \varepsilon < 0.$$

This is the desired inequality. As to the bankruptcy probability note that it is evaluated as

$$\begin{aligned} \mathcal{P}_b &= \mathbf{E}^P[\frac{1}{B_d(T)} \mathbf{1}(A(T) < D(T)) | \mathcal{F}_0] \\ &= \int_{\mathcal{O} > \frac{1}{L}} d\mathcal{Q}(y). \end{aligned}$$

Increasing L certainly makes $\mathcal{P}_b$ increase for a positive $\mathcal{O}$. This establishes the latter part. ||

Associated with this lemma we can derive an allowable range of the marginal option value. Taking into account of $\int_{O<1/L} OdQ(y) < \int OdQ(y) = \sum_i x_i D_i^0(0)$ in the intermediate derivation of above lemma 5 we get a lower bound of the marginal call option value included in relevent contingent claims. Its lower and upper bounds are given by

$$-\sum_i x_i D_i^0(0) \leq \frac{\partial C^0(0)}{\partial L} < 0. \qquad (B \cdot 4)$$

This is of course valid in the multiple asset and debt case. It is easily seen from $l \geq 0$ that the lower bound is realized at $L = 0$ and the upper bound at $L \to \infty$.

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