Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework Part II: Securities and Derivatives Markets

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Abstract. Asset/Liability management, optimal fund design and optimal portfolio selection have been key issues of interest to the (re)insurance and investment banking communities, respectively, for some years - especially in the design of advanced risk-transfer solutions for clients in the Fortune 500 group of companies. AFIR 1996 publications dealing with these topics were, e.g., *Optimal Fund Design for Investors with Holding Constraints* (Vol. I, p. 245), *Optimal Portfolio and Optimal Trading in a Dynamic Continuous Time Framework* (Vol. I, p. 275), *Mean-Variance Portfolio Selection under Portfolio Insurance* (Vol. I, p. 347), *Baseline for Exchange Rate - Risks of an International Reinsurer* (Vol. I., p. 395), *Optimizing Investment and Contribution Policies of a Defined Benefit Pension Fund* (Vol. I, p. 593), *Continuous-Time Pension Fund Modelling* (Vol. I, p. 609), *Linear Approach for Solving Large-Scale Portfolio Optimization Problems in a Lognormal Market* (Vol. II, p. 1019), *Options as an Asset Class* (Vol. II, p. 1413), *Optioned Portfolios: The Trade-off between Expected and Guaranteed Returns* (Vol. II, p. 1443), *Optimal Optioned Portfolios with Confidence Limits on Shortfall Constraints* (Vol. II, p. 1497), among others. Taking up some of the new ideas and approaches in this literature we introduce the concept of *limited risk arbitrage investment management in a general diffusion type securities and derivatives market* and characterize the corresponding trading and portfolio management strategies (*risk/arbitrage strategies*) as the solutions of a stochastic control problem with constraints on instantaneous investment risk, future portfolio risk dynamics, portfolio time decay dynamics and portfolio value appreciation dynamics. *Part II: Securities and Derivatives Markets* shows how an efficient allocation of risk can be achieved by investing in a portfolio of securities and non-redundant (i.e., securities market completing) financial options.

Keywords. Limited risk arbitrage investment management, instantaneous investment risk, future risk dynamics, time decay dynamics, value appreciation dynamics, risk/arbitrage strategies, strategy space transformation, market parametrization, maximum principle, convex duality, Markovian characterization, market prices of risk, completion premium, target wealth, required wealth, transformation premium, canonical solution, CS (constraint set) projection, DC (drawdown control) transform.

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4. Derivatives Risk Exposure Assessment and Control

<u>Market Completion With Options</u>. Recall from *Part I: Securities Markets*: Within the class of general assets [i.e., assets with RCLL semimartingale price processes $dV(t) = \omega(t)dW(t) + dA(t)$ where $\omega \in \Omega$, $\Omega = K(\sigma) \oplus K^{\perp}(\sigma)$ and $A(t)$ is of class VF (variation finie)] in the incomplete securities market model

$$\alpha(t) = \sigma(t)^T \kappa(t)^{-1} [\mu(t) - r(t)1_M]$$

$$\beta(t) = \exp(-\int_0^t \alpha(s)^T dW(s) - \frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds) \qquad E[\exp(\frac{1}{2}\int_0^t \|\alpha(s)\|^2 ds)] < \infty \qquad (4.1)$$

$$\xi(t) = \beta(t)D(t)$$

considered here contingent claims are compatible with the equilibrium bond and stock prices [i.e., after discounting with the risk-neutral discount factor $D(t)$ their value processes are continuous martingales under the minimax local martingale measure $\tilde{\pi}_{\hat{a}_{\gamma v}}$] and can therefore be written in the form

$$dV(t) = \; + \; \frac{[\theta(t)^T[\mu(t) - r(t)1_M] + V(t)r(t)]dt + \theta(t)^T\sigma(t)dW(t)}{\omega_1(t)^T\omega_2(t)dt + \omega_1(t)^T dW(t)} \qquad (4.2)$$

with the (partial) replication strategy $\theta \in K^{\perp}(\sigma)$ and the additional parameters $\omega_1, \omega_2 \in K(\sigma)$. The first term in this representation is the value process of a purely tradable (attainable) asset while the second term characterizes a totally non-tradable asset in this financial economy. The above decomposition is unique up to constants. Furthermore, every compatible asset (with non-constant associated value process) is either purely tradable (attainable) or totally non-tradable. It also follows that the minimal equivalent martingale measure is necessarily

$$\tilde{\pi}(\varphi) = E[\beta(H)1_{\varphi}], \; \varphi \in \Phi, \text{ and } \tilde{W}(t) = W(t) + \int_0^t \alpha(s)ds \qquad (4.3)$$

and that any equivalent martingale measure $\tilde{\psi}$ for the equilibrium security price processes is related to this minimal (in terms of its distance to the reference probability π measured in units of relative entropy) martingale measure via

$$E[\frac{d\tilde{\psi}}{d\pi}\Big|F_t] = \exp(-\int_0^t \gamma(s)^T dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds) \quad \gamma(t) = \alpha(t) + \omega(t), \; \omega \in K(\sigma). \qquad (4.4)$$

[We therefore call the quantity

$$v_{\hat{a}_{\gamma v}} = \frac{B_Q^T \nabla_Q V}{X \nabla_X V} = b^T \frac{I_Q \nabla_Q V}{X \nabla_X V}, \; v_{\hat{a}_{\gamma v}} \in K(\sigma)$$

$$E[\frac{d\tilde{\pi}_{\hat{a}_{\gamma v}}}{d\pi}\Big|F_t] = \exp(-\int_0^t \hat{\alpha}_{\hat{a}_{\gamma v}}(s)^T dW(s) - \frac{1}{2}\int_0^t \|\hat{\alpha}_{\hat{a}_{\gamma v}}(s)\|^2 ds) \quad \hat{\alpha}_{\hat{a}_{\gamma v}}(t) = \alpha(t) + v_{\hat{a}_{\gamma v}}(t)$$

a completion premium.] Moreover, among all equivalent martingale measures for the equilibrium bond and stock price processes the minimal equivalent martingale measure is characterized by the property that for every totally non-tradable asset the stochastic differential equation

$$dV(t) = \omega_1(t)^T\omega_2(t)dt + \omega_1(t)^T dB(t) \qquad (4.5)$$

holds with the same coefficients $\omega_1, \omega_2 \in K(\sigma)$ and any (related to each other via the Girsanov theorem) π - and $\tilde{\pi}$ - Brownian motions $B(t)$. Note that all the results about option pricing and hedging derived above in the securities market completion $\hat{\xi}_{\hat{a}_{\gamma v}}(t)$ also apply in

this originally given incomplete market model $\xi(t)$ [with respect to the minimal equivalent martingale measure $\tilde{\pi}$ and the corresponding Girsanov transform $\tilde{W}(t)$ of the exogenous sources $W(t)$ of securities market uncertainty], i.e., we have the derivatives value processes

$$V^{vc}(t) = \frac{1}{D(t)} \tilde{E}[D(T)C|F_t] \quad \text{(European options)} \quad (4.6a)$$

$$V^{vc}(t) = \frac{1}{D(t)} \tilde{E}[D(\tau_0)C(\tau_0)|F_t] \quad \text{(American options)} \quad (4.6b)$$

with associated risk minimizing replication costs

$$dC^{vc}(t) = dL_C(t), \quad C^{vc}(0) = V^{vc}(0), \quad (4.7)$$

and their Markovian characterization

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V) - \nabla V^T B\alpha - Vr = 0 \qquad B\alpha = A - rP$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} tr(BB^T \nabla^2 V) + r(P^T \nabla V - V) = 0 \qquad (4.8)$$

with corresponding boundary conditions $V(T,P) = h(P)$ for European options and $V(\tau_0, P) = h(\tau_0, P)$ for American options where

$$A = \begin{bmatrix} P_1\mu_1 \\ \vdots \\ P_M\mu_M \end{bmatrix} \qquad B = \begin{bmatrix} P_1\sigma_{11} & \cdots & P_1\sigma_{1N} \\ \vdots & & \vdots \\ P_M\sigma_{M1} & \cdots & P_M\sigma_{MN} \end{bmatrix} \qquad \nabla = \begin{bmatrix} \dfrac{\partial}{\partial P_1} \\ \vdots \\ \dfrac{\partial}{\partial P_M} \end{bmatrix} \quad (4.9)$$

as well as the associated unique risk minimizing replication strategies

$$v_C = \nabla V, \quad (4.10)$$

but the obtained results are not necessarily consistent with the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$. Note also that not attainable (i.e., totally non-tradable with respect to a given incomplete set of underlying bonds and stocks) contingent claims are suitable candidates for a securities market completion in which subsequently an optimal (in the sense of Arrow and Debreu) allocation of investment risk is feasible.

Part II: Securities and Derivatives Markets: Ross [43], Arditti and John [44] and John [45, 46] have analyzed the properties of such securities market completions with derivatives and the structure of the associated options markets in a finite state space setting: (1) A complete (fully efficient) financial market can be achieved by writing ordinary call and put options on a single portfolio of the basic securities - bonds and stocks. (2) Almost any portfolio formed with these basic securities in the market can be chosen for this purpose, i.e., the set of such (maximally) efficient funds is open and dense in portfolio space. The results of a similar analysis carried out by Green and Jarrow [47], Nachman [48] and Duan, Moreau and Sealey [49] for general state spaces are: (1) Ordinary call and put options on the bonds and stocks in the market and on portfolios of such ordinary call and put bond and stock options are sufficient to provide a complete financial market (in the limit). (2) If in a large underlying securities market returns are generated by a linear factor structure, then ordinary call and put options on a well-diversified securities market index and on portfolios of such ordinary call and put index options provide a complete financial market (in the limit).

If the prices of the underlying securities - bonds and stocks - follow a diffusion process, then the price processes of corresponding contingent claims are also of the diffusion type. Moreover, Krasa and Werner [6] have recently implemented a general equilibrium model which includes options markets and shown that because in a Pareto efficient equilibrium only relative spot prices are determined an appropriate choice of the levels of absolute prices makes contingent claims to real-world substitutes for Arrow securities. Equilibrium diffusion type price processes for such a heterogeneous securities and derivatives market can then be derived along the lines of Huang [3], Duffie [4], Duffie and Huang [5], Huang [7] and Karatzas, Lehoczky and Shreve [8]. Note however that even if the corresponding market structure is dynamically complete (i.e., the number of traded assets including derivatives equals the number of exogenous sources of market uncertainty) there are always multiple inefficient equilibria in which some of the contingent claims are redundant in addition to a Pareto efficient equilibrium (indeterminacy of equilibria for a fixed securities market configuration). Furthermore, Zipkin [50] has recently reinforced the diffusion process model for equilibrium securities prices from a risk exposure assessment point of view. Using the total positivity property of a diffusion money market rate process $r(t)$ he has been able to show that in such an equilibrium model the Cox, Ingersoll and Ross [51, 52] measure

$$-\frac{1}{B}\frac{\partial B}{\partial r} \quad (4.11)$$

of interest rate risk exposure for discount bonds increases with the time $T-t$ to maturity (i.e., long bonds are riskier than short bonds) whereas in equilibrium models with state processes that are not of the diffusion type there may be discount bonds which do not have this property. In these diffusion type equilibrium models interest rate risk exposure is therefore characterized by the time to maturity of the traded securities (discount bonds) and independent of the details of a particular model specification.

<u>Limited Risk Arbitrage</u>. The value process $V(t,S(t))$ of a general contingent claim in a Markovian securities market setting with state variables

$$dS(t) = A(t)dt + B(t)dW(t)$$

$$A(t,S) \in R^N \qquad \nabla = \nabla_S = \left[\frac{\partial}{\partial S_1} \quad \cdots \quad \frac{\partial}{\partial S_N}\right]^T \quad (4.12)$$
$$B(t,S) \in R^{N \times N}$$

[i.e., the prices $P(t)$ of the basic traded assets - bonds and stocks - and in the case of an incomplete securities market the additional market state variables $Q(t)$] follows the stochastic differential equation

$$dV = [\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V)]dt + \nabla V^T BdW \quad (4.13)$$

and is therefore completely characterized by the corresponding sensitivities (derivatives risk parameters)

$$\Theta = \frac{\partial V}{\partial t} \text{ (theta)}, \quad \Delta = \nabla V \text{ (delta) and } \Gamma = \nabla^2 V \text{ (gamma)} \quad (4.14)$$

with respect to the underlying securities market variables and time. The (conditional) first and second order moments of an infinitesimal option value increment are then

$$E[dV|F_t] = [\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V)]dt$$
$$V[dV|F_t] = \left\|\nabla V^T B\right\|^2 dt \quad (4.15)$$

whereas for the underlying securities market variables

$$E[dS|F_t] = A dt \quad \text{and} \quad V[dS|F_t] = BB^T dt \quad (4.16)$$

holds. In addition, we define the option value appreciation rate

$$\Lambda = \lim_{\Delta t \to 0} E[\frac{\Delta V}{\Delta t}\Big|F_t] = \frac{E[dV|F_t]}{dt}$$
$$= \frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V) \quad (4.17)$$

(lambda) which shows how fast the option value changes in an infinitesimal trading interval $[t, t+dt]$ with an associated rate of change

$$\lim_{\Delta t \to 0} E[\frac{\Delta S}{\Delta t}\Big|F_t] = \frac{E[dS|F_t]}{dt} = A \quad (4.18)$$

in the underlying securities market variables. Defining the above derivatives risk/return characteristics in the obvious way for the underlying bonds and stocks we are now going to focus on general investment management and asset allocation problems

$$\max_{(c,\theta) \in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))]$$
$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v \quad (4.19)$$

$[\varsigma(t) = \xi(t) \text{ or } \varsigma(t) = \hat{\xi}_{\hat{a}_{\gamma_v}}(t)]$ with limited risk arbitrage objectives

$$|v(t)^T \Delta(t)| \le \delta, \ 0 \le t \le H \quad \text{(instantaneous investment risk)} \quad (4.20a)$$
$$|v(t)^T \Gamma(t)| \le \gamma, \ 0 \le t \le H \quad \text{(future portfolio risk dynamics)} \quad (4.20b)$$
$$v(t)^T \Theta(t) \ge \vartheta, \ 0 \le t \le H \quad \text{(portfolio time decay dynamics)} \quad (4.20c)$$
$$v(t)^T \Lambda(t) \ge \lambda, \ 0 \le t \le H \quad \text{(portfolio value appreciation dynamics)} \quad (4.20d)$$

$[\theta_i(t) = v_i(t)X_i(t)]$ involving portfolios $v(t) = [v_1(t) \ \cdots \ v_i(t) \ \cdots \ v_L(t)]^T$ of bonds and stocks as well as options, i.e., at the general task of achieving an efficient [in terms of an investor's utility functions $U^c(t,c)$ and $U^V(V)$ and risk/arbitrage parameters δ, γ, ϑ and λ] allocation of risk in a (diffusion type) securities and derivatives market

$$dX(t) = C(t)dt + D(t)dW(t)$$

$$C(t) = \begin{bmatrix} X_1(t)M_1(t) \\ \vdots \\ X_L(t)M_L(t) \end{bmatrix} \quad D(t) = \begin{bmatrix} X_1(t)\Sigma_{11}(t) & \cdots & X_1(t)\Sigma_{1N}(t) \\ \vdots & & \vdots \\ X_L(t)\Sigma_{L1}(t) & \cdots & X_L(t)\Sigma_{LN}(t) \end{bmatrix} \quad (4.21)$$

with associated [expressed in terms of the underlying securities market model $(t,S(t))$] instantaneous investment risk, future derivatives risk dynamics, options time decay dynamics and asset value appreciation dynamics

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) \\ \vdots \\ \Delta_L(t) \end{bmatrix}, \ \Gamma(t) = \begin{bmatrix} \Gamma_1(t) \\ \vdots \\ \Gamma_L(t) \end{bmatrix}, \ \Theta(t) = \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_L(t) \end{bmatrix} \text{ and } \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ \vdots \\ \Lambda_L(t) \end{bmatrix} \quad (4.22)$$

[where $\Delta_i(t) = \nabla_S X_i(t,S(t))$ is the delta (N-vector), $\Gamma_i(t) = \nabla_S^2 X_i(t,S(t))$ the gamma ($N \times N$-matrix), etc. of traded asset $X_i(t,S(t))$ in the market, $1 \le i \le L$, *asset/liability management* - with the focus on the asset side in this paper, see also *Baseline for Exchange Rate - Risks of an International Reinsurer*, AFIR 96, Vol. I, p. 395].

<u>Complete Securities Markets</u>. In the special case of a complete underlying securities market

$$dP = Adt + BdW \quad (4.23)$$

with additional (redundant) options

$$dX_i = [\Theta_i + A^T\Delta_i + \frac{1}{2}\text{tr}(BB^T\Gamma_i)]dt + \Delta_i^{\ T}BdW \quad (4.24)$$

$$= \Lambda_i dt + \Delta_i^{\ T}BdW$$

$[N + 1 \le i \le L]$ where we have $L > N$ we can write the above portfolios in the form

$$v = \begin{bmatrix} v_P \\ v_X \end{bmatrix} \quad (4.25)$$

and thus derive the representation

$$dV = [v^T[C - rX] + Vr - c]dt + v^T DdW$$

$$= [v_P^{\ T}[A - rP] + v_X^{\ T}\Delta_X^{\ T}[A - rP] + Vr - c]dt + [v_P^{\ T}B + v_X^{\ T}\Delta_X^{\ T}B]dW \quad (4.26)$$

$$= [(v^T\Delta)^T[A - rP] + Vr - c]dt + (v^T\Delta)^T BdW$$

for the portfolio value dynamics. In this (in practical applications important) case the general investment management and asset allocation problem

$$\max_{(c,\theta)\in A(v)} E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\theta}(H))]$$

$$E[\int_0^H \varsigma(t)c(t)dt + \varsigma(H)V_v^{c\theta}(H)] \le v \quad (4.27)$$

with a corresponding instantaneous investment risk control objective

$$\left|v(t)^T\Delta(t)\right| \le \delta, \ 0 \le t \le H, \quad (4.28)$$

[where the feasible trading strategies $v(t)$ also have to satisfy the other limited risk arbitrage constraints] hence leads to an equality constraint

$$v(t)^T\Delta(t) = \delta_{U^cU^V}(t), \ 0 \le t \le H, \quad (4.29)$$

for the portfolio delta. The individual portfolio positions can then for instance be determined as the solutions of an additional (independent) optimization problem with linear constraints

$$v(t)^T\Delta(t) = \delta_{U^cU^V}(t), \ 0 \le t \le H \quad (4.30a)$$

$$\left|v(t)^T\Gamma(t)\right| \le \gamma, \ 0 \le t \le H \quad \text{(future portfolio risk dynamics)} \quad (4.30b)$$

$$v(t)^T\Theta(t) \ge \vartheta, \ 0 \le t \le H \quad \text{(portfolio time decay dynamics)} \quad (4.30c)$$

$$v(t)^T\Lambda(t) \ge \lambda, \ 0 \le t \le H \quad \text{(portfolio value appreciation dynamics)} \quad (4.30d)$$

in strategy space. We shall therefore from now on assume that $L \le N$ and the volatility matrix $\Sigma(t)$ has full rank which allows us to define the associated market prices of risk

$$A(t) = \Sigma(t)^T K(t)^{-1}[M(t) - r(t)1_L] \quad (4.31)$$

with the asset price covariance matrix $K(t) = \Sigma(t)\Sigma(t)^T$.

<u>Options Portfolio Characteristics</u>. While the important idea of constraining investment risk and at the same time maximizing portfolio return has been successfully applied in the securities or traditional financial markets current research in quantitative finance seems to be focused more on the issue of eliminating or reducing derivatives risk exposure by hedging contingent claims on a single-instrument basis than on the task of optimally solving derivatives risk exposure assessment and control problems in a portfolio context. Among the exceptions in the recent finance literature Sears and Trennepohl [53] is an early attempt to analyze option risk characteristics from a portfolio (consisting of stocks and corresponding European call options) point of view. They combine a simple CAPM framework with the

Black & Scholes option pricing model to derive option betas (systematic risk coefficients) and study the effects of diversification when an investor's portfolio includes long positions in call options. Modulo leverage and intrinsic value effects, i.e., higher expected levels of risk as well as higher variance of portfolio risk about these expectations and much greater importance of the cross-sectional component (see Elton and Gruber [54]) of expected risk, the well-known properties of diversified stock or bond portfolios are shared by portfolios that include options. In this context note that the return process of a contingent claim is

$$dR = \frac{dV}{V} = d[\log(V)]$$

$$= [\frac{\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V)}{V} - \frac{\left\| \nabla V^T B \right\|^2}{2V^2}]dt + \frac{\nabla V^T B}{V} dW \qquad (4.32)$$

$[\, dS = Adt + BdW \,]$ with first and second order moments

$$E[dR|F_t] = [\frac{\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V)}{V} - \frac{\left\| \nabla V^T B \right\|^2}{2V^2}]dt \qquad (4.33)$$

$$V[dR|F_t] = \frac{\left\| \nabla V^T B \right\|^2}{V^2} dt.$$

Furthermore, if $c(t)$ is a fund withdrawal rate and $v(t)$ a portfolio of securities and corresponding contingent claims, then the associated return process is

$$dR = \frac{dV}{V} = d[\log(V)]$$

$$= [\frac{v^T(C - rX) + Vr - c}{V} - \frac{\left\| v^T D \right\|^2}{2V^2}]dt + \frac{v^T D}{V} dW \qquad (4.34)$$

$[\, dX = Cdt + DdW \,]$ with first and second order moments

$$E[dR|F_t] = [\frac{v^T(C - rX) + Vr - c}{V} - \frac{\left\| v^T D \right\|^2}{2V^2}]dt \qquad (4.35)$$

$$V[dR|F_t] = \frac{\left\| v^T D \right\|^2}{V^2} dt = \frac{\left\| (v^T \Delta)^T B \right\|^2}{V^2} dt.$$

Therefore the above constraints

$$\left| v(t)^T \Delta(t) \right| \le \delta \text{ and } \left| v(t)^T \Gamma(t) \right| \le \gamma, \ 0 \le t \le H, \quad (4.36)$$

on both the instantaneous investment risk and the future portfolio risk dynamics very effectively control the cumulative (conditional) variance

$$\int_0^H V[dR|F_t] = \int_0^H \frac{\left\| (v^T \Delta)^T B \right\|^2}{V^2} dt \qquad (4.37)$$

of the portfolio return process over the entire investment horizon. Booth, Tehranian and Trennepohl [55] later analyze the effectiveness of securities and derivatives portfolio selection rules and the relative importance of efficient (under these rules) options portfolios versus pure stock and fixed income investments. They find that stochastic dominance algorithms, i.e., SSD, TSD, SSDR and TSDR (see Whitmore [56], Levy and Kroll [57, 58] and Jean [59]), which place only few restrictions on an investor's utility functions and on asset return distributions are suitable general options portfolio selection rules and illustrate the importance of investments in heterogeneous derivatives portfolios by the large relative proportion of portfolios that include contingent claims in the dominant (efficient) set under these rules.

Hedging With Options. More recently Lamberton and Lapeyre [60] have included derivatives into their basket of securities used to hedge index options with few underlying assets. In our setting the price process of these hedging assets would be $X(t)$, the discount factor $\varsigma(t)$, the risk-neutral probability measure $\widetilde{\psi}$ and the associated Brownian motion $\widetilde{B}(t)$ where

$$\varsigma(t) = \xi(t), \quad \widetilde{\psi} = \widetilde{\pi} \text{ and } \widetilde{B}(t) = \widetilde{W}(t) \quad (4.38a)$$

in a complete and

$$\varsigma(t) = \widehat{\xi}_{\widehat{a}_{yv}}(t), \quad \widetilde{\psi} = \widetilde{\pi}_{\widehat{a}_{yv}} \text{ and } \widetilde{B}(t) = \widetilde{W}_{\widehat{a}_{yv}}(t) \quad (4.38b)$$

in an incomplete underlying securities market [and the relevant securities market completion parameter $\widehat{a}_{yv}(t)$ is uniquely determined by the investor's overall risk management objectives $U^c(t,c)$ and $U^V(V)$]. The discounted value process associated with a self-financing trading strategy $v(t)$ is then

$$\begin{aligned} D(t)V_c^v(t) &= c + \int_0^t d[D(s)G^v(s)] \\ &= c + \int_0^t v(s)^T d[D(s)X(s)] \end{aligned} \quad (4.39)$$

and given an index option with value process $I(t)$ the corresponding lack of hedging at time t is

$$L_c^v(t) = D(t)[I(t) - V_c^v(t)]. \quad (4.40)$$

If we now set $c = I(0)$, we have $L_c^v(0) = 0$. Furthermore, note that $L_c^v(t)$ is a square-integrable martingale under $\widetilde{\psi}$ and $\widetilde{E}[L_c^v(T)^2] = \widetilde{E}[< L_c^v >_T]$ holds for the residual risk at option maturity (see also Bouleau and Lamberton [42]). The unique risk minimizing (partial) replication strategy $v_I(t)$ for the index option then minimizes the quadratic variation $< L_c^v >_t$ over the hedge horizon $[0,T]$ and has the representation

$$v_I(t) = A(t)[A(t)^T A(t)]^{-1} b(t) \quad (4.41)$$

where

$$D(t)X(t) = x + \int_0^t A(s)d\widetilde{B}(s) \text{ and } D(t)I(t) = c + \int_0^t b(s)^T d\widetilde{B}(s) \quad (4.42)$$

$[A(s) = D(s)I_x(s)\Sigma(s)]$. The above approach to hedging with options however does not take into consideration the derivatives risk characteristics (inherent in the included contingent claims) of the applied hedging strategies. This could be done in our framework by allowing only replication strategies $v(t)$ that satisfy appropriate constraints

$$|v(t)^T \Delta(t)| \leq \delta \text{ and } |v(t)^T \Gamma(t)| \leq \gamma, \, 0 \leq t \leq T, \quad (4.43)$$

on instantaneous investment risk and future risk dynamics.

Finally, instead of requiring our investment management and asset allocation strategies $v(t)$ to be of the strictly limited risk type, i.e.,

$$|v(t)^T \Delta(t)| \leq \delta \text{ and } |v(t)^T \Gamma(t)| \leq \gamma, \, 0 \leq t \leq H, \quad (4.44)$$

with narrow bands δ and γ it may be advantageous in some situations to allow wider bands and to apply an impulse control technique similar to Jeanblanc [61] that keeps the derivatives risk parameters within the narrow bands at a certain cost. Moreover, Grossman and Zhou [62] point out another constraint on portfolio management strategies $v(t)$ that is important when the investments include contingent claims in addition to stocks and bonds (leverage effects). They define the stochastic process

$$M^{v}(t) = \max_{0 \le s \le t} D(s)V^{v}(s) \quad (4.45)$$

where

$$1 - \frac{D(t)V^{v}(t)}{M^{v}(t)} \quad (4.46)$$

is the time t drawdown resulting from an application of portfolio management strategy $v(t)$. Drawdown control at a level $1 - \alpha$, $0 < \alpha < 1$, can then be implemented by the constraint

$$D(t)V^{v}(t) > \alpha M^{v}(t), \ 0 \le t \le H, \quad (4.47)$$

on admissible trading and portfolio management strategies.

5. Risk/Arbitrage Strategies

Limited Risk Arbitrage Investment Management. Risk/Arbitrage strategies are trading or portfolio management strategies in the securities and derivatives markets that guarantee (with probability one) a limited risk exposure over the entire investment horizon and at the same time achieve a maximum (with guaranteed floor) rate of portfolio value appreciation over each individual trading period. They ensure an efficient allocation of investment risk in these integrated financial markets and are the solutions of the general investment management and asset allocation problem

$$\max_{(c,\theta) \in A(v)} E[\int_{0}^{H} U^{c}(t,c(t))dt + U^{V}(V_{v}^{c\theta}(H))] \quad (5.1)$$
$$E[\int_{0}^{H} \varsigma(t)c(t)dt + \varsigma(H)V_{v}^{c\theta}(H)] \le v$$

with limited risk arbitrage objectives

$$\begin{aligned}
|v(t)^{T}\Delta(t)| &\le \delta(t) \\
|v(t)^{T}\Gamma(t)| &\le \gamma(t) \\
v(t)^{T}\Theta(t) &\ge \vartheta(t) \\
v(t)^{T}\Lambda(t) &\ge \lambda(t)
\end{aligned} \quad (5.2)$$

$[\theta(t) = I_{X}(t)v(t)]$ and additional inequality and equality constraints

$$g(t,X(t),D(t),\varsigma(t),v(t)) \le 0 \quad (5.3a)$$
$$h(t,X(t),D(t),\varsigma(t),v(t)) = 0 \quad (5.3b)$$

(e.g., market frictions, etc.) in a securities and derivatives market

$$dX(t) = I_{X}(t)[M(t)dt + \Sigma(t)dW(t)]$$
$$dD(t) = -D(t)r(t)dt \qquad d\varsigma(t) = -\varsigma(t)[r(t)dt + A(t)^{T}dW(t)] \quad (5.4)$$

$$M(t) = \begin{bmatrix} M_{1}(t) \\ \vdots \\ M_{L}(t) \end{bmatrix} \quad \Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \cdots & \Sigma_{1N}(t) \\ \vdots & & \vdots \\ \Sigma_{L1}(t) & \cdots & \Sigma_{LN}(t) \end{bmatrix}$$

with associated [in general expressed in terms of any underlying Markov risk exposure assessment and control model $(t,S(t))$ in which $S(t)$ is an N-vector of state variables and $A(t) = \Sigma(t)^{T}K(t)^{-1}[M(t) - r(t)1_{L}]$ with $K(t) = \Sigma(t)\Sigma(t)^{T}$] instantaneous investment risk, future derivatives risk dynamics, options time decay dynamics and asset value appreciation dynamics

$$\Delta(t) = \begin{bmatrix} \Delta_1(t) \\ \vdots \\ \Delta_L(t) \end{bmatrix}, \ \Gamma(t) = \begin{bmatrix} \Gamma_1(t) \\ \vdots \\ \Gamma_L(t) \end{bmatrix}, \ \Theta(t) = \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_L(t) \end{bmatrix} \text{ and } \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ \vdots \\ \Lambda_L(t) \end{bmatrix} \quad (5.5)$$

[where in the case of an underlying securities market $(t, S(t))$ for the market prices of risk $A(t)$ associated with the exogenous sources $W(t)$ of securities and derivatives market uncertainty alternatively either $A(t) = \alpha(t)$ (dynamic completeness) or $A(t) = \hat{\alpha}_{\bar{a}_{\gamma_v}}(t)$ (dynamic incompleteness) holds]. As in the securities market case considered earlier it follows immediately that the continuous and strictly decreasing function

$$H(y) = E[\int_0^H \varsigma(t) I^c(t, y\varsigma(t)) dt + \varsigma(H) I^V(y\varsigma(H))] \quad (5.6)$$

has an inverse $[H(y)]^{-1}$ that maps $v \xrightarrow{1\text{-}1} y_v$. The corresponding fund withdrawal rate $c_v(t) = I^c(t, y_v\varsigma(t))$ and final wealth $V_H^v = I^V(y_v\varsigma(H))$ then have the property

$$E[\int_0^H \varsigma(t) c_v(t) dt + \varsigma(H) V_H^v] = v. \quad (5.7)$$

Furthermore, we have the familiar upper bound

$$E[\int_0^H U^c(t, c(t)) dt + U^V(V_v^{c\theta}(H))] \le E[\int_0^H U^c(t, c_v(t)) dt + U^V(V_H^v)] \quad (5.8)$$

for admissible pairs $(c, \theta) \in A(v)$ with $v(t) \in N_t$, $0 \le t \le H$, where the sets

$$N_t = \left\{ n \in R^L: \begin{array}{ll} |n^T \Delta(t)| \le \delta(t) & |n^T \Gamma(t)| \le \gamma(t) \\ n^T \Theta(t) \ge \vartheta(t) & n^T \Lambda(t) \ge \lambda(t) \\ g(t, X(t), D(t), \varsigma(t), n) \le 0 & h(t, X(t), D(t), \varsigma(t), n) = 0 \end{array} \right\} \quad (5.9)$$

in $v(t)$- strategy space are assumed to be closed (with non-empty interior) and convex. This assumption is fulfilled under the usual conditions that have to be imposed on the given constraints $g(t, X, D, z, n)$ and $h(t, X, D, z, n)$ in non-linear optimization problems. Note that in the absence of these additional constraints the origin $n = 0$ is feasible if and only if the guaranteed floors on portfolio time decay and value appreciation are non-positive.

Strategy Space Transformation. Because of these constraints on the admissible trading strategies [for which we shall use the notations $v(t)$ (number of assets), $\theta(t)$ (amount invested in assets) and $\bar{\theta}(t)$ (proportion of wealth invested in assets),

$$\theta_i(t) = v_i(t) X_i(t) \text{ and } \bar{\theta}_i(t) = \frac{\theta_i(t)}{V_v^{c\theta}(t)}, \quad (5.10)$$

interchangeably from now on] we can again not hope to directly find an asset allocation $\theta_v(t)$ that has the properties $(c_v, \theta_v) \in A(v)$, $v_v(t) \in N_t$, $0 \le t \le H$, and $V_v^{c, \theta_v}(H) = V_H^v$. The main idea is therefore once more to consider variations of the given Arrow-Debreu price system $\varsigma(t)$ that minimize the right-hand side of the above inequality. In order to apply the corresponding techniques of Cvitanic and Karatzas [63] we first transform our stochastic control problem into an equivalent one in $\bar{\theta}(t)$- strategy space

$$\max_{(c, \bar{\theta}) \in A(v)} E[\int_0^H U^c(t, c(t)) dt + U^V(V_v^{c\bar{\theta}}(H))]$$

$$E[\int_0^H \varsigma(t) c(t) dt + \varsigma(H) V_v^{c\bar{\theta}}(H)] \le v \quad (5.11)$$

$$\bar{\theta}(t) \in K_t$$

where $\overline{\theta}(t) = \overline{\theta}_t^{vc}(v)$ are the transformation rules [parametrized by the available choices of initial wealth $v > 0$ and intertemporal fund consumption $c(t)$]

$$
\begin{array}{ccc}
v & \xrightarrow{\overline{\theta}^{vc}} & \overline{\theta} \\
\downarrow & & \uparrow \\
\theta & \xrightarrow[v,c]{} & V_v^{c\theta}
\end{array}
\qquad (5.12)
$$

and the associated constraint sets

$$
K_t = \bigcup_{v,c} \overline{\theta}_t^{vc}(N_t)
$$

$$
\overline{\theta}_t^{vc}(N_t) = \left\{ \frac{I_x(t)v(t)}{V_v^{cv}(t)} : (c,v) \in A(v), \forall s[v(s) \in N_s] \right\}
\qquad (5.13)
$$

are also assumed to be closed (with non-empty interior) and convex. Note that this assumption is fulfilled if there are no constraints on initial wealth $v > 0$ and fund withdrawal rates $c(t)$ because then it is easily seen from

$$
\begin{aligned}
V_v^{cv}(t) &= \frac{1}{D(t)} \left[v + \int_0^t D(s)[\theta(s)^T[M(s) - r(s)1_L] - c(s)]ds + \int_0^t D(s)\theta(s)^T \Sigma(s)dW(s) \right] \\
&= \frac{1}{D(t)}[v - \int_0^t D(s)c(s)ds] + G^v(t) = \frac{v - c\Delta t}{D(t)} + G^v(t), \ c \geq 0
\end{aligned}
\qquad (5.14)
$$

$[c(s) = \dfrac{c}{D(s)}$ for $t - \Delta t \leq s \leq t$ and $c(s) = 0$ everywhere else] where

$$
G^v(t) = \frac{1}{D(t)} \left[\int_0^t D(s)\theta(s)^T[M(s) - r(s)1_L]ds + \int_0^t D(s)\theta(s)^T \Sigma(s)dW(s) \right] \quad (5.15)
$$

that for any given time and state (t,ω) and trading strategy $v(t)$ $[\forall t : v(t) \in N_t]$ as inital wealth $v > 0$ and intertemporal fund consumption $c(t)$ vary $[(c,v) \in A(v)]$ the corresponding portfolio value $V_v^{cv}(t,\omega)$ takes on every real number in the interval $(0,\infty)$ which means that the constraint set $K_t = K(t,\omega)$ is a closed convex cone [the point $k = 0$ may be missing depending on whether $n = 0$ is feasible or not]. We then define the support functions

$$
\delta(r|K_t) = \sup_{k \in K_t}(-k^T r) \quad (5.16)
$$

with effective domains

$$
\overline{K}_t = \left\{ r \in R^L : \delta(r|K_t) < \infty \right\} \quad (5.17)
$$

which are closed, positively homogeneous and properly convex functions on R^L with the subadditivity property $\delta(a + b|K_t) \leq \delta(a|K_t) + \delta(b|K_t)$. The family of these support functions $\delta(r|K_t)$, $0 \leq t \leq H$, is assumed to be uniformly bounded from below on R^L and continuous on its common effective domain $\overline{K} = \overline{K}_t$ $[= R^L$ for bounded $K_t]$, $0 \leq t \leq H$, which is a convex cone.

Market Parametrization. The variation parameter space is now taken to be the Hilbert space Ω of progressively measurable stochastic processes

$$
\omega(t) = \begin{bmatrix} \omega_1(t) & \cdots & \omega_L(t) \end{bmatrix}^T \text{ with } E[\int_0^H \|\omega(t)\|^2 dt] < \infty \quad (5.18)
$$

where the associated inner product and norm are

$$(\omega_1, \omega_2) = E[\int_0^H \omega_1(t)^T \omega_2(t)dt] \text{ and } \|\omega\| = \sqrt{(\omega, \omega)} . \quad (5.19)$$

The combined stochastic processes $\delta(\omega(t)|K_t)$, $\omega \in \Omega$, are assumed to also be progressively measurable whenever $\omega(t) \in \overline{K}$, $0 \le t \le H$, holds. Moreover, all constant stochastic processes $\omega(t) = \omega$ with this property are assumed to belong to the set

$$\overline{\Omega} = \left\{ \omega \in \Omega : E[\int_0^H \delta(\omega(t)|K_t)dt] < \infty \right\} . \quad (5.20)$$

For a variation parameter $\omega \in \overline{\Omega}$ we next consider the financial market with interest rates

$$r_\omega(t) = r(t) + \delta(\omega(t)|K_t) \quad (5.21)$$

and securities and derivatives price processes

$$dX_i^\omega(t) = dX_i(t) + X_i(t)[\omega_i(t) + \delta(\omega(t)|K_t)]dt . \quad (5.22)$$

If $c(t)$ is a fund withdrawal rate and $\overline{\theta}(t)$ a trading strategy, then with these asset price dynamics and money market rates the associated portfolio value process is

$$\begin{aligned} dV_\omega &= [V_\omega \overline{\theta}^T[M - r1_L + \omega] + V_\omega[r + \delta(\omega|K)] - c]dt + V_\omega \overline{\theta}^T \Sigma dW \\ &= [V_\omega r - c]dt + V_\omega[\overline{\theta}^T \omega + \delta(\omega|K)]dt + V_\omega \overline{\theta}^T \Sigma d\widetilde{B} \end{aligned} \quad (5.23)$$

[where the factorization with respect to the portfolio value in the second drift term of the above sum is crucial in the following considerations and the reason for choosing $\overline{\theta}(t)$ - strategy space]. Note also that with our different notations for trading and portfolio management strategies [i.e., $v(t)$, $\theta(t)$ and $\overline{\theta}(t)$] we have the equivalent representations

$$\begin{aligned} dV &= v^T dX + [V - v^T X]rdt - cdt \\ &= [v^T[I_X M - rX] + Vr - c]dt + v^T I_X \Sigma dW \\ &= [\theta^T[M - r1_L] + Vr - c]dt + \theta^T \Sigma dW \quad (5.24) \\ &= [V\overline{\theta}^T[M - r1_L] + Vr - c]dt + V\overline{\theta}^T \Sigma dW \\ &= [Vr - c]dt + V\overline{\theta}^T \Sigma d\widetilde{B} \end{aligned}$$

of the associated portfolio value process in the originally given securities and derivatives market. Let for an initial wealth $v > 0$ now $A_\omega(v)$ be the set of all pairs $(c, \overline{\theta})$ consisting of a fund withdrawal rate $c(t)$ and a trading strategy $\overline{\theta}(t)$ which are admissible in the financial market parametrized by $\omega(t)$ and $\overline{A}(v)$ be the set of all feasible pairs, i.e., originally admissible pairs $(c, \overline{\theta}) \in A(v)$ that have the property $\overline{\theta}(t) \in K_t$, $0 \le t \le H$. Then we have

$$\overline{A}(v) \subseteq A_\omega(v) \quad (5.25)$$

and for every feasible pair $(c, \overline{\theta}) \in \overline{A}(v)$,

$$V_{\omega v}^{c\overline{\theta}}(t) \ge V_v^{c\overline{\theta}}(t) , \quad (5.26)$$

because of the fact that $\overline{\theta}(t)^T \omega(t) + \delta(\omega(t)|K_t) \ge 0$.

Unconstrained Solutions. Furthermore, the general investment management and asset allocation problem

$$\max_{(c,\overline{\theta}) \in A_\omega(v)} E[\int_0^H U^c(t, c(t))dt + U^V(V_{\omega v}^{c\overline{\theta}}(H))]$$

$$E[\int_0^H \varsigma_\omega(t)c(t)dt + \varsigma_\omega(H)V_{\omega v}^{c\overline{\theta}}(H)] \le v \quad (5.27)$$

in the $\omega(t)$-parametrized securities and derivatives market

$$D_\omega(t) = \exp(-\int_0^t r_\omega(s)ds)$$

$$M_\omega(t) = M(t) + \omega(t) + \delta(\omega(t)|K_t)1_L$$

$$\alpha_\omega(t) = \Sigma(t)^T K(t)^{-1}[M_\omega(t) - r_\omega(t)1_L] \qquad (5.28)$$

$$\beta_\omega(t) = \exp(-\int_0^t \alpha_\omega(s)^T dW(s) - \frac{1}{2}\int_0^t \|\alpha_\omega(s)\|^2 ds)$$

$$\varsigma_\omega(t) = \beta_\omega(t)D_\omega(t)$$

has a solution

$$V_{\omega v}^{c_v^\omega \bar{\theta}_v^\omega}(t) = \frac{1}{\varsigma_v^\omega(t)} E[\int_t^H \zeta_v^\omega(s)c_v^\omega(s)ds + \zeta_v^\omega(H)V_H^{\omega v}\Big|F_t] \qquad (5.29)$$

where

$$c_v^\omega(t) = I^c(t, H_v^{\omega-1}(v)\varsigma_v^\omega(t))$$
$$V_H^{\omega v} = I^V(H_v^{\omega-1}(v)\varsigma_v^\omega(H)) \qquad H_v^\omega(y) = E[\int_0^H \varsigma_v^\omega(t)I^c(t, y\varsigma_v^\omega(t))dt + \varsigma_v^\omega(H)I^V(y\varsigma_v^\omega(H))] \qquad (5.30)$$

[and in the incomplete case the associated Arrow-Debreu price system $\varsigma_v^\omega(t)$ and minimax local martingale measure $\widetilde{\pi}_v^\omega$ are again uniquely determined by the investor's utility functions $U^c(t,c)$ and $U^V(V)$]. If for this optimal $\omega(t)$- asset allocation $\bar{\theta}_v^\omega(t)$ (which is unconstrained) furthermore

$$\bar{\theta}_v^\omega(t) \in K_t \text{ and } \bar{\theta}_v^\omega(t)^T\omega(t) + \delta(\omega(t)|K_t) = 0 \qquad (5.31)$$

holds, then the feasible pair $(c_v^\omega, \bar{\theta}_v^\omega) \in \overline{A}(v)$ is a solution of the constrained stochastic control problem in the originally given securities and derivatives market and in familiar control theory notation

$$J(v,c,\bar{\theta}) = E[\int_0^H U^c(t,c(t))dt + U^V(V_v^{c\bar{\theta}}(H))]$$
$$\overline{V}(v) = \sup_{(c,\bar{\theta}) \in \overline{A}(v)} J(v,c,\bar{\theta}) \qquad (5.32)$$

and

$$J_\omega(v,c,\bar{\theta}) = E[\int_0^H U^c(t,c(t))dt + U^V(V_{\omega v}^{c\bar{\theta}}(H))]$$
$$V_\omega(v) = \sup_{(c,\bar{\theta}) \in A_\omega(v)} J_\omega(v,c,\bar{\theta}) \qquad (5.33)$$

satisfies

$$J(v,c_v^\omega,\bar{\theta}_v^\omega) = J_\omega(v,c_v^\omega,\bar{\theta}_v^\omega) = \min_{\bar{\omega}\in\Omega}J_{\bar{\omega}}(v,c_v^\omega,\bar{\theta}_v^\omega) = \overline{V}(v) = V_\omega(v) = \min_{\bar{\omega}\in\Omega}V_{\bar{\omega}}(v) \qquad (5.34)$$

whereas in general only

$$J(v,c,\bar{\theta}) \le \inf_{\bar{\omega}\in\Omega}J_{\bar{\omega}}(v,c,\bar{\theta}), \ (c,\bar{\theta}) \in \overline{A}(v), \text{ and } \overline{V}(v) \le \inf_{\bar{\omega}\in\Omega}V_{\bar{\omega}}(v) \qquad (5.35)$$

holds.

<u>Maximum Principle</u>. If on the other hand $c(t)$ is a fund withdrawal rate and V_H a final wealth target such that for a market parametrization $\omega(t)$

$$\max_{\bar{\omega}\in\Omega} E[\int_0^H \varsigma_{\bar{\omega}}(t)c(t)dt + \varsigma_{\bar{\omega}}(H)V_H] = E[\int_0^H \varsigma_\omega(t)c(t)dt + \varsigma_\omega(H)V_H] = v \qquad (5.36)$$

holds, then there exists a trading strategy $\bar{\theta}_v^\omega(t)$ that satisfies $(c,\bar{\theta}_v^\omega) \in A_\omega(v)$ and $V_{\omega v}^{c\bar{\theta}_v^\omega}(H) = V_H$ as well as $\bar{\theta}_v^\omega(t) \in K_t$ and $\bar{\theta}_v^\omega(t)^T\omega(t) + \delta(\omega(t)|K_t) = 0$ (see [63] for details).

Therefore the corresponding asset allocation is feasible, i.e., $(c, \overline{\theta}_v^\omega) \in \overline{A}(v)$, and sufficient in the originally given securities and derivatives market, i.e., $V_v^{c\overline{\theta}_v^\omega}(H) = V_H$. It is optimal if

$$c(t) = I^c(t, H_{\overline{\omega}}^{-1}(v)\varsigma_{\overline{\omega}}(t)) \text{ and } V_H = I^V(H_{\overline{\omega}}^{-1}(v)\varsigma_{\overline{\omega}}(H)) \quad (5.37)$$

holds for some variation parameter $\overline{\omega}(t)$ because then

$$J(v, c, \overline{\theta}_v^\omega) = V_{\overline{\omega}}(v). \quad (5.38)$$

Convex Duality. As in the securities market case the above maximum principle can be invoked (and the constrained investment management and asset allocation problem for heterogeneous derivatives portfolios solved) after considering a suitable dual optimization problem. Given any $y > 0$ the convex minimization program

$$\min_{\overline{\omega} \in \Omega} E[\int_0^H U_\bullet^c(t, y\varsigma_{\overline{\omega}}(t))dt + U_\bullet^V(y\varsigma_{\overline{\omega}}(H))] \quad (5.39)$$

with the continuous, strictly decreasing and continuously differentiable duals

$$U_\bullet^c(t,c) = U^c(t, I^c(t,c)) - cI^c(t,c) \text{ and } U_\bullet^V(V) = U^V(I^V(V)) - VI^V(V) \quad (5.40)$$

of the investor's utility functions has a unique solution $\omega_y(t)$. With this market variation parameter we then define

$$v_y = H_{\omega_y}(y), \ c_y(t) = I^c(t, y\varsigma_{\omega_y}(t)) \text{ and } V_H^y = I^V(y\varsigma_{\omega_y}(H)) \quad (5.41)$$

and have

$$\max_{\overline{\omega} \in \Omega} E[\int_0^H \varsigma_{\overline{\omega}}(t)c_y(t)dt + \varsigma_{\overline{\omega}}(H)V_H^y] = E[\int_0^H \varsigma_{\omega_y}(t)c_y(t)dt + \varsigma_{\omega_y}(H)V_H^y] = v_y \quad (5.42)$$

and therefore a solution of the constrained investment management and asset allocation problem in the originally given securities and derivatives market that is related to the initial wealth v_y. Furthermore, any initial wealth $v > 0$ can be reached with the mapping $y \to v_y$. The corresponding optimal (constrained) fund withdrawal rate $c_{y_v}(t)$ and trading strategy $\overline{\theta}_{y_v}(t)$, $(c_{y_v}, \overline{\theta}_{y_v}) \in \overline{A}(v)$, in the given securities and derivatives market coincide with the solution of the unconstrained investment management and asset allocation problem

$$\max_{(c,\theta) \in A_{\omega_{y_v}}(v)} E[\int_0^H U^c(t, c(t))dt + U^V(V_{\omega_{y_v},v}^{c\overline{\theta}}(H))]$$

$$E[\int_0^H \varsigma_{\omega_{y_v}}(t)c(t)dt + \varsigma_{\omega_{y_v}}(H)V_{\omega_{y_v},v}^{c\overline{\theta}}(H)] \le v \quad (5.43)$$

in its $\omega_{y_v}(t)$-parametrization and therefore the associated portfolio value process has the representation

$$V_v^{c_{y_v}, \overline{\theta}_{y_v}}(t) = V_{\omega_{y_v}, v}^{c_{y_v}, \overline{\theta}_{y_v}}(t) = \frac{1}{\varsigma_v^{\omega_{y_v}}(t)} E[\int_t^H \varsigma_v^{\omega_{y_v}}(s)c_{y_v}(s)ds + \varsigma_v^{\omega_{y_v}}(H)V_H^y \big| F_t]. \quad (5.44)$$

Markovian Characterization. If this $\omega_{y_v}(t)$-parametrized financial market is dynamically complete, then the portfolio value function

$$V_v^{c_{y_v}, \overline{\theta}_{y_v}}(t) = V_{\omega_{y_v}, v}^{c_{y_v}, \overline{\theta}_{y_v}}(t) = V(t, X(t), Z(t)) \quad (5.45)$$

satisfies the linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} tr(BB^T \nabla^2 V) - \nabla V^T B\alpha_{\omega_{y_v}} - Vr_{\omega_{y_v}} + I^c(t, Z) = 0 \quad (5.46)$$

with boundary conditions $V(0, X, Z) = v$ and $V(H, X, Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ \vdots \\ X_N M_N^{\omega_{y_v}} \\ -Z r_{\omega_{y_v}} \end{bmatrix}, \ B = \begin{bmatrix} X_1 \Sigma_{11} & \cdots & X_1 \Sigma_{1N} \\ \vdots & & \vdots \\ X_N \Sigma_{N1} & \cdots & X_N \Sigma_{NN} \\ -Z \alpha_1^{\omega_{y_v}} & \cdots & -Z \alpha_N^{\omega_{y_v}} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \dfrac{\partial}{\partial X_1} \\ \vdots \\ \dfrac{\partial}{\partial X_N} \\ \dfrac{\partial}{\partial Z} \end{bmatrix} = \begin{bmatrix} \nabla_X \\ \nabla_Z \end{bmatrix}. \quad (5.47)$$

The optimal asset allocation is

$$\overline{\theta}_{y_v} = \frac{[B\Sigma^{-1}]^T \nabla V}{V} = \frac{I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N] Z \nabla_Z V}{V}. \quad (5.48)$$

In the incomplete case we have the quasi-linear partial differential equation

$$\frac{\partial V}{\partial t} + A^T \nabla V + \frac{1}{2} \text{tr}(BB^T \nabla^2 V) - \nabla V^T B \hat{\alpha}_{\hat{a}_{z_v}}^{\omega_{y_v}} - V r_{\omega_{y_v}} + I^c(t, Z) = 0 \quad (5.49)$$

for the portfolio value function

$$V_v^{c_{y_v}, \overline{\theta}_{y_v}}(t) = V_{\omega_{y_v} v}^{c_{y_v}, \overline{\theta}_{y_v}}(t) = V(t, X(t), Y(t), Z(t)) \quad (5.50)$$

with boundary conditions $V(0, X, Y, Z) = v$ and $V(H, X, Y, Z) = I^V(Z)$ where

$$A = \begin{bmatrix} X_1 M_1^{\omega_{y_v}} \\ \vdots \\ X_L M_L^{\omega_{y_v}} \\ Y_1 \hat{a}_1^{z_v} \\ \vdots \\ Y_{N-L} \hat{a}_{N-L}^{z_v} \\ -Z r_{\omega_{y_v}} \end{bmatrix}, \ B = \begin{bmatrix} X_1 \Sigma_{11} & \cdots & X_1 \Sigma_{1N} \\ \vdots & & \vdots \\ X_L \Sigma_{L1} & \cdots & X_L \Sigma_{LN} \\ Y_1 b_{11} & \cdots & Y_1 b_{1N} \\ \vdots & & \vdots \\ Y_{N-L} b_{N-L1} & \cdots & Y_{N-L} b_{N-LN} \\ -Z \hat{\alpha}_1^{\omega_{y_v} \hat{a}_{z_v}} & \cdots & -Z \hat{\alpha}_N^{\omega_{y_v} \hat{a}_{z_v}} \end{bmatrix} \text{ and } \nabla = \begin{bmatrix} \nabla_X \\ \nabla_Y \\ \nabla_Z \end{bmatrix}. \quad (5.51)$$

In addition, we find that $\theta_{y_v}^T \Sigma = \nabla V^T B$ holds and therefore the completion premium $v_{\hat{a}_{z_v}} \in K(\Sigma)$ [associated with the market prices of risk $\alpha_{\omega_{y_v}} \in K^{\perp}(\Sigma)$] satisfies

$$v_{\hat{a}_{z_v}} = \frac{B_Y^T \nabla_Y V}{Z \nabla_Z V} = b^T \frac{I_Y \nabla_Y V}{Z \nabla_Z V} \quad (5.52)$$

and the optimal asset allocation is

$$\overline{\theta}_{y_v} = \frac{[B \Sigma^T K^{-1}]^T \nabla V}{V} = \frac{I_X \nabla_X V - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L] Z \nabla_Z V}{V}. \quad (5.53)$$

Risk/Arbitrage Strategies. Given an initial wealth $v > 0$ (target wealth) we have above been able to derive an optimal solution

$$\left(c_{y_v}, \overline{\theta}_{y_v}, V_v^{c_{y_v}, \overline{\theta}_{y_v}} \right) \quad (5.54)$$

of the limited risk arbitrage investment management problem in $\overline{\theta}(t)$ - strategy space with associated risk/arbitrage strategy

$$\overline{\theta}_{y_v} = \begin{cases} \dfrac{I_X \nabla_X V_v^{c_{y_v}, \overline{\theta}_{y_v}} - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N]Z\nabla_Z V_v^{c_{y_v}, \overline{\theta}_{y_v}}}{V_v^{c_{y_v} \overline{\theta}_{y_v}}} \\[3mm] \dfrac{I_X \nabla_X V_v^{c_{y_v}, \overline{\theta}_{y_v}} - K^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L]Z\nabla_Z V_v^{c_{y_v}, \overline{\theta}_{y_v}}}{V_v^{c_{y_v} \overline{\theta}_{y_v}}} \end{cases} \quad (5.55)$$

[where $\overline{\theta} = \overline{\theta}^{vc}(v)$, $(c,v) \in A(v)$]. The corresponding trading strategy

$$v_{y_v} = \overline{\theta}^{vc_{y_v}-1}(\overline{\theta}_{y_v}) = I_X^{-1}[V_v^{c_{y_v}, \overline{\theta}_{y_v}} \overline{\theta}_{y_v}]$$

$$= \begin{cases} \nabla_X V_v^{c_{y_v}, \overline{\theta}_{y_v}} - [KI_X]^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_N]Z\nabla_Z V_v^{c_{y_v}, \overline{\theta}_{y_v}} \\[2mm] \nabla_X V_v^{c_{y_v}, \overline{\theta}_{y_v}} - [KI_X]^{-1}[M_{\omega_{y_v}} - r_{\omega_{y_v}} 1_L]Z\nabla_Z V_v^{c_{y_v}, \overline{\theta}_{y_v}} \end{cases} \quad (5.56)$$

$[V_v^{c_{y_v} v_{y_v}} = V_v^{c_{y_v}, \overline{\theta}_{y_v}}]$ in $v(t)$- strategy space does however not necessarily satisfy

$$v_{y_v}(t) \in N_t, \quad 0 \le t \le H, \quad (5.57)$$

i.e., the originally given limited risk arbitrage objectives and additional inequality and equality constraints. [By recalling the applied strategy space transformation

$$\overline{\theta}(t) = \overline{\theta}_t^{vc}(v) = \frac{I_X(t)v(t)}{V_v^{cv}(t)} \qquad [(c,v) \in A(v)] \quad (5.58)$$

$$K_t = \bigcup_{v,c} \overline{\theta}_t^{vc}(N_t) \qquad \overline{\theta}_t^{vc}(N_t) = \left\{ \frac{I_X(t)v(t)}{V_v^{cv}(t)} : (c,v) \in A(v), \forall s[v(s) \in N_s] \right\}$$

we can of course find an initial wealth $v' > 0$, an intertemporal fund withdrawal rate $c'(t)$ and a trading strategy $v'(t)$, $(c',v') \in A(v')$ and $v'(t) \in N_t$, $0 \le t \le H$, such that

$$\overline{\theta}_t^{v'c'}(v') = \frac{I_X(t)v'(t)}{V_{v'}^{c'v'}(t)} = \overline{\theta}_{y_v}(t) = \overline{\theta}^{vc_{y_v}}(v_{y_v}) = \frac{I_X(t)v_{y_v}(t)}{V_v^{c_{y_v}, v_{y_v}}(t)} \quad (5.59)$$

holds but this is not necessarily optimal in terms of the given investment management objectives $U^c(t,c)$ and $U^V(V)$.] The main idea therefore now is for a trading strategy $v'(t)$ that satisfies the limited risk arbitrage constraints [i.e., for which $\forall t : v'(t) \in N_t$ holds] to require the associated stochastic process

$$x(t) = x_v^{v'}(t) = x_v^{\alpha'}(t) = V_v^{c_{y_v} v'}(t) - V_v^{c_{y_v}, v_{y_v}}(t)$$

$$dx(t) = x(t)r(t)dt + (\alpha'(t) - 1)\theta_{y_v}(t)^T \Sigma(t)d\widetilde{B}(t), \quad x(0) = v' - v \quad (5.60)$$

[where we more specifically choose the portfolio

$$\theta'(t) = I_X(t)v'(t) = \alpha'(t)\theta_{y_v}(t), \quad \alpha'(t) \ge 0 \quad (5.61)$$

on the $\theta_{y_v}(t)$-ray emanating from the origin] to satisfy

$$x(t) \ge 0, \quad 0 \le t \le H. \quad (5.62)$$

This [admissibility, i.e., after a time scale adjustment

$$t \to \langle M \rangle_t, \qquad M(t) = \int_0^t \theta_{y_v}(s)^T \Sigma(s)d\widetilde{B}(s) \qquad \left[\langle M \rangle_t = \int_0^t \left\| \theta_{y_v}(s)^T \Sigma(s) \right\|^2 ds \right] \quad (5.63)$$

we can consider the constrained asset allocation

$$dV(t) = V(t)r(t)dt + \theta(t)d\widetilde{z}(t) \qquad a_t \le \theta(t) \le b_t \quad (5.64)$$

in which the risky asset's price process is

$$dP(t) = P(t)[r(t)dt + d\widetilde{z}(t)] \quad (5.65)$$

$[\sigma(t) = 1]$ under the equivalent martingale measure $\widetilde{\psi}$] condition ensures that risk/arbitrage trading in $v(t)$- strategy space performs at least as well as the above derived optimal asset

allocation in $\overline{\theta}(t)$ - strategy space [in terms of the given investment management objectives $U^c(t,c)$ and $U^V(V)$]. The quantity

$$\hat{v} = \inf\left\{v' \geq v : \exists v'[(c_{y_v}, v') \in A(v') \wedge \forall t[v'(t) \in N_t] \wedge \forall t[x_{v'}^{v}(t) \geq 0]]\right\} \quad (5.66)$$

is then the required wealth for limited risk arbitrage investment management in $v(t)$ - strategy space [with associated $\overline{\theta}(t)$ - strategy space wealth target $v > 0$] and $\hat{v} - v$ is the corresponding transformation premium. Any two respective solutions

$$\left(v, c_{y_v}, \overline{\theta}_{y_v}, V_v^{c_{y_v}, \overline{\theta}_{y_v}}\right) \text{ and } \left(\hat{v}_{v'}, c_{y_v}, v', V_{\hat{v}_{v'}}^{c_{y_v}, v'}\right) \quad (5.67)$$

of the limited risk arbitrage investment management and asset allocation problem where

$$\hat{v}_{v'} = \inf\left\{v' \geq v : (c_{y_v}, v') \in A(v') \wedge \forall t[x_{v'}^{v}(t) \geq 0]\right\} \qquad \left[\forall t : v'(t) \in N_t\right] \quad (5.68)$$

can now be considered equivalent [at an initial wealth premium $\hat{v}_{v'} - v$ for a corresponding increase

$$E[U^V(V_{\hat{v}_{v'}}^{c_{y_v}, v'}(H)) - U^V(V_v^{c_{y_v}, \overline{\theta}_{y_v}}(H))] \quad (5.69)$$

in expected utility of final wealth]. Alternatively, if $\hat{v}(t)$ is a risk/arbitrage strategy associated with the required wealth $\hat{v}$ [i.e., a CS projection of the optimal asset allocation $\overline{\theta}_{y_v}(t)$], then the two solutions

$$\left(v, c_{y_v}, \overline{\theta}_{y_v}, V_v^{c_{y_v}, \overline{\theta}_{y_v}}\right) \text{ and } \left(\hat{v}, c_{y_v}, \hat{v}, V_{\hat{v}}^{c_{y_v}, \hat{v}}\right) \quad (5.70)$$

of the limited risk arbitrage investment management and asset allocation problem can be considered equivalent. In either case

$$\left(v, c_{y_v}, \overline{\theta}_{y_v}, V_v^{c_{y_v}, \overline{\theta}_{y_v}}\right) \quad (5.71)$$

characterizes the equivalence class of corresponding solutions and we shall therefore always refer to this canonical solution when we discuss limited risk arbitrage investment management.

Dynamic Programming. Chow [64] points out an interesting alternative (and complementary method) to the above martingale representation approach which uses an algebraic solution technique involving KKT first order optimality conditions and Lagrange multipliers instead of solving the HJB dynamic programming equation for a (constrained) stochastic control problem. In his setting the investment management and asset allocation problem that characterizes risk/arbitrage strategies is

$$J(v, c, v) = E[\int_0^H U^c(t, c(t))dt + U^V(V_v^{cv}(H))]$$
$$\overline{V}(v) = \sup_{(c,v) \in \overline{A}(v)} J(c, c, v) \quad (5.72)$$

with controlled state dynamics (portfolio value function)

$$dV(t) = [v(t)^T[C(t) - r(t)X(t)] + V(t)r(t) - c(t)]dt + v(t)^T D(t)dW(t)$$
$$C(t) = I_X(t)M(t) = C(t, V(t)) \qquad D(t) = I_X(t)\Sigma(t) = D(t, V(t)) \quad (5.73)$$
$$V(0) = v$$

control variables $c(t)$ and $v(t)$, risk/arbitrage objectives

$$\left|v(t)^T \Delta(t)\right| \le \delta(t)$$
$$\left|v(t)^T \Gamma(t)\right| \le \gamma(t)$$
$$v(t)^T \Theta(t) \ge \vartheta(t) \qquad (5.74)$$
$$v(t)^T \Lambda(t) \ge \lambda(t)$$

and additional inequality and equality constraints

$$g(t, V(t), c(t), v(t)) \le 0$$
$$h(t, V(t), c(t), v(t)) = 0. \qquad (5.75)$$

Feasibility of a control $u(t) = (c(t), v(t))$, i.e., $(c, v) \in \overline{A}(v)$, is defined in the obvious way and here an optimal solution of the above stochastic control problem can be derived directly in $v(t)$ - strategy space. Using the notation

$$J(t, V(t), u) = E[\int_t^H U^c(s, c(s))ds + U^V(V_{v(t)}^{cv}(H)) \Big| F_t]$$

$$\overline{V}(t, V(t)) = \sup_{u \in \overline{A}(t, V(t))} J(t, V(t), u)$$

$$a(t, V(t), u(t)) = v(t)^T[C(t) - r(t)X(t)] + V(t)r(t) - c(t) \qquad (5.76)$$

$$b(t, V(t), u(t)) = v(t)^T D(t)$$

$$f(t, u(t)) = U^c(t, c(t))$$

and applying Ito's formula we have [assuming $\overline{V}(t, V) \in C^{1,2}$]

$$d\overline{V} = [\frac{\partial \overline{V}}{\partial t} + a\frac{\partial \overline{V}}{\partial V} + \frac{1}{2}\|b\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}]dt + \frac{\partial \overline{V}}{\partial V}bdW \qquad (5.77)$$

with corresponding (conditional) first and second order moments

$$E[d\overline{V}|F_t] = [\frac{\partial \overline{V}}{\partial t} + a\frac{\partial \overline{V}}{\partial V} + \frac{1}{2}\|b\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}]dt$$
$$\qquad (5.78)$$
$$V[d\overline{V}|F_t] = [\frac{\partial \overline{V}}{\partial V}]^2\|b\|^2 dt.$$

If we now consider an infinitesimal trading period $[t, t + dt]$, then according to the dynamic programming principle the optimal value function $\overline{V}(t, V)$ of the limited risk arbitrage control problem satisfies

$$\overline{V}(t, V(t)) = \sup_{(c,v) \in \overline{A}(t, V(t))} E[\int_t^H U^c(s, c(s))ds + U^V(V_{v(t)}^{cv}(H)) \Big| F_t]$$

$$= \sup_{u \in \overline{A}(t, V(t))} E[f(t, u(t))dt + \overline{V}(t + dt, V(t + dt)) | F_t]$$

$$= \sup_{u \in \overline{A}(t, V(t))} E[f(t, u(t))dt + \overline{V}(t, V(t)) + d\overline{V}(t) | F_t] \qquad (5.79)$$

$$= \overline{V}(t, V(t)) + \sup_{u \in \overline{A}(t, V(t))} \Big[f(t, u(t))dt + E[d\overline{V}(t) | F_t] \Big]$$

thus

$$0 = \sup_{u \in \overline{A}(t, V(t))} \Big[f(t, u(t))dt + E[d\overline{V}(t) | F_t] \Big]$$

$$= \sup_{u \in \overline{A}(t, V(t))} \Bigg[f(t, u(t)) + \frac{E[d\overline{V}(t) | F_t]}{dt} \Bigg] \qquad (5.80)$$

and therefore the Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_{u \in \overline{A}(t, V(t))} [f + \frac{\partial \overline{V}}{\partial t} + a \frac{\partial \overline{V}}{\partial V} + \frac{1}{2} \|b\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}] = 0 \quad (5.81)$$

with boundary condition $\overline{V}(H, V) = U^V(V)$. If this second order, non-linear PDE is uniformly parabolic, i.e.,

$$\|b\|^2 \geq \epsilon \quad (5.82)$$

for some $\epsilon > 0$, then the associated boundary value problem has a unique classical solution (in the degenerate parabolic case viscosity solutions can be considered). Hence, using the verification theorem of stochastic dynamic programming the standard approach to determining an optimal investment strategy in these limited risk arbitrage control problems would be as follows. Consider the maximization program

$$\max_u f(t, u) + \frac{\partial \overline{V}}{\partial t}(t, V(t)) + a(t, V(t), u) \frac{\partial \overline{V}}{\partial V}(t, V(t)) + \frac{1}{2} \|b(t, V(t), u)\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}(t, V(t))$$

$$RA(t, u) \leq 0 \quad (5.83)$$

$$g(t, V(t), u) \leq 0$$

$$h(t, V(t), u) = 0$$

where the risk/arbitrage objectives are

$$RA(t, u(t)) \leq 0 \quad \leftrightarrow \quad \begin{aligned} |v(t)^T \Delta(t)| &\leq \delta(t) \\ |v(t)^T \Gamma(t)| &\leq \gamma(t) \\ v(t)^T \Theta(t) &\geq \vartheta(t) \\ v(t)^T \Lambda(t) &\geq \lambda(t) \end{aligned} \quad (5.84)$$

With

$$F = f + \frac{\partial \overline{V}}{\partial t} + a \frac{\partial \overline{V}}{\partial V} + \frac{1}{2} \|b\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}$$

$$u = \begin{bmatrix} u_0 & \cdots & u_L \end{bmatrix}^T \quad \nabla = \begin{bmatrix} \dfrac{\partial}{\partial u_0} & \cdots & \dfrac{\partial}{\partial u_L} \end{bmatrix}^T \quad (5.85)$$

the associated Karush-Kuhn-Tucker (KKT) first order necessary conditions for a maximum at $\overline{u}$ are

$$\nabla F(t, V(t), \overline{u}) - \ell_{RA}^T \nabla RA(t, \overline{u}) - \ell_g^T \nabla g(t, V(t), \overline{u}) - \ell_h^T \nabla h(t, V(t), \overline{u}) = 0$$

$$\ell_i^{RA} RA_i(t, \overline{u}) = 0$$

$$\ell_j^g g_j(t, V(t), \overline{u}) = 0 \quad (5.86)$$

$$\ell_i^{RA} \geq 0$$

$$\ell_j^g \geq 0$$

and if $\overline{u} = \overline{u}(t, V(t))$ is a corresponding optimal solution, then the value function $\overline{V}(t, V)$ associated with limited risk arbitrage control can be obtained by solving the second order, linear PDE

$$f(t, \overline{u}(t, v)) + \frac{\partial \overline{V}}{\partial t}(t, v) + a(t, v, \overline{u}(t, v)) \frac{\partial \overline{V}}{\partial V}(t, v) + \frac{1}{2} \|b(t, v, \overline{u}(t, v))\|^2 \frac{\partial^2 \overline{V}}{\partial V^2}(t, v) = 0 \quad (5.87)$$

[where the state variable v - instead of V - has been chosen for notational convenience in the following linearizations]. If however instead of solving for the optimal value function we introduce the two Lagrangians

$$\ell_t^{\overline{V}} = \frac{\partial \overline{V}}{\partial t} \qquad \ell_v^{\overline{V}} = \frac{\partial \overline{V}}{\partial V} \qquad (5.88)$$

and use the linearizations

$$\ell_v^{\overline{V}}(t,v) = L^{\ell_v^{\overline{V}}}v + 1^{\ell_v^{\overline{V}}} \qquad\qquad \nabla F(t,v,u) = L_v^{VF}v + L_u^{VF}u + 1^{VF}$$

$$\nabla f(t,u) = L^{Vf}u + 1^{Vf} \qquad\qquad \nabla RA(t,u) = L^{VRA}u + 1^{VRA}$$

$$a(t,v,u) = L_v^a v + L_u^a u + 1^a \qquad\qquad \nabla g(t,v,u) = L_v^{Vg}v + L_u^{Vg}u + 1^{Vg} \qquad (5.89)$$

$$\nabla[\tfrac{1}{2}\|b(t,v,u)\|^2] = L_v^{\nabla[\frac{1}{2}\|b\|^2]}v + L_u^{\nabla[\frac{1}{2}\|b\|^2]}u + 1^{\nabla[\frac{1}{2}\|b\|^2]} \qquad \nabla h(t,v,u) = L_v^{Vh}v + L_u^{Vh}u + 1^{Vh}$$

about $(V(t),\overline{u})$, then

$$L_v^{VF} = (L_u^a + L_v^{\nabla[\frac{1}{2}\|b\|^2]})L_v^{\ell_v^{\overline{V}}}$$

$$L_u^{VF} = L^{Vf} + L_u^{\nabla[\frac{1}{2}\|b\|^2]}L_v^{\ell_v^{\overline{V}}} \qquad\qquad (5.90)$$

$$1^{VF} = 1^{Vf} + L_u^a 1^{\ell_v^{\overline{V}}} + 1^{\nabla[\frac{1}{2}\|b\|^2]}L_v^{\ell_v^{\overline{V}}}$$

and from the above KKT conditions and the corresponding PDE at optimality it follows that

$$L_v^{KKT}v + L_u^{KKT}\overline{u} + 1^{KKT} = 0 \qquad (5.91)$$

$$\overline{u} = L^{\overline{u}}v + 1^{\overline{u}}$$

with

$$L_v^{KKT} = L_v^{VF} - \ell_g{}^T L_v^{Vg} - \ell_h{}^T L_v^{Vh}$$

$$L_u^{KKT} = L_u^{VF} - \ell_{RA}{}^T L^{VRA} - \ell_g{}^T L_u^{Vg} - \ell_h{}^T L_u^{Vh} \qquad (5.92)$$

$$1^{KKT} = 1^{VF} - \ell_{RA}{}^T 1^{VRA} - \ell_g{}^T 1^{Vg} - \ell_h{}^T 1^{Vh}$$

as well as

$$L^{PDE}v + 1^{PDE} = 0 \qquad (5.93)$$

with

$$L^{PDE} = 2(L_v^a + L_u^a L^{\overline{u}})L_v^{\ell_v^{\overline{V}}} \qquad\qquad\qquad (5.94)$$

$$1^{PDE} = L^f L^{\overline{u}} + L_t^{\ell_t^{\overline{V}}} + (L_v^a + L_u^a L^{\overline{u}})1^{\ell_v^{\overline{V}}} + (L_u^a 1^{\overline{u}} + 1^a + L_v^{\frac{1}{2}\|b\|^2} + L_u^{\frac{1}{2}\|b\|^2}L^{\overline{u}})L_v^{\ell_v^{\overline{V}}}$$

where we have also used the linearizations

$$\ell_t^{\overline{V}}(t,v) = L_t^{\ell_t^{\overline{V}}}v + 1^{\ell_t^{\overline{V}}} \qquad\qquad \tfrac{1}{2}\|b(t,v,u)\|^2 = L_v^{\frac{1}{2}\|b\|^2}v + L_u^{\frac{1}{2}\|b\|^2}u + 1^{\frac{1}{2}\|b\|^2}. \qquad (5.95)$$

$$f(t,u) = L^f u + 1^f$$

These two equations can now be used to numerically (Riccati-iterations) solve for the optimal control $\overline{u} = \overline{u}(t, V(t))$ and the determinants of the above Lagrangian functions $\ell_t^{\overline{V}}$ and $\ell_v^{\overline{V}}$. Note that this approach does not necessarily derive the optimal value function $\overline{V}(t, V)$ of the risk/arbitrage control problem itself [the portfolio value process $V(t)$ is the more interesting quantity and can be calculated by applying the optimal control $\overline{u}(t, V(t))$ to the state dynamics with a suitable discrete approximation of the corresponding linear SDE].

Drawdown Control. The limited risk arbitrage investment management and asset allocation problem which also incorporates drawdown control

$$D(t)V^v(t) > \alpha M^v(t) \qquad \left[M^v(t) = \max_{0 \le s \le t} D(s)V^v(s) \right] \qquad (5.96)$$

at a level $1-\alpha$, $0 < \alpha < 1$, can be solved by applying a simple strategy space transformation. Following Cvitanic and Karatzas [65] given such a control level $1-\alpha$ we consider the set $\overline{A}_\alpha^\bullet(v)$ of feasible pairs $(c, \theta^\bullet)$ such that the constrained stochastic functional equation

$$dV^\bullet = [(V^\bullet - \frac{\alpha M^\bullet}{D})\theta^{\bullet T}[M - rl_L] + V^\bullet r - c]dt + (V^\bullet - \frac{\alpha M^\bullet}{D})\theta^{\bullet T}\Sigma dW \tag{5.97}$$
$$DV^\bullet > \alpha M^\bullet \qquad V^\bullet(0) = v$$

has a unique strong (non-negative) solution. Note that for any fund withdrawal rate $c(t)$ and any R^N-valued progressively measurable stochastic process $\rho(t)$ with

$$\theta^\bullet(t) = K(t)^{-1}\Sigma(t)\rho(t) \tag{5.98}$$

the above SFE has a unique strong solution

$$V_v^{\bullet c\theta^\bullet}(t) = \frac{1}{D(t)}\left[-\int_0^t D(s)c(s)ds + v\exp[(1-\alpha)\eta(t)][\alpha + (1-\alpha)\exp(\xi(t)-\eta(t))]\right] \tag{5.99}$$
$$\xi(t) = \int_0^t \rho(s)^T d\widetilde{B}(s) - \int_0^t \|\rho(s)\|^2 ds \qquad \eta(t) = \max_{0\le s\le t}\xi(s)$$

with the property

$$M_v^{\bullet c\theta^\bullet}(t) = \max_{0\le s\le t} D(s)V_v^{\bullet c\theta^\bullet}(s) = -\int_0^t D(s)c(s)ds + v\exp[(1-\alpha)\eta(t)] \tag{5.100}$$

[and that the optimal trading strategies for limited risk arbitrage investment management without drawdown control satisfy $\theta_{y_{,}}(t) = K(t)^{-1}\Sigma(t)B(t)^T\nabla V(t)$]. Furthermore, if we compare the stochastic functional equation related to drawdown control with the evolution equation

$$dV = [\theta^T[M - rl_L] + Vr - c]dt + \theta^T\Sigma dW, \ V(0) = v, \tag{5.101}$$

for the portfolio value process used so far, then with a fund withdrawal rate $c(t)$ and a trading strategy $\theta^\bullet(t)$ such that $(c, \theta^\bullet) \in \overline{A}_\alpha^\bullet(v)$ by applying the transformation

$$\theta(t) = (V_v^{\bullet c\theta^\bullet}(t) - \frac{\alpha M_v^{\bullet c\theta^\bullet}(t)}{D(t)})\theta^\bullet(t) \tag{5.102}$$

we find

$$V_v^{c\theta}(t) = V_v^{\bullet c\theta^\bullet}(t). \tag{5.103}$$

Since the risk/arbitrage constraint sets K_t in $\overline{\theta}(t)$ - strategy space are convex cones it is also clear that $(c, \theta) \in \overline{A}(v)$. If we therefore (using our equivalent notations for trading strategies) consider the convex cones

$$R_t = P_t^{-1}(K_t) \qquad P_t(\rho) = K(t)^{-1}\Sigma(t)\rho(t) \tag{5.104}$$

$[K_t = \{\cdots \ \overline{\theta}_{vc}(t) \ \cdots\}$ with $\overline{\theta}_{vc}(t) = \overline{\theta}_t^{vc}(v)$], then for a progressively measurable process $\rho_{vc}^\bullet(t)$ with $\rho_{vc}^\bullet(t) \in R_t$, $0 \le t \le H$, the stochastic processes

$$\overline{\theta}_{vc}^\bullet(t) = K(t)^{-1}\Sigma(t)\rho_{vc}^\bullet(t)$$
$$\overline{\theta}_{vc}(t) = (V_v^{\bullet c\overline{\theta}_{vc}^\bullet}(t) - \frac{\alpha M_v^{\bullet c\overline{\theta}_{vc}^\bullet}(t)}{D(t)})\overline{\theta}_{vc}^\bullet(t) \qquad \rho_{vc}(t) = (V_v^{\bullet c\overline{\theta}_{vc}^\bullet}(t) - \frac{\alpha M_v^{\bullet c\overline{\theta}_{vc}^\bullet}(t)}{D(t)})\rho_{vc}^\bullet(t) \tag{5.105}$$

have the properties

$$\overline{\theta}_{vc}^\bullet(t) \in K_t \qquad \overline{\theta}_{vc}(t) \in K_t \qquad \rho_{vc}(t) \in R_t$$
$$\overline{\theta}_{vc}(t) = K(t)^{-1}\Sigma(t)\rho_{vc}(t) \tag{5.106}$$
$$V_v^{c\overline{\theta}_{vc}}(t) = V_v^{\bullet c\overline{\theta}_{vc}^\bullet}(t)$$

and consequently

$$(c, \overline{\theta}_{vc}^{\bullet}) \in \overline{A}_{\alpha}^{\bullet}(v) \qquad (c, \overline{\theta}_{vc}) \in \overline{A}(v)$$

$$\overline{\theta}_{vc}^{\bullet}(t) = \frac{\overline{\theta}_{vc}(t)}{V_v^{c\overline{\theta}_{vc}}(t) - \dfrac{\alpha M_v^{c\overline{\theta}_{vc}}(t)}{D(t)}} \qquad (5.107)$$

[$V_v^{\bullet c\overline{\theta}_{vc}^{\bullet}}(t) \geq 0, \; 0 \leq t \leq H$]. We can thus now define the set $\overline{A}_{\alpha}(v) \subseteq \overline{A}(v)$ of feasible pairs $(c, \overline{\theta})$ for which $(c, \overline{\theta}_{vc}^{\bullet}) \in \overline{A}_{\alpha}^{\bullet}(v)$ holds with the DC transform

$$\overline{\theta}_{vc}^{\bullet}(t) = \frac{\overline{\theta}_{vc}(t)}{V_v^{c\overline{\theta}_{vc}}(t) - \dfrac{\alpha M_v^{c\overline{\theta}_{vc}}(t)}{D(t)}} \qquad (5.108)$$

of trading strategy $\theta(t)$ [given initial wealth $v > 0$ and fund withdrawal rate $c(t)$] and solve the limited risk arbitrage investment management and asset allocation problem with these admissible controls. If $(c_{y,v}^{\alpha}, \theta_{y,v}^{\alpha}) \in \overline{A}_{\alpha}(v)$ is a corresponding optimal solution, then the associated portfolio value process satisfies the drawdown constraint at level $1 - \alpha$. Note that the relevant constraint sets in $\overline{\theta}(t)$ - strategy space are

$$\begin{aligned} K_t^{\alpha} &= T_t^{\alpha}(K_t) \\ &= [T_{\alpha} \circ P]_t(R_t) \end{aligned} \qquad T_t^{\alpha}(\overline{\theta}_{vc}^{\bullet}) = (V_v^{\bullet c\overline{\theta}_{vc}}(t) - \frac{\alpha M_v^{\bullet c\overline{\theta}_{vc}^{\bullet}}(t)}{D(t)})\overline{\theta}_{vc}^{\bullet}(t) \qquad (5.109)$$

which can be seen to be closed (with non-empty interior) and convex subsets of the risk/arbitrage constraint cones K_t. This is the case because for fixed $\theta^{\bullet}(t)$ such that $\forall t: v^{\bullet}(t) \in N_t$ holds as initial wealth $v > 0$ and intertemporal fund consumption $c(t)$ vary [$(c, \theta^{\bullet}) \in \overline{A}_{\alpha}^{\bullet}(v)$] the term

$$\frac{M_v^{\bullet c\theta^{\bullet}}(t)}{D(t)V_v^{\bullet c\theta^{\bullet}}(t)} = \max_{0 \leq s \leq t} \frac{D(s)V_v^{\bullet c\theta^{\bullet}}(s)}{D(t)V_v^{\bullet c\theta^{\bullet}}(t)} \qquad (5.110)$$

takes on every value in the interval $[1, \dfrac{1}{\alpha})$. Consequently, the norms $\|\overline{\theta}_{vc}(t)\|$ of the corresponding trading strategies

$$\overline{\theta}_{vc}(t) = (1 - \frac{\alpha M_v^{\bullet c\theta^{\bullet}}(t)}{D(t)V_v^{\bullet c\theta^{\bullet}}(t)})\theta^{\bullet}(t) \qquad (5.111)$$

[$V_v^{\bullet c\theta^{\bullet}}(t) = V_v^{\bullet c\overline{\theta}_{vc}^{\bullet}}(t)$ and $M_v^{\bullet c\theta^{\bullet}}(t) = M_v^{\bullet c\overline{\theta}_{vc}^{\bullet}}(t)$] take on every value in the interval $(0, (1 - \alpha)\|\theta^{\bullet}(t)\|]$. Since the set $I_X(t)(N_t)$ of these trading strategies $\theta^{\bullet}(t)$ is closed (with non-empty interior) and convex the above conclusion follows.

Partial Observation. In a scenario where the information available to investors in the securities and derivatives market (i.e., the observed asset prices) only partially reflects the history of this market (Karatzas and Xue [66]) we consider a probability space (Ω, Φ, π) with two (augmented) filtrations $\{F_t\}_{t \geq 0}$ and $\{G_t\}_{t \geq 0}$ that satisfy the usual conditions. The exogenous sources $W(t)$ of market uncertainty are adapted to the filtration $\{G_t\}$ which therefore records the market history and the coefficients $r(t)$, $M(t)$ and $\Sigma(t)$ of the securities and derivatives market model are assumed to be progressively measurable with respect to $\{G_t\}$. The filtration $\{F_t\}$ is generated by the asset prices $X(t)$ and thus (in a

Markovian setting) records the market information available to investors. Fund withdrawal rates $c(t)$ and trading strategies $\theta(t)$ are assumed to be progressively measurable with respect to $\{F_t\}$ which means that the associated portfolio value processes $V_v^{c\theta}(t)$ are also $\{F_t\}$-adapted. This follows from the fact that with $\hat{M}(t) = E[M(t)|F_t]$ and a square root $\hat{\Sigma}(t,X)$ of the asset price covariance matrix $K(t,X) = \Sigma(t,X)\Sigma(t,X)^T$ the $\{F_t\}$-adapted stochastic processes

$$\hat{X}_i(t) = X_i(t) - X_i(0) - \int_0^t X_i(s)\hat{M}_i(s)ds \quad (5.112)$$

are continuous, square-integrable $\{F_t\}$-martingales that can be represented in the form

$$\hat{X}_i(t) = \int_0^t X_i(s)\hat{\Sigma}_{i.}(s,X(s))d\hat{W}(s) \quad (5.113)$$

with an $\{F_t\}$- Brownian motion $\hat{W}(t)$. The asset price processes are then

$$dX_i(t) = X_i(t)[\hat{M}_i(t)dt + \sum_{j=1}^{L} \hat{\Sigma}_{ij}(t,X(t))d\hat{W}_j(t)] \quad (5.114)$$

and the value process of an investor's securities and derivatives portfolio evolves according to the linear stochastic differential equation

$$dV(t) = [\theta(t)^T[\hat{M}(t) - r(t)1_L] + V(t)r(t) - c(t)]dt + \theta(t)^T\hat{\Sigma}(t,X(t))d\hat{W}(t). \quad (5.115)$$

After this reduction of the partial observations setting to a market model with complete observations [i.e., all relevant stochastic processes are adapted to the filtration $\{F_t\}$ representing the market information available to investors] we can apply a Girsanov transformation of probability measure

$$\hat{A}(t) = \hat{\Sigma}(t,X(t))^{-1}[\hat{M}(t) - r(t)1_L]$$

$$\hat{B}(t) = \exp(-\int_0^t \hat{A}(s)^T d\hat{W}(s) - \frac{1}{2}\int_0^t \|\hat{A}(s)\|^2 ds) \qquad E[\exp(\frac{1}{2}\int_0^t \|\hat{A}(s)\|^2 ds)] < \infty$$

$$\zeta(t) = \hat{B}(t)D(t) \tag{5.116}$$

$$\tilde{\psi}(\varphi) = E[\hat{B}(H)1_\varphi], \quad \varphi \in \Phi \qquad \tilde{B}(t) = \hat{W}(t) + \int_0^t \hat{A}(s)ds$$

and then solve the limited risk arbitrage investment management and asset allocation problem (with drawdown control) in a dynamically complete financial market.

Final Remark

With our papers *Risk/Arbitrage Strategies: A New Concept for Asset/Liability Management, Optimal Fund Design and Optimal Portfolio Selection in a Dynamic, Continuous-Time Framework Part I: Securities Markets* and *Part II: Securities and Derivatives Markets* we have outlined a very general and powerful, yet still implementable asset/liability management (ALM) model which focusses on the asset (corporate and investment banking) side. In the paper *Baseline for Exchange Rate - Risks of an International Reinsurer*, AFIR 1996, Vol. I, p. 395, the liability (reinsurance) side was given equal attention. However, to do so, a significant model extension (optimal control of Markov jump diffusion processes) was necessary. *We plan to present such a more sophisticated ALM framework in two forthcomming AFIR publications along the following lines*:

During the construction process that led to the optimal solutions of the above outlined ALM (stochastic control) problem for strictly limited risk investments in (highly geared) derivative financial products several assumptions about an investor's utility functions $U^c(t,c)$ and $U^V(V)$ had to be made, especially

$$R^c(t,c) = -\frac{c\dfrac{\partial^2 U^c(t,c)}{\partial c^2}}{\dfrac{\partial U^c(t,c)}{\partial c}} \le 1 \text{ and } R^V(V) = -\frac{V\dfrac{d^2 U^V(V)}{dV^2}}{\dfrac{dU^V(V)}{dV}} \le 1$$

for the associated coefficients of relative risk aversion. In a general dynamic programming framework [*planned future AFIR publication no. 1*] all these restrictions (beyond the standard differentiability and boundedness assumptions) on the investor's overall risk management objectives can be removed and furthermore efficient alternative numerical solution methods obtained: We consider a discrete-time Markov chain approximation $x_h(t)$ (derived with the explicit finite difference method in a viscosity solution setting) of the controlled continuous-time state dynamics

$$dx(t) = a(t,x(t),u(t))dt + b(t,x(t),u(t))dW(t)$$

$$a(t) = \begin{bmatrix} \theta(t)^T[M(t)-r(t)1_L] \\ +V(t)r(t)-c(t) \\ X_1(t)M_1(t) \\ \vdots \\ X_L(t)M_L(t) \end{bmatrix} \quad b(t) = \begin{bmatrix} \theta(t)^T\Sigma(t) \\ X_1(t)\Sigma_{11}(t) & \cdots & X_1(t)\Sigma_{1N}(t) \\ \vdots & & \vdots \\ X_L(t)\Sigma_{L1}(t) & \cdots & X_L(t)\Sigma_{LN}(t) \end{bmatrix}$$

[where $x(t) = (V(t),X(t))$, $u(t) = (c(t),\theta(t))$ are the controls and the coefficients $a(t,x,u) \in R^{L+1}$ and $b(t,x,u) \in R^{(L+1)\times N}$ satisfy the usual conditions that guarantee a unique strong solution of the associated evolution equation with bounded absolute moments] with one step transition probabilities

$$\pi_t^u(x, x\pm\delta e_1) = \frac{h}{2\delta^2}\left[c_{11}(t,x,u) - \sum_{\substack{k=0 \\ k\ne 1}}^{L}|c_{k1}(t,x,u)| + 2\delta a_1^\pm(t,x,u)\right]$$

$$\pi_t^u(x, x+\delta e_k \pm\delta e_1) = \frac{h}{2\delta^2}\cdot c_{k1}^\pm(t,x,u) \quad \pi_t^u(x, x-\delta e_k \pm\delta e_1) = \frac{h}{2\delta^2}c_{k1}^\mp(t,x,u) \quad [k\ne 1]$$

$$\pi_t^u(x,x) = 1 - \frac{h}{\delta^2}\sum_{l=0}^{L}\left[c_{11}(t,x,u) - \frac{1}{2}\sum_{\substack{k=0 \\ k\ne l}}^{L}|c_{kl}(t,x,u)| + \delta|a_1(t,x,u)|\right]$$

[where $c(t,x,u) = b(t,x,u)b(t,x,u)^T$ is the diffusion process covariance matrix, $e_0,..,e_L$ is the standard basis in R^{L+1} and $\pi_t^u(x,y) = 0$ for all other grid points y on the associated lattice structure $X_h \subseteq R^{L+1}$]. The discrete-time HJB dynamic programming equation is then

$$V_h(t,x) = \max_{u\in A_{tx}}\left[\sum_{y\in X_h}\pi_t^{u(t)}(x,y)V_h(t+h,y) + hL(t,x,u(t))\right]$$

with boundary condition $V_h(H,x) = \psi(x)$ [where

$$L(t,x(t),u(t)) = U^c(t,c(t)) \text{ and } \psi(x(H)) = U^V(V(H))$$

are the investor's overall risk management objectives and $\overline{A}_{tx}$ is the set of all feasible controls $u(s)$ on the time interval $[t, H]$ when the time t state is x] and an associated optimal Markov control policy $\overline{u}_h(t, x)$ maximizes the expression

$$\sum_{y \in X_h} \pi_t^{u(t)}(x, y) V_h(t + h, y) + hL(t, x, u(t))$$

in $\overline{A}_{tx}$ [backwards in time from $H - h$ to 0]. Furthermore, we have the uniform convergence

$$\lim_{\substack{(\tau, \xi) \to (t, x) \\ h \downarrow 0}} V_h(\tau, \xi) = V(t, x)$$

[where

$$J(t, x, u) = E_{tx}[\int_t^H L(s, x(s), u(s)) ds + \psi(x(H))]$$

$$V(t, x) = \sup_{u \in A_{tx}} J(t, x, u)$$

is the corresponding continuous-time expected utility maximization criterion] of the discrete-time Markov chain control problem to the continuous-time diffusion process control problem. If we now however instead of requiring the securities and derivatives investment management strategies $\theta(t)$ to be of the risk/arbitrage (without drawdown control) type, i.e.,

$$\overline{\theta}(t) \in K_t, \ 0 \le t \le H,$$

for an arbitrarily given trading strategy $\theta(t)$ consider the stochastic evolution of the corresponding portfolio value and sensitivities

$$\left(V_v^\theta(t), \Delta_v^\theta(t), \Gamma_v^\theta(t), \Theta_v^\theta(t), \Lambda_v^\theta(t)\right)$$

over the investment horizon $[0, H]$, then the limited risk arbitrage objectives can periodically be enforced at a certain cost by using impulsive controls that keep the portfolio value and sensitivities within a specified tolerance band

$$\overline{G} = (0, \delta, \gamma, \vartheta, \lambda)$$

[*planned future AFIR publication no. 2*]: With a discrete-time Markov chain approximation

$$x_h(t) = x + \begin{bmatrix} \int_0^t \overline{a}(s, x_h(s), u_h(s)) ds + \\ \int_0^t b(s, x_h(s)) dW_h(s) \end{bmatrix} + J_h(t) + F_h(t) + \widetilde{F}_h(t) + z_h(t) + \widetilde{z}_h(t) + \varepsilon_4^h(t)$$

$$E[\sup_{0 \le t \le H} \|\widetilde{F}_h(t)\|^2] \xrightarrow{h \downarrow 0} 0 \quad E[\sup_{0 \le t \le H} \|\widetilde{z}_h(t)\|^2] \xrightarrow{h \downarrow 0} 0$$

$$E[\sup_{0 \le t \le H} \|\varepsilon_4^h(t)\|] \xrightarrow{h \downarrow 0} 0$$

[that is locally consistent with the controlled continuous-time reflected jump diffusion state dynamics

$$x(t) = \left(X(t), V_v^\theta(t), \Delta_v^\theta(t), \Gamma_v^\theta(t), \Theta_v^\theta(t), \Lambda_v^\theta(t)\right)$$

where $X(t)$ is the price process of the traded assets - bonds, stocks and options - spanning the securities and derivatives market] an approximation $V_h(t, x)$ of the value function

$$J(t, x, m, f) = E_{tx} \begin{bmatrix} \int_t^H \int_U L(s, x(s), u) m_s(du) ds + \\ \int_t^H M(s, x(s))^T df(s) + \\ \int_t^H N(s, x(s))^T dz(s) \end{bmatrix} \quad V(t, x) = \inf_{AS} J(t, x, m, f)$$

[where the continuous reflection part of the bounded and continuous total risk exposure control costs (L, M, N) satisfies $N(t, y)^T r \geq 0$, $r \in R(y)$, on ∂G and the infimum is taken over all admissible (relaxed/singular) control systems] of such a singular (or impulse) control approach

$$x(t) = x + \begin{bmatrix} \int_0^t \int_U \overline{a}(s, x(s), u) m_s(du) ds + \\ \int_0^t b(s, x(s)) dW(s) \end{bmatrix} + J(t) + F(t) + z(t)$$

$$J(t) = \int_0^t \int_{R^a} q(s, x(s-), y) N(ds\,dy) \quad x + q(t, x, y) \in \overline{G}, \ x \in \overline{G}$$

$$F(t) = \sum_{i=1}^l r_i f_i(t) \quad r_i \in R(y_i), \ y_i \in \partial G \quad f_i(t) \geq 0, \ df_i(t) \geq 0$$

$$z(t) = \int_0^t r(s) d|z|(s), \ r(s) \in R(x(s)) \ [\mu_z(ds) \ \text{a.e.}] \quad |z|(t) = \int_0^t 1_{x(s) \in \partial G} d|z|(s)$$

[which is based on a Lipschitz continuous solution mapping in the Skorokhod problem for (G, R) and under standard conditions on the diffusion coefficients and the usual compact control space has a unique strong solution $x(t) \in \overline{G}$, $0 \leq t \leq H$, for every $x \in G$] to limited risk arbitrage investment management satisfies the discrete-time HJB dynamic programming equation

$$V_h(t, x) = \min \left\{ \begin{array}{c} \min_{u \in U} \begin{bmatrix} (1 - \lambda \Delta t_{txu}^h - \delta_{txu}^h) \sum_{y \in X_h} \pi_{txu}^h(y|D) V_h(t + \Delta t_{txu}^h, y) + \\ (\lambda \Delta t_{txu}^h + \delta_{txu}^h) \int V_h(t + \Delta t_{txu}^h, x + q_h(t, x, y)) \Pi(dy) + \\ \Delta t_{txu}^h L(t, x, u) \end{bmatrix}, \\ \min_{1 \leq i \leq l} \begin{bmatrix} \sum_{y \in X_h} \pi_{tx}^h(y|r_i^h) V_h(t, y) + h M_i(t, x) \end{bmatrix} \end{array} \right\}$$

$$V_h(t, x) = \sum_{y \in X_h} \pi_{tx}^h(y|R) V_h(t, y) + N(t, x)^T \Delta z_h(t, x), \ x \in \partial G_h$$

$$V_h(H, x) = 0.$$

By solving this equation backwards in time from $t_{K_h-1}^h$ [$t_{K_h}^h = H$] to 0 we can determine an optimal Markov control policy $\overline{u}_h(t, x)$ [which we denote by $\overline{m}_{tx}^h(du)$ in relaxed control notation], the corresponding (optimal) intertemporal singular control impulses $\Delta \overline{f}_h(t, x)$ and the necessary reflection impulses $\Delta \overline{z}_h(t, x)$ at the boundary ∂G_h (where $\overline{G}_h$ is a corresponding discretization of the given risk/arbitrage tolerance band $\overline{G}$), i.e., an optimal discrete-time impulsive risk exposure control strategy $(\overline{u}_h = \overline{m}_h, \overline{f}_h, \overline{z}_h)$ and the associated state evolution $\overline{x}_h(t) \in \overline{G}_h$, $0 \leq t \leq H$. A weak convergence argument [with an embedded time scale adjustment to ensure tightness of the singular control and boundary reflection parts of the approximating discrete-time state dynamics $x_h(t)$] then also establishes an optimal solution of the initially given continuous-time impulsive limited risk arbitrage investment management problem, i.e., a continuous-time optimal impulsive risk exposure control strategy $(\overline{m}, \overline{f}, \overline{z})$ and the associated state evolution $\overline{x}(t) \in \overline{G}$, $0 \leq t \leq H$. The above outlined HJB dynamic programming framework (based on the Markov chain approximation technique) also turns out to be a very convenient setting for a full extension of the concept of

limited risk arbitrage investment management to a financial market with discontinuous securities and derivatives price processes. Such a generalization and extension then implements an optimal combination of continuous risk/arbitrage control (i.e., expected utility maximization under given risk exposure constraints) and impulsive risk exposure control (i.e., minimization of the exposure control costs associated with a given risky trading or portfolio management strategy).

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