FINANCIAL RISK AND THE MARKOWITZ
AND BLACK-SCHOLES WORLDS

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ABSTRACT

The actuarial profession world-wide is at an important cross-roads. Some of the risk management tools that led to our supremacy in the fields of life assurance and general insurance are of only limited relevance in other areas of finance, and in recent years the methodologies of modern finance theory have been seen, even by many actuaries, as superior to the traditional actuarial approach in providing a framework for expert and relevant solutions to financial problems involving future uncertainty and risk. The choice now facing the profession is whether or not to abandon the conceptual approach that led to our past success and replace it with a more mathematical approach based on utility theory, mean-variance analysis, stochastic calculus, and similar methodologies. This paper investigates the conceptual foundations of these modern methodologies, with particular emphasis on the hypothetical Markowitz and Black-Scholes worlds within which influential theories of portfolio selection and option pricing have been formulated. The main conclusion of the paper is that modern methodologies are fundamentally unsound as regards frameworks for the recognition, measurement and control of risk in the real financial world. It is suggested that the actuarial profession should as a matter of urgency develop a new and more realistic theory of financial risk that is consistent with traditional actuarial principles, and the outline of a possible new framework is described.

KEYWORDS

1 INTRODUCTION

1.1 The actuarial profession world-wide is at an important cross-roads. Some of the risk management tools that led to our supremacy in the fields of life assurance and general assurance are of only limited relevance in other areas of finance, and in recent years the methodologies of modern finance theory have been seen, even by many actuaries, as superior to the traditional actuarial approach in providing a framework for expert and relevant solutions to financial problems involving future uncertainty and risk. The choice now facing the profession is whether or not to abandon the conceptual approach that led to our past success and replace it with a more mathematical approach based on utility theory, mean-variance analysis, stochastic calculus, and similar methodologies.

1.2 Although a formal approach to the management of risk in general insurance can be stated very succinctly as "maximise profits subject to the probability of ruin not exceeding some very small specified value", there is no obvious framework for risk management in the area of life assurance. Actuaries, unlike financial economists, do not yet have a formalised and unified framework for financial risk that can be tailored to meet the specific requirements of particular application areas such as investment management and asset/liability modelling.

1.3 The methodologies of modern finance theory seem to permit rigorous quantification, starting from a supposedly rational set of axiomatic principles, in a manner that might appear impossible within traditional actuarial approaches, and it is the lure of this quantification and rationalism that leads some to believe that modern finance theory offers the more scientific framework. Quantification, however, is often possible only at the expense of having to make numerous simplifying assumptions, many of which are seen by those building the theories (but perhaps not by those who then teach these theories) as being unrealistic in the extreme. Also, the rationalism approach has been fiercely criticised by many eminent economists. In particular, Hayek (1988) explains how the real promoters of the rationalism approach are not the distinguished scientists (who "have better things to do") that may have investigated and developed the underlying concepts:

> "They rather tend to be the so-called 'intellectuals' that I have elsewhere unkindly called professional 'second-hand dealers in ideas': teachers, journalists and 'media representatives' who, having absorbed rumours in the corridors of science, appoint themselves as representatives of modern thought, as persons superior in knowledge and moral virtue to any who retain a high regard for traditional values, as persons whose very duty it is to offer new ideas to the public - and who must, in order to make their wares seem novel, deride whatever is conventional. For such people, due to the positions in which they find themselves, 'newness' or 'news' and not truth, becomes the main value."

1.4 The present author has endeavoured to apply in Clarkson (1981, 1997) the traditional actuarial approach to the management of equity portfolios and to option pricing. In both cases, however, it was necessary to explain at some length why the theoretical

constructs of modern finance, namely the Markowitz and Black-Scholes worlds, appear to be seriously deficient as conceptual frameworks within which to formulate strategies for prudent financial management. In each case, the question of "truth" relates crucially to the appropriateness or otherwise of the underlying framework for the recognition, measurement and control of financial risk. This paper aims to initiate a debate as to whether the risk methodologies of modern finance theory are adequate for the purposes of the actuarial profession in the next century or whether some new framework, possibly along the lines first sketched out in Clarkson (1989) using parallels with physical risk in sports, should be developed instead. The main conclusion of the paper is that modern methodologies are fundamentally unsound as regards frameworks for the recognition, measurement and control of risk in the real financial world. It is suggested that the actuarial profession should as a matter of urgency develop a new and more realistic theory of financial risk that is consistent with traditional actuarial principles, and the outline of a possible new framework is described.

2   PARALLELS WITH ASTRONOMY AND MATHEMATICAL LOGIC

2.1   *Ptolemaic Astronomy*

Although Ptolemaic astronomy was based on the erroneous causal mechanism of all heavenly bodies rotating around the Earth, it provided for some two thousand years a very useful mathematical framework for important applications such as navigation. However, more and more arbitrary adjustments had to be made to explain anomalies that emerged as measurements increased steadily in accuracy, until the system was overly cumbersome.   Similarly, in modern finance theory, more and more complexities such as ARCH effects and "jumps" have been introduced in attempts to rectify some of the obvious inconsistencies of the initial simple theories.

2.2   Although Aristarchos of Samos had suggested a heliocentric system in the third century BC, it was only when Copernicus pointed out nearly two thousand years later that if all planets move around the sun and each planet spins about its axis, then those simple mechanisms can explain in general terms all observed behaviour within the solar system.   Similarly, an approach to risk roughly along the lines first set out in Clarkson (1989, 1990) appears to give a better framework for the real financial world than the standard theories such as utility theory.

2.3   The heliocentric system based on Kepler's three laws of planetary motion seemed, even to proponents of the new approach, to be arbitrary in some respects and lacking in cohesion. The real breakthough only came when Newton introduced his universal law of gravitation, which immediately justified existing results and paved the way for pioneering practical work such as Halley's discovery of his periodic comet. A parallel concept introduced in this paper is the risk weighting function described in Section 6.2 as the unifying feature of human behaviour under conditions of uncertainty and risk.

2.4   The acute controversy over the "dangerous" new ideas propounded by Copernicus and Galileo show that the establishment will vehemently oppose any new system that

threatens existing self-interests. Similar controversy has already begun within the U.K. actuarial profession in discussions on papers such as Clarkson (1996, 1997).

2.5   Some may argue that "modern" mathematical methods, where the underlying logic is apparently beyond question, are far superior to "traditional" approaches where success is dependent on the experience and judgement of the practitioner. Again the comments in Hayek (1988) are relevant; those who take mathematical methods "off the shelf" and apply them unthinkingly to fields quite different from those in which they were first developed will be oblivious to the "truth" or otherwise of these methodologies in the attempted applications. Furthermore, history shows that genuine progress in mathematical approaches may take many decades if not centuries. The Ancient Greeks, for instance, made massive strides in two-dimensional geometry and the theory of rational numbers, but they did not have access to the very powerful methodology of the differential and integral calculus, which was only "discovered" some two thousand years later.

2.6   A classic example of the purely temporary success of the "rationalism" approach was the rigorous quantification of mathematical logic in Frege (1879). As Van Heijenoort (1970) observes, "Frege's booklet brought to the world the theory of quantification and thus opened up a new epoch in the history of logic". Many eminent mathematicians, and in particular Von Neumann, followed in the footsteps of Frege and set up comprehensive formalised frameworks for reducing mathematical proofs to a few axioms and rules of inference. It might then have seemed reasonable to conjecture that the truth or otherwise of any mathematical results could be established by this rationalist approach. Much of the modern theory of finance is based on a similar rationalist approach using, as two key axiomatic frameworks, utility theory and stochastic calculus.

2.7   The illusion of invincibility as regards the rationalism approach to mathematical logic was shattered in 1931 by Gödel's famous incompleteness paper in which, by careful analysis of the statement "This assertion cannot be proved", he showed that in any formal axiomatic system there are relatively simple problems involving elementary arithmetic that cannot be proved to be either true or false on the basis of the axioms. Similarly, it is my strongly held belief that in the modern theory of finance the "equilibrium", "rational behaviour" and "risk equals variance of return" paradigms are not only seriously incomplete as a description of the real financial world but also that it is in precisely those instances where these paradigms break down that the highest inherent levels of financial risk arise. Should this indeed be the case, the teachings of modern finance theory would be dangerously unsound as a framework for prudent financial behaviour.

2.8   Proponents of modern finance theory often dismiss any serious criticism of its conceptual foundations by suggesting that, if a better theory existed, then it would have been discovered long ago. Developments in the realm of mathematical logic suggest otherwise, in that it took 52 years for the apparent "truth" of the axiomatic approach pioneered by Frege to be invalidated by the work of Gödel. A similar period of 52 years from the initial quantification of financial risk as the variance of

return in Markowitz (1952) takes us to 2004, suggesting that the scientific clock may now be right for a new leap forward in our understanding of financial risk.

3 PHYSICAL RISK IN SPORTS

3.1 *The Equivalent Probability of Death*

3.1.1 Since an adverse occurrence (such as a capsize in rapid river canoeing or an avalanche in off-piste skiing) can result in quite different undesirable outcomes from minor injury to death, we require to define an equivalence measure for the undesirability of outcomes before we can construct an additive measure of physical risk.

3.1.2 An important actuarial analogy is compound interest, where "present value" defined as the discounted value at the appropriate rate of interest provides an intuitively obvious measure that possesses the crucial property of additivity. The dimension of time is thereby taken into account through the equivalence measure of present value. In this case, the discount factor tends to 1 as the future time period tends to zero.

3.1.3 Suppose now that we classify undesirable outcomes from physical risk into the following degrees of severity:

Severity	Description
1	Minor injury
2	Moderate injury
3	Severe injury
4	Very severe injury
5	Permanent incapacity
6	Death

Then by analogy with compound interest, we can obtain a plausible numerical scale by giving a value of 1 to death and giving values of a, a^2, a^3, a^4 and a^5 (where a is positive and less than 1) respectively to permanent incapacity, very severe injury, severe injury, moderate injury and minor injury.

3.1.4 By summing over all possible scenarios the risk elements consisting of the product of the "outcome undesirability" and the associated probability we obtain as the basic measure of physical risk the "equivalent probability of death". Additivity is made possible by incorporating the dimension of "outcome undesirability" in terms of an equivalence measure identical in concept to that used in compound interest.

3.2 *Estimated Risk Values*

3.2.1 In Clarkson (1989) this general approach is used to estimate, for typical daily participation, the physical risk of certain potentially dangerous sports. In the case of ski mountaineering, for instance, two quite distinct risks are considered - the risk of death or injury from a bad fall on unpisted snow, and the risk of death or injury from an avalanche.

3.2.2   Given that an "adverse occurrence" (such as an avalanche) arises, the resulting outcome in terms of death or severity of injury will depend on many factors, including the skill and calmness of the individual in the face of acute danger, the type and quality of safety equipment (such as avalanche transmitters), and sheer luck. A Poisson distribution with appropriate parameter gives a convenient statistical description of this range of outcomes.

3.2.3   The values of the various parameters required were estimated from a combination of personal experience, often over a long period of years, and observation of others, and were intended to reflect an "intermediate" level of technical competence in each sport.

3.2.4   The resulting values of risk, calculated on the basis of the daily equivalent probability of death multiplied by one million, are as below:

Sport	Risk
Wind surfing	0.5
Skiing	3.1
Rapid river canoeing	13.8
Ski mountaineering	106.4
Hang gliding	142.6

3.3   *Observed Risk Values*

3.3.1   Some comparable observed rates of physical risk were subsequently published in the 1992 report of a Royal Society study group. Using a very similar definition of risk, namely the number of deaths per million days' participation, the relevant values are as below:

Sport	Risk	
Skiing	{3.5	USA, 1967-68
	{6.5	France, 1974-76
Canoeing	2.5	UK, 1960-62
Hang gliding	{100-325	USA, 1978
	{375	UK, 1977-79

3.2.2   Given that rapid river canoeing is obviously much more dangerous than canoeing mainly on flat water, these empirical values agree surprisingly well with the estimated values, thereby confirming that the new framework described above provides a powerful method for estimating risk values.

3.4   *A Personal Threshold of Maximum Risk*

3.4.1   In assessing whether a particular activity is acceptable in terms of the attendant risk, an individual will compare the perceived risk with some threshold of risk based on his personal preferences and experience. A risk value of around 5, which corresponds to one chance in a thousand of serious injury or worse for typical daily participation, appears to be a plausible maximum risk threshold for many people who have little or

no first hand experience of potentially dangerous sports. This threshold would accommodate skiing, whereas rapid river canoeing would be perceived as too risky.

3.4.2   A crucial feature of this personal threshold is that it will increase over time with increasing levels of technical competence and practical experience. For example, only the most foolhardy novice skiers would attempt very steep and icy slopes, whereas experienced skiers would derive great personal satisfaction from tackling these challenging descents. Since, after many years' experience of piste skiing, I now enjoy ski mountaineering but regard hang gliding as too risky, my personal threshold must now be of the order of 125, which is very considerably higher than the suggested starting point of around 5 for individuals with no first hand experience of potentially dangerous sports.

3.5   *A Paradox in Probability*

3.5.1   Suppose now that I am offered the choice between a 1 in 2 chance of a day's ski mountaineering and a 1 in 3 chance of a day's hang gliding. I have no hesitation whatsoever in rejecting the latter alternative on risk grounds and opting instead for the former.

3.5.2   On a look-through basis (and taking the risk values as 100 and 150 respectively per day to simplify the arithmetic), a 1 in 2 chance of ski mountaineering has exactly the same risk value as a 1 in 3 chance of hang gliding. If we denote by $R[(x)(y)]$ the psychological impact of the perceived risk of participating with probability x in a sport where the daily equivalent probability of death is y, then $R[(ab)(c)]$ is totally different from $R[(a)(bc)]$, where in this case $a = 1/3$, $b = 3/2$, and $c = 10^{-4}$.

3.5.3   This contrasts vividly with the situation in the branches of pure and applied mathematics relevant to models of the physical world. For real and complex numbers, the axiom of associatively gives:

$$(ab)c = a(bc) = abc,$$

and similarly for the multiplication of compatible matrices we have:

$$(AB)C = A(BC) = ABC.$$

3.5.4   I now have the choice between ski mountaineering with certainty and hang gliding with probability 2/3. Again I have no hesitation in choosing the former on the grounds that I would not contemplate any meaningful probability of participating in a sport that I regarded as too risky. If we put $a = 1$ and write $R[(1)(y)]$ simply as $R[(y)]$, then in functional terms we have:

$$R[(x)(y)] \neq R[(xy)].$$

In other words, the seemingly obvious look-through property does not hold for probabilities in these high risk situations.

3.6   *A Binary Neural Mechanism*

3.6.1   Far from indicating a fundamental flaw in the workings of the human mind, this apparent breakdown in the axioms of probability can be seen as an exceptionally effective neural mechanism for human choice under conditions of uncertainty and risk. In an essentially subconscious pattern recognition manner, the brain processes within a very short period of time the available data (which may be very scanty) and other information relevant to the situation to arrive at a yes/no decision as to whether or not the level of risk is acceptable in the context of the personal threshold of maximum risk.

3.6.2   Crucial components in this assessment process are the completeness of the background data and the competence at that particular instant of the individual to cope with the consequences of an adverse occurrence. For instance, someone at the "beginner" level of technical competence may have only a very limited knowledge of the background data, and it will obviously be unsound in the extreme to base his decision in a slavishly computational manner on a combination of incomplete historic data and subjective estimates of the present and future that are almost certainly erroneous. Also, a particular situation may be either within or outwith the risk tolerance appropriate to his current level of technical competence. For example, an experienced canoeist who would be perfectly happy to tackle a long and very difficult set of rapids in the company of other experienced canoeists when wearing a life-jacket and safety helmet would not consider tackling these rapids either alone or when not wearing the usual safety gear.

3.6.3   Another way in which the dimension of time enters is the degree of difficulty of a particular challenge, since guide books can only give benchmark gradings of difficulty in "normal" conditions. For example, a fairly straightforward set of rapids in a deep gorge may represent a serious threat to life when the river is in spate.

3.6.4   This binary neural mechanism is consistent with the compelling evidence in Penrose (1989) to the effect that the human mind does not, as had previously been generally thought, act in a slavishly computational manner along the lines of some very powerful digital computer. As discussed above, most decisions involving physical risk involve so many degrees of freedom in areas where statistical data is not available that a purely numerical approach is not only theoretically impracticable but also completely outwith the ability of the human mind to process within a reasonable timescale.

3.7   *A General Pattern of Indifference Curves*

3.7.1   Suppose that we plot in a chart the risk value R against a measure of "satisfaction" S, where this latter measure relates to the strength of the feeling of satisfaction derived from accomplishing some challenge involving physical risk. For values of risk above the personal threshold of maximum risk there is no meaningful trade-off between an increase in risk and an increase in satisfaction, and accordingly this region of the chart can be characterised by risk-only indifference curves which are *horizontal straight lines*.

3.7.2   Practical experience of different sports suggests that, for a given value of satisfaction, the "enjoyment" level while actually participating in a sporting challenge decreases as the level of risk increases, and in particular decreases very rapidly as the risk value increases towards the personal threshold. Also, for very low levels of physical risk (as, for instance, in the choice between making a journey by air or by car) risk is generally disregarded in the choice between the alternatives. Accordingly, for values of risk below the personal threshold, we can postulate "equal enjoyment" indifference curves which are asymptotic to vertical straight lines as the risk tends to zero, decrease in gradient as risk increases, and are asymptotic to the personal threshold. It must be stressed that this general pattern of indifference curves is a snapshot relevant only to a given individual at a given instant in time, and that in general the risk inherent in a particular challenge will decrease over time as the technical competence of the individual increases.

3.8   *Parallels with Financial Risk*

3.8.1   Since the perception of risk is quite literally "all in the mind", it would be very surprising if the psychological distress associated with highly adverse financial consequences of either the individual or corporate level gave rise to different neural responses from the psychological distress associated with the highly undesirable consequences of injury or death that result from situations involving physical risk. Indeed, there are strong parallels between the binary neural response described above in the context of physical risk and the behaviour of an individual who at present regards stockmarket investment as "too risky" for him. Such an individual will typically not invest even a small amount of his assets in such investments. Also, later in life, perhaps after obtaining first hand experience of financial markets through his professional training or employment, he may no longer regard stockmarket investment as inappropriate on risk grounds. There is very strong evidence throughout the financial and actuarial literature in support of this psychological equivalence between financial risk and the basic features of physical risk as described above, and three such parallels are now described.

3.8.2   In the highly perceptive Chapter 12 of his "General Theory", Keynes (1936) suggests that "lack of confidence" will result in no investment being made, no matter how high the expected outcome. Since "lack of confidence" is clearly equivalent to the existence of significant uncertainty and hence risk, the behaviour described by Keynes corresponds precisely to that implied by the binary neural mechanism described above.

3.8.3   In the discussion on two investment papers in 1953, the then President of the Faculty of Actuaries, R.L. Gwilt, drew attention to the dangers of assessing risk simply in terms of subjective estimates as to the future:

> "If you will forgive a brief presidential platitude, I think it is wise in the investment of the assets of a life office to resist any temptation to strive for a spectacular profit if there is any possibility of a serious loss, if one's assumptions should prove to be wrong".

The crucial point, as discussed in Section 3.6.2, is that any assessment of risk which ignores the likely errors in subjective estimates, whether single point forecasts or probability distributions, may be dangerously unsound.

3.8.4 Tversky (1990) questions the fundamental assumption that individuals are "risk averse" in the generally accepted sense and cites a simple example to illustrate his point. An investor has the choice between a certain loss of £85,000 or an 85% chance of losing £100,000 and a 15% chance of losing nothing. In this situation Tversky suggests that most people will "gamble" and choose the latter, despite the significant chance of further loss. Such behaviour is inconsistent with the assumption of economic agents being "risk-averse". Consider now the new approach to risk described above. Taking both £85,000 and £100,000 losses as leading to financial ruin, the "equivalent probability of financial ruin" (the measure of financial risk in the new approach) is 1 under the first alternative but only 0.85 under the second, and accordingly the latter alternative will be chosen, precisely as Tversky suggests.

4 UTILITY THEORY AND THE VON NEUMANN AND MORGENSTERN AXIOMS

4.1 *Bernoulli (1738)*

4.1.1 The origins of what we now know as utility theory can be traced back to Daniel Bernoulli's seminal 1738 paper. He rejects the previous "expectations" approach to financial risk, in which no characteristic of the individual is taken into account, by citing the following counterexample:

> "Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats?"

Bernoulli suggests that the poor man would be ill-advised not to sell it for nine thousand ducats, while the rich man would be ill-advised to refuse to buy it for nine thousand ducats, and concludes that the previous paradigm, in which all men use the same rule to evaluate the risky situation, must be discarded.

4.1.2 Bernoulli then generalises the argument to say that the value of an item must not be based on its price, but rather on the "utility" it yields. The price of an item is dependent only on the thing itself and is equal for everyone, whereas the "utility" is dependent on the particular circumstances of the individual making the assessment.

4.1.3 By making a number of very sweeping simplifying assumptions, such as a strong symmetry in the nature of preferences above and below the point of current wealth, Bernoulli uses his "moral expectation" approach (what today we would call a logarithmic utility function) to "solve" the St Petersburg Paradox.

4.1.4 This strong symmetry of preferences is inconsistent with the general pattern of indifference curves described in Section 3.7.2. Also, Keynes (1921), after discussing the very strong elements of psychological doubt that arise from the totally unrealistic

nature of the financial background to the St Petersburg Paradox, observes that "the theoretical dispersal of what element of paradox remains must be brought about, I think, by a development of the theory of risk".

4.2   *Von Neumann & Morgenstern (1944)*

4.2.1   Bernoulli's entirely new hypothesis, namely that "no valid measurement of a risk can be obtained without consideration being given to ... the utility ... to the individual", remained essentially dormant until it was supposedly formulated in precise mathematical terms by Von Neumann & Morgenstern (1944). The great enthusiasm with which this highly mathematical approach was received can be seen from the following two review comments from highly respected quarters:

> "Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science - the science of economics. The foundation which they have laid is extremely promising."
> The Bulletin of the American Mathematical Society

> "The main achievement of the book lies, more than in its concrete results, in its having introduced into economics the tools of modern logic and in using them with an astonishing power of generalisation."
> The Journal of Finance

4.2.2   The mathematical approach draws heavily on parallels with "natural" operations in the physical sciences such as obtaining the centre of gravity of a body as a weighted average. An individual's preferences for events C, A and B (ranked in that order) are introduced by postulating the existence of $\propto$, a real number between 0 and 1, such that event A is exactly as desirable as the compound event consisting of event B with probability 1 - $\propto$ and event C with probability $\propto$. The most fundamental result (which is very far from obvious in that its proof requires ten pages of highly complex mathematics) is that, if a measure of utility satisfying their axioms exists, it is unique up to a linear transformation.

4.2.3   The practical decision rule that emerges is to maximise "expected utility" on a probability-weighted basis. If, for example, the utilities attached to outcomes of 10, 20 and 30 are 0, 0.6 and 1 respectively, a certain outcome of 20 has a utility of 0.6 and is therefore preferred to a 50:50 chance of either 10 or 30, which has a utility of 0.5.

4.2.4   It is important to note that this highly mathematical approach (much of which was developed by Von Neumann in the 1920s before Gödel demonstrated the incompleteness of all axiomatic systems dependent on mathematical logic) takes no explicit account of risk. Also, it is stated early in the book that "we have practically defined numerical utility as being that thing for which the calculus of mathematical expectations if legitimate". This observation has two profound implications as regards the real world relevance of the Von Neumann & Morgenstern approach. First, rather than building on Bernoulli's pioneering work more than two centuries earlier, it essentially "turns the clock back" to the pre-Bernoulli expectations paradigm.

Second, it cannot cope with the much more subtle system of psychology in high risk situations as described in Section 3.

4.3   *Markowitz (1952, 1959)*

4.3.1   Let us suppose that for all individual securities in a particular set the expected return and variance of return, together with the covariances of return between each pair of securities, are known exactly, then for any portfolio selected from these individual securities it is theoretically possible to calculate the expected return E and the variance of return V. Markowitz (1952) observes that, on these assumptions, the utility U(E,V) of the portfolio increases as E increases for given V and decreases as V increases for given E. This gives rise to the concept of the "efficient frontier", namely the locus of points in the E-V diagram with maximum E for given V and minimum V for given E. Markowitz then suggests that an investor should select that portfolio on the efficient frontier which best meets his specific risk/return preferences.

4.3.2   In Markowitz (1959) there is a detailed discussion of which measure of risk is most appropriate. The conclusion is that, while semi-variance of return (i.e. the probability-weighted sum of the squared deviations below the mean) is the more plausible on common sense grounds, variance of return is preferable for two reasons - it is much more familiar to mathematicians and statisticians, and it reduces the computational workload considerably.

4.3.3   Markowitz (1959) also contains a detailed discussion of whether, despite the criticisms of Allais and other eminent economists, utility theory as formulated in terms of the Von Neumann & Morgenstern axioms provides a satisfactory framework for "rational" behaviour.   Markowitz concludes that utility theory is indeed appropriate, but warns of the limitations of his approach:

> "The study of rational behaviour has produced only general principles to be kept in mind as guides. Even the significance of some of these principles is subject to controversy. The value of the study of rational behaviour is that it supplies us with a new viewpoint on problems of criteria - a viewpoint to be added to common sense to serve as a basis of good judgement."

4.3.4   There are two potentially significant aspects of financial risk that are completely ignored in the Markowitz mean-variance framework.   First, although Markowitz accepts that the behaviour of apparently "reasonable" men is often inconsistent with the expected utility maxim, no mention is made of the "financial collisions" that might occur if a large number of individuals do not act "rationally".   A parallel involving physical risk is that driving on the left and driving on the right are both eminently satisfactory conventions, provided that everyone in a particular country follows the convention of that country.   Second, in the Markowitz world risk is a function of an investor's subjective estimates of probability distributions of future returns with no allowance for the possibility that these estimates may be grossly inaccurate.   Accordingly, the "true" risk could be considerably higher than that implied by the Markowitz approach.

4.4 *Comparisons of Axioms*

4.4.1 In Clarkson (1989) the new framework for financial risk developed using parallels with physical risk in potentially dangerous sports is summarised in terms of eight axioms, 1C to 8C. When the Markowitz (1959) approach is summarised in terms of axioms 1M to 7M which are conceptionally similar, a remarkably close correspondence between the two approaches can be seen:

Clarkson (1989) axiom	Corresponding Markowitz (1959) axiom
1C	1M
2C	2M
3C	3M
4C	No equivalent
5C	4M
6C	5M
7C	6M
8C	7M

4.4.2 The axiom 4C which has no equivalent in the Markowitz framework is:

"Each investor has a threshold of maximum risk and will not make an investment which involves a value of risk higher than this threshold."

It is difficult to see how a theoretical framework that does not incorporate this common sense axiom, or one very similar, can be regarded as a satisfactory guide to prudent real world behaviour.

4.4.3 Markowitz (1959) purports to give a rigorous justification of the validity of utility theory by taking each of the Von Neumann & Morgenstern axioms in turn and showing that it is a very plausible formulation of some particular aspect of human choice under uncertainty. However, when the Von Neumann & Morgenstern axioms 3:A:a, 3:A:b, 3:B:a, 3:B:b, 3:B:c, 3:C:a and 3:C:b are matched with the axioms Ia, Ib, II and III that Markowitz discusses, the correspondence is found to be incomplete:

Von Neumann & Morgenstern axiom	Corresponding Markowitz axiom
3:A:a	Ia
3:A:b	Ib
3:B:a	II
3:B:b	II
3:B:c	III
3:C:a	No equivalent
3:C:b	No equivalent

4.4.4 Axiom 3:C:a is the statement that it is irrelevant in which order the two constituents u and v of a combination are named. This assumption is innocuous in the extreme, and

the omission of any reference to it is not suprising. Axiom 3:C:b is the statement that it is irrelevant whether a combination of two constituents is obtained in two successive steps, first the probabilities a and 1-a then the probabilities b and 1-b; or in one operation using the probabilities c and 1-c where c = ab. Although seemingly innocuous to someone with no first hand experience of human behaviour in very high risk situations, this assumption, which is equivalent to a "look-through" rule for compound probabilities, is inconsistent with the real world behaviour discussed in Section 3. Accordingly, the real world relevance or otherwise of utility theory and hence of the Markowitz mean-variance approach is very closely linked to the real world relevance or otherwise of axiom 3:C:b.

4.5   *Axiom 3:C:b*

4.5.1   Von Neumann & Morgenstern observe that it may be that axiom 3:C:b "has a deeper significance" in that it is the one which comes closest to excluding a specific "utility of gambling". However, they conclude that it seems to be plausible and legitimate, unless "a much more refined system of psychology is used than the one now available for the purpose of economics." Since the expression "utility of gambling" can be regarded as being equivalent to "prescription of human behaviour in situations involving a dangerously high level of risk", these highly perceptive comments are entirely consistent with the suggestion in Section 3 that the look-through rule for compound probabilities implied by axiom 3:C:b is inappropriate in high risk situations.

4.5.2   At the end of the Appendix on the axiomatic treatment of risk that was added to the 1946 second edition, Von Neumann & Morgenstern comment as follows:

> "It seems probable that the really critical group of axioms is 3:C, or, more specifically, the axiom 3:C:b. This axiom expresses the combination rule for multiple chance alternatives and it is plausible that a specific utility or disutility of gambling can only exist if this simple combination rule is abandoned. Some change of the system 3:A-3:C, at any rate involving the abandonment or at least a radical modification of 3:C:b, may perhaps lead to a mathematically complete and satisfactory calculus of utilities, which allows for the possibility of a specific utility or disutility of gambling. It is hoped that a way will be found to achieve this, but the mathematical difficulties seem to be considerable. Of course, this makes the fulfilment of the hope of a successful approach by purely verbal means appear even more remote."

4.5.3   In the preface to the 1953 third edition, Von Neumann & Morgenstern respond as follows to the criticisms of their axiomatic approach that had recently been made by many eminent economists:

> "In connection with the methodological critique exercised by some of the contributions to the Symposium on Cardinal Utilities in Econometrica, Volume 20, (1952), we would like to mention that we applied the axiomatic method in the customary way, with the customary precautions .... In particular, our discussion and selection of 'natural operations' ... covers what

seems to us the relevant substrate of the Samuelson-Malinvaud 'independence axiom'."

4.5.4   The above reservations on the part of Von Neumann & Morgenstern and the subsequent criticisms by eminent economists are all fully consistent with the arguments in Section 3 to the effect that the look-through property for probability does not hold in high risk situations. Accordingly, there seems no alternative but to abandon axiom 3:C:b completely.

4.6   *Implications*

The abandonment of axiom 3:C:b and its replacement by the general pattern of indifference curves described in Section 3.7.2 has two crucial implications. First, as shown in Clarkson (1990), a utility function of the type generally employed is valid only in comparisons of points in the risk-satisfaction diagram where the gradients of the "equal enjoyment" indifference curves are equal. For most practical purposes, this is equivalent to comparisons of situations with very similar levels of risk. Second, the proof given by Von Neumann & Morgenstern that a utility measure satisfying their axioms is unique up to a linear transformation breaks down completely. This shatters not only the theoretical foundations of utility theory as currently formulated but also the concept of expected utility as a measure of overall attractiveness.

4.7   *Philosophical Inconsistencies*

4.7.1   A very detailed refutation of the conceptual foundations of utility theory is given in Allais (1953). Although the paper is in French, two passages from the English summary not only demonstrate the severity of his criticisms but are also consistent with the comments and conclusions set out in Section 4.6:

> "Whatever their attraction might be, none of the fundamental postulates leading to the Bernoulli principle as formulated by the American school can withstand analysis. All are based on false evidence."

> "For the rational man, there does not exist in general an indicator $B(x)$ such that the optimum situation could be defined by maximising $B(x)$."

4.7.2   Sharpe (1970) investigates utility theory as a plausible framework for the implementation of mean-variance analysis and discovers a serious inconsistency:

> "If portfolios with radically different prospects are considered by an investor, too much reality may be omitted if his decision is assumed to depend only on expected return and standard deviation of return."

Despite this serious inconsistency, Sharpe suggests that the use of a utility curve may still be justified if it is assumed that investors choose between portfolios of roughly similar risk. This is equivalent to saying, precisely as suggested in Section 4.6, that, contrary to the general belief, utility theory has no relevance in comparisons of situations involving materially different levels of financial risk.

4.8 *Textbook Inconsistencies*

4.8.1   Even in modern textbooks, serious inconsistencies are found in attempting to apply utility theory to real world behaviour. In particular, even although Sharpe (1970) shows that only a quadratic utility function is completely consistent with choices based solely on expected return and standard deviation of return, it is well known that the use of a quadratic utility function often leads to nonsensical results. For instance, in Example 1.3 of Bowers et al (1986), an actuarial textbook on risk theory and other topics, it is shown that the maximum premium that a decision maker should pay for insurance against a random loss increases with the wealth of the decision maker.

4.8.2   This and similar blatant absurdities can be readily explained by the analysis in Section 4.6, which suggests that there is no justification whatsoever for the assumed existence, in comparisons between situations involving differing levels of risk, of what is generally called a utility function.

5   CONTINUOUS TIME FINANCE AND STOCHASTIC CALCULUS

5.1   *Bernstein (1992)*

5.1.1   Some very informative general comments about the continuous time approach to finance are contained in Bernstein (1992):

> "Merton used a concept known as 'continuous time analysis' to transform the Capital Asset Pricing Model into a description of what happens over a sequence of time periods during which conditions are changing rather than standing still..... It was this shift from static to dynamic modelling that had led Modigliani to discourage Treynor from pursuing his explorations into asset pricing any further, given Treynor's limitations in mathematics. Merton had no such limitations. In the spring of 1969, he decided to incorporate Ito's lemma 'and all that stuff' as he puts it, into his intertemporal model of portfolio selection. Merton first applied it to the valuation of warrants and options. When he applied it to the Capital Asset Pricing Model, he was able to write out the dynamics of the whole process. Unfortunately, nobody in economics had ever heard of Ito's lemma. According to Merton, Samuelson himself 'could not tell whether the mathematics was right or wrong' ".

5.1.2   Two immediate observations can be made. First, Ito's lemma is nothing more than the stochastic calculus equivalent of Taylor's Theorem, which allows a function to be expressed as a power series expansion involving its derivatives. Second, the inability of Samuelson, a Nobel prizewinner who made extensive use of advanced mathematical techniques in his economic theories, to vouch for the appropriateness of the continuous time approach raises exceptionally strong doubts as to the conceptual validity of this approach.

5.1.3   Bernstein also makes the following observations as to the profound difficulties that Markowitz experienced in striving to come to terms with the continuous time approach to finance:

"In a letter to Samuelson in October 1985, Markowitz wrote: 'Ito's lemma turned out to be a cornucopia of interesting results, and Bob's work has become central to much of the modern theory of finance. The one thing that bothers me about continuous portfolio selection is that I don't really understand it.' In a note written shortly after receiving the Nobel Prize in October 1990, Markowitz told me that 'after much intense study, I am almost mediocre in the mathematics of continuous time models'."

5.1.4   It is difficult to avoid the conclusion that most mathematicians who apply the methodologies of the continuous time approach to financial behaviour will have little or no true understanding of the validity or otherwise of the "results" that they obtain.

5.2   *Risk*

5.2.1   Consider first of all the very much simplified example of a financial company which at present has assets of 100, has a liability of 108 in one year's time, and has to choose between two investment strategies. The first strategy involves investment in default-free bonds maturing in one year's time and gives with certainty a return of 5% over the year. The second involves investment in stockmarket securities that will give a return over the year of either 5%, 10% or 15% with probabilities of 0.25, 0.5 and 0.25 respectively. The financial risk inherent in the first strategy, which will lead with certainty to insolvency, is clearly far higher than with the second strategy, under which there is only a 1 in 4 chance of insolvency.

5.2.2   Consider now the manner in which risk is taken into account in Lamberton & Lapeyre (1996), a typical and very recently published textbook on stochastic calculus as applied to finance. On p.1 the "riskless asset" is defined as that asset on which the return is known with certainty, and a "risky asset" is then defined as any asset which is not the "riskless asset". There is no further discussion of risk whatsoever. In the above example, the default-free bond would be the "riskless asset", which is clearly nonsensical. Throughout modern finance theory, "risk-free" has come to mean "uncertainty-free", to the exclusion of any consideration of the adverse real world consequences that may arise from this so-called "riskless" rate of return.

5.3   *Viability and Completeness*

5.3.1   The assumption that, at any instant in time, the underlying pattern of market prices can be modelled as a smooth (ie. continuously differentiable) function of certain key attributes is generally quite innocuous and often leads to highly effective models. The gilts model described in Clarkson (1978), for example, is the general solution of a second order partial differential inequality derived on "no arbitrage" principles by the application of Taylor's Theorem to the observed price structure at a particular instant in time. The corresponding assumptions for the Black-Scholes world, in which it is assumed that prices are twice continuously differentiable, with time being one of the independent variables, are much more restrictive. However, the stochastic calculus framework within which the Black-Scholes world is formulated incorporates two even stronger assumptions, namely a very strict "no arbitrage" assumption and an even more restrictive "completeness" assumption.

5.3.2   Lamberton & Lapeyre (1996) define a market as "viable" if "there is no arbitrage opportunity". Since this is equivalent to saying that "price" is always exactly equal to "value", it is a very strong assumption that is inconsistent with many comprehensive empirical studies such as Shiller (1989) and Peters (1991) as well as with the day-to-day practical experience of most investment professionals. In essence, "viability" is in the eyes of the mathematician, not the practitioner; "viability" can be seen as a necessary condition for a simple mathematical approach to be possible, but there is no discussion of whether this condition is consistent with reality. The level of mathematical sophistication is then increased significantly by introducing a theorem to the effect that a market is viable if and only if certain results relating to equivalent probability measures and martingales hold. Virtually no professional investors are aware of what an equivalent probability measure or a martingale means in terms of real world behaviour, while virtually no mathematicians conversant with these theoretical concepts are competent to judge their real world relevance. Moreover, there is an extensive body of evidence in the UK actuarial literature to the effect that security prices cannot be represented satisfactorily in terms of discounted values of expected future benefits. In particular, Clarkson (1978) identifies significant and exploitable non-linearity of price as a function of coupon within the UK gilts market, whereas any discounted value approach implies that price is a linear function of coupon.

5.3.3   It is sometime argued that the absence of convincing evidence in refutation of the Efficient Market Hypothesis first formulated by Fama (1970) is sufficient justification for the "viability" assumption within stochastic calculus. However, this assumption acts to the severe disadvantage of competent practitioners in three ways. First, it discourages any attempt to identify and exploit more realistic models of the structure of security prices such as those described in Weaver & Hall (1967) and Clarkson (1981). Second, it distracts attention from the obvious fact that better than average forecasts of the future (which are possible, but at a cost) can enhance the likely investment return significantly. Third, the seemingly new and possibly revolutionary concept of "strategy investment" described in O'Shaughnessy (1996) and Slater (1996) can translate the unintelligent and time-invariant behavioural patterns of imperfect human beings into significant enhancements in achievable investment performance.

5.3.4   Supporters of efficiency and equilibrium will argue that in the absence of "viability" there would be a very rapid transfer of wealth from "unintelligent" players to "intelligent" players which would destroy the financial viability of the "unintelligent" players and thereby restore both efficiency and equilibrium. First hand experience of the non-linear gilts model described in Clarkson (1978) showed that, as the result of numerous elements of inertia at both the personal and the corporate level, the permeation of the new ideas was so slow that very large arbitrage profits could be made by the original users of the model for fifteen years after its general structure was widely known.

5.3.5   Lamberton & Lapeyre then define a market as complete if "every contingent claim is attainable". The underlying real world interpretation (which is not discussed) is that markets are always "orderly" in the sense that no impediments ever exist to the

orderly execution of predetermined trading strategies that are contingent on the prices ruling at any instant in time.

5.3.6   This assumption that financial behaviour is never "disorderly" is blatantly inconsistent with observed real world behaviour. From time to time, often when there are acute economic or political uncertainties, market computer systems are unable to keep up with very fast moving prices. In such circumstances, current dealing prices (if available at all) will be subject to extreme uncertainty, and it will generally be impossible to implement predetermined trading strategies. Very occasionally there may be a state of such acute disorder that the authorities halt trading completely in certain securities or derivatives markets, in which case there is a total breakdown of the completeness assumption.

5.3.7   Lamberton & Lapeyre comment as follows on the completeness assumption:

> "To assume that a financial market is complete is a rather restrictive assumption that does not have such a clear economic justification as the no-arbitrage assumption. The interest of complete markets is that it allows us to derive a simple theory of contingent claim pricing and hedging."

In plain English, a tractable mathematical "solution" is impossible unless we make the very strong assumption of completeness, regardless of whether the resulting theoretical structure bears any resemblance to the real financial world.

5.3.8   Many mathematicians, particularly those who do not have to take responsibility for the potentially grave financial consequences that could arise if real world behaviour diverges from that implied by their models, will claim that it is better to have a simple model, albeit one where some simplifying assumptions have to be relaxed in certain practical applications, than to have no model at all. The above discussion suggests a quite different conclusion, namely that such models may do much more harm than good. Stochastic calculus contains no framework for the recognition, let alone the measurement of financial risk on any common sense basis. Furthermore, the totally unrealistic assumption of completeness discourages any meaningful analysis of disorderly market behaviour by suggesting that such behaviour will never arise. The comments by Soros (1994) in evidence to the US House Banking Committee on the very severe damage that can be done by "dynamic hedging" strategies based on the mathematics of the continuous time approach are especially relevant in this context:

> "If there is an overwhelming amount of dynamic hedging to be done in the same direction, price movements may become discontinuous. This raises the specter of financial dislocation. Those who need to engage in dynamic hedging, but cannot execute their orders, may suffer catastrophic losses... That is what happened in the stock market crash of 1987. In short, attempts to "rebalance" portfolios on either a sharp fall or a sharp rise in the market could shatter the theoretical "equilibrium" on which the rebalancing strategy was based."

5.4   *The Black-Scholes Option Pricing Formula*

5.4.1   By far the most widely taught application of stochastic calculus is the derivation, using Ito's lemma, of the formula for the pricing of European options first described in Black & Scholes (1973). The formula represented a very important breakthrough at the time, in that it gave practitioners a one-parameter graduation formula (the only unknown parameter being the volatility of return on the underlying security) very much akin to the Gompertz law of mortality formulated in 1825. The general comments in Lamberton & Lapeyre (1996) are illuminating in the extreme:

> "One of the main features of the Black-Scholes model (and one of the reasons for its success) is the fact that the pricing formulae, as well as the hedging formulae we will give later, depend on only one non-observable parameter, called 'volatility' by practitioners. (The drift parameter disappears by change of probability)."

Two methods of evaluating volatility are then described - the historical method of estimation by statistical means using the asset prices observed in the past, and the "implied" method of inverting the Black-Scholes formula to associate a volatility to the quoted market price of an option. Lamberton & Lapeyre then comment as follows on the serious practical difficulties that arise:

> "In those problems concerning volatility, one is soon confronted with the imperfections of the Black-Scholes model. Important differences between historical volatility and implied volatility are observed, the latter seeming to depend upon the strike price and the maturity. In spite of these incoherences, the model is considered as a reference by practitioners."

5.4.2   Although many mathematicians with little or no practical experience of investment and finance have unthinkingly accepted the many simplifying assumptions required to derive the Black-Scholes formula, the late Fischer Black had no such illusions. In particular, in Black (1989) the opening summary is as follows:

> "The Black-Scholes formula is still around, even although it depends on at least ten unrealistic assumptions. Making the assumptions more realistic hasn't produced a formula that works better across a wide range of circumstances. In special cases, though, we can improve the formula. If you think investors are making an unrealistic assumption like one of those we used in deriving the formula, there is a strategy you may want to follow that focuses on that assumption."

One crucial feature of the real world that Black describes is a "jump" in the security price, which results in a total breakdown of the continuous time finance assumptions of "viability" and "completeness":

> "A major news development may cause a sudden large change in the stock price, often accompanied by a temporary suspension of trading in the stock ... When the big news, if it comes, is sure to be good, or is sure to be bad, the

> resulting jump is not like a change in volatility. Up jumps and down jumps have different effects on option values than symmetric jumps, where there is an equal chance of an up jump or a down jump."

5.4.3   Clarkson (1997) describes an actuarial approach to option pricing which takes explicit account of the expected return on the underlying security and of the observed cyclical variation of security prices around "central values" that are consistent with consensus views of fundamental variables such as earnings per share and the expected future growth rate of profits. The resulting model not only explains the well known "smile" and "skew" effects of implied volatility varying with strike price and maturity, but also provides a structured framework for exploiting precisely the divergencies that Black describes between the real financial world and the unrealistic simplifying assumptions of modern finance theory.

5.4.4   While "jump risk" may not lead to serious losses in the context of a diversified portfolio of options relating to different securities, the situation is quite different in the context of market risk. Black (1989) explains how the unthinking translation of the Black-Scholes assumptions into the concept of "portfolio insurance" may have led to the "Crash of 1987":

> "The same unrealistic assumptions that led to the Black-Scholes formula are behind some versions of "portfolio insurance". As people have shifted to more realistic assumptions, they have changed the way they use portfolio insurance. Some people have dropped it entirely, or have switched to the opposite strategy. People using incorrect assumptions about market conditions may even have caused the rise and sudden fall in stocks during 1987."

These comments, together with the observations by Soros cited in Section 5.3.8, suggest that attempted applications of stochastic calculus to market trading strategies have increased, not decreased, the inherent level of systemic risk in present day financial markets.

6   THE NEW APPROACH

6.1   *General Remarks*

This section represents only the briefest of outlines of the suggested new approach, and hence is similar in style to Markowitz (1952) rather than Markowitz (1959). With the exception of the following paragraph, all the basic concepts have already been described in varying degrees of detail in Clarkson (1989, 1990, 1996, 1997).

6.2   *The Risk Weighting Function*

By analogy with physical risk in sports and with probability as a real number in the range 0 to 1, we can define the risk weighting function f(x) in terms of some appropriate measure of financial outcome x as follows:

1   Let L_0 be the threshold value of x above which no adverse effects occur, and let L_1 be the value of x below which "financial ruin" occurs.

2   The function f(x) is constant at 1 for values of X below L_1, decreases smoothly (and is concave upwards) from 1 at L_1 to 0 at L_0, and is constant at 0 above L_0.

6.3   *The Risk Measure*

The risk measure R is the integral (or summation) over all possible outcomes of f(x)p(x), where p(x) is the probability associated with outcome x. The generalisation to two or more independent variables is obvious.

6.4   *Indifference Curves*

The pattern of indifference curves described in Section 3 leads to the following two decision rules once the "satisfaction" values (normally a monetary amount or a relative performance measure) have been determined:

1   In respect of points for which risk is below the maximum risk threshold R_0, choose that point of possible outcomes which lies on the "enjoyment" indifference curve lying furthest to the right.

2   If there is no point representing a possible outcome with a value of risk below the maximum risk threshold R_0, choose the point with minimum risk.

6.5   *Four Stereotypes of Behaviour*

This behaviour can be described as "intelligent". However, as described in Clarkson (1996), some people may behave "unintelligently" in that their choices are "worse" than what is achievable. "Optimal" behaviour can then be taken as that strategy which combines to best effect the combination of "intelligent" and "unintelligent" behaviour on the part of others. Finally, regarding "optimal" behaviour as relating to current levels of skill and expertise, we can define "rational" behaviour as the combination of "optimal" behaviour and a continuing attempt to improve these levels of skill and expertise by relevant training of specific practical experience. In general, there will of course be a gradation of levels of competence rather than simply these four stereotypes.

6.6   *Four Kinds of Uncertainty*

It may be helpful in practical applications to adopt a structured approach to different elements of uncertainty by considering four quite distinct types. "First kind" involves identifying homogeneous subgroups (eg "select" lives) for further study, "second type" involves obtaining estimates (generally of expected values) by statistical analysis and graduation techniques, "third kind" involves variation of observed values

by pure random chance, and "fourth type" involves elements of real world behaviour that have not been included in the mathematical model.

6.7 *Incompleteness*

Since there is obviously a stage beyond which it is pointless to go on cost-benefit grounds to obtain more and more information, this incompleteness has to be taken into account in a pragmatic way when assessing courses of action suggested by the mathematical approach sketched out above. Furthermore, our understanding of the motives and competence of different types of economic agent may also be seriously incomplete, and, in particular, attempts by regulators or governments to prevent financial accidents of types that have occurred in the past could modify the characteristics of the system significantly and invalidate conclusions drawn from historical data.

6.8 *Final Remarks*

Clearly much further work remains to be done to develop the above approach to its full potential. However, it appears to have distinct advantages over the highly axiomatic and much less practical approach of utility theory, which remains a cornerstone of modern finance theory.

7 PARAMETRIC AND NON-PARAMETRIC RISK

7.1 *Parametric Risk*

A concave upwards risk weighting function of the type shown between L, and Lo in Figure 6.1 is generally called a parametric risk function, since the curve is often represented by x^a, where a is a positive number. Markowitz (1959) suggested a = 2, Bawa & Lindenberg (1977) allow a to be any positive integer (normally 1 or 2), whereas Clarkson (1989) allows a to take any value greater than 1, with 1.5, 2, 2.5 and 3 corresponding respectively to low, moderate, high and very high degrees of aversion to risk.

7.2 The straight line value of 1 gives the constant risk weighting function of non-parametric risk. This produces the well-known "probability of ruin" which is important in non-life insurance and other high risk applications.

7.3 The compound risk weighting function described in Section 6.2 combines parametric risk (which is appropriate in low and medium risk situations) and non-parametric risk (which is appropriate for high risk situations) in a way that has not yet been introduced into modern finance theory. Whereas Markowitz originally applied parametric risk to the essentially low risk situation of portfolio selection where diversification limits the risk, others have unthinkingly applied it to medium or high risk situations such as asset allocation or futures and options, where the upward continuation of the parametric risk function could seriously distort the "true" risk value.

7.4 While the advantages and disadvantages of parametric and non-parametric risk have been investigated in detail by Booth (1997) and Chadburn (1996) in important application areas such as the solvency of life offices, the current literature does not discuss the possibility of unifying the two approaches. The unification of parametric and non-parametric risk sketched out above would appear to offer an improvement over currently accepted formal frameworks such as utility theory and the "risk equals variance (or semi-variance) of return" paradigm of risk.

8   CONCLUSIONS

8.1 In a written contribution to the discussion on Clarkson (1996), Professor Gordon Pepper, who has very extensive experience of both the real financial world and the academic world, comments as follows on the unsoundness of much of modern finance theory:

> There is a tendency to build layer upon layer of complication on top of basic assumptions that are approximations. The result is an inverted pyramid on top of dubious foundations. As an intellectual exercise it may be very clever, but it has lost touch with reality. The academics would do much better to re-examine the foundations, secure them, and rebuild the pyramid."

The present paper follows, in the field of investment risk, precisely this suggested approach of first of all identifying which of the simplifying assumptions are untenable, then finding sound conceptual principles to replace them, and only thereafter proceeding to rebuild the pyramid from the ground upwards.

8.2 In the context of the Markowitz mean-variance framework, which combines utility theory with the "risk equals variance (or semi-variance) of return" paradigm, two far-reaching conclusions can be drawn. First, the untenability of axiom 3:C:b of the Von Neumann & Morgenstern set results in utility theory, as it is currently taught and applied, being potentially unsound in all real world applications involving any meaningful level of financial risk. Second, the parametric risk measures of variance and semi-variance are unsatisfactory in situations where financial ruin, however defined, is a possibility. In short, while the Markowitz mean-variance approach may be a useful aid to logical thought in essentially low risk application areas such as stock selection within diversified investment portfolios, its attempted application to situations involving higher levels of inherent financial risk is unsound in the extreme.

8.3 As discussed in detail in Section 5, stochastic calculus does far more harm than good as regards providing a plausible mathematical framework within which to formulate prudent financial and investment strategies. Not only has it no underlying mechanism for the recognition or measurement of financial risk, but it also promotes, through the totally unrealistic completeness assumption, the dangerously naive belief that there is no meaningful likelihood of precisely the disorderly market behaviour that has led to numerous derivatives-based financial disasters in recent years.

8.4 The main conclusion of the paper is that the risk methodologies of modern finance theory, being fundamentally unsound as regards frameworks for the recognition,

measurement and control of risk in the real financial world, are seriously inadequate for the purposes of the actuarial profession in the next century. Accordingly, it is suggested that the actuarial profession should take the lead in developing, and promoting for general use throughout the financial world, a new and more realistic theory of financial risk that is consistent with traditional actuarial principles. While much more work clearly remains to be done, the new framework outlined in Section 6 appears to offer useful insights as to how best to rebuild the pyramid of the theory of financial risk.

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