Abstract
Several issues concerning the pricing and hedging of derivative instruments are analyzed under diverse versions of the CAPM. Our study is divided into three parts of related but independent interest.

A class of perfectly hedged equity-linked contracts is introduced and characterized by several properties. In the framework of "premium calculation principles" a CAPM "arbitrage-free" pricing method is applied to get the involved derivative prices. In particular the "martingale approach" is recovered as a special case of this method.

The relationship between the guaranteed rate of return of some capital protection option strategies and the market rate of return is studied. A "risk adjusted" CAPM is considered, for which budget equations and a OPT compatibility condition are derived. Conditions under which the "original" CAPM is consistent with OPT are obtained. As by-product an OPT based test of the CAPM is suggested. Furthermore a stability criterion, which helps find "good" capital protection option strategies, is formulated and illustrated at some examples.

A coefficient of risk exposure, which generalizes the concept of beta factor in CAPM theory, is defined. It measures the deviation between future values of assets and liabilities relative to a reference investment. Applied to OPT, our approach allows for simultaneous hedging under different hedging techniques encountered in the literature. Interesting results about the rate of return of financial assets are derived.

Keywords: CAPM, OPT, arbitrage, premium calculation principle, derivatives, hedging, guaranteed rate of return
1. **Perfectly hedged asset-liability strategies in Insurance and Finance.**

Consider a one-periodic financial model in which the random variables $A$, $L$ represent accumulated asset values respectively liability values taken end of period. A main goal of ALM (= asset and liability management) consists to meet $L$ given $A$ in some "optimal" way. Classical optimality criteria may be quite divers, including minimization of the expected square surplus $E[(A-L)^2]$, or its conditional version $E[(A-L)^2|\mathcal{F}]$ given some information $\mathcal{F}$. Modern ALM is characterized by the use of derivative instruments to achieve its goal. A simple method to meet $L$ given $A$ consists to find transformed random variables $f(A,L)$, $g(A,L)$, representing end of period values of derivative instruments based on $A$, $L$, such that with probability one

$$A + f(A,L) = L + g(A,L).$$

(1.1)

Appropriate choices allow to guarantee with certainty the liability value $L$.

**Example 1.1** (portfolio insurance) Let $S_t$ be the value of a risky fund at time $t$. If an investor wants to protect its asset value at the level $K$, he may buy a put-option with strike price $K$ such that at time $t$ one has

$$S_t + (K - S_t)_+ = K + (S_t - K)_+.$$

(1.2)

This equation expresses the well-known economic equivalence between the "hold fund - buy put" and the "hold cash - buy call" capital protection option strategies. Combined with mortality risk, portfolio insurance leads to equity-linked life insurance contracts with an asset value guarantee studied by Brennan and Schwartz(1976/79), Delvaux and Magnée(1991), Bacinello and Ortu(1993), Aase and Persson(1994) and others. To get (1.2), one has to set $A=S_t$, $f(A,L)=(K-A)_+$, $g(A,L)=(A-K)_+$ in (1.1).

**Example 1.2** (excess-of-loss and stop-loss reinsurance) Let $X_t$ represent accumulated insurance claims at time $t$, which should be paid out by an insurer. To guarantee the liability payment $X_t$, an insurer can choose to pay claims up to a deductible $d$ and to reinsurer the excess claims. Setting in (1.1) $A=d$, $L=X_t$, $f(A,L)=(L-d)_+$, $g(A,L)=(d-L)_+$, one gets

$$d + (X_t - d)_+ = X_t + (d - X_t)_+.$$

(1.3)
The design and premium rating of insurance contracts based on the relation

\[(1.4) \quad d + Z_t = X_t + D_t,\]

where \(d\) represents a maximum deductible, \(Z\) the payment from a reinsurance form or risk-exchange, and \(D\) a perfectly hedged experience rated bonus, has been discussed recently in Hürlimann (1994a/94b/94c/94d).

The formal process of exchanging the role of the variables \(A, L,\) and making the needed reinterpretations, leads to similar conclusions valid in a finance context. It is our aim to exploit the obtained duality between Insurance and Finance. In the present work indications for the adaptation of our previous insurance based results to the finance context are given. In particular an interesting CAPM approach to derivative pricing is proposed. This approach includes as special case the ubiquitous arbitrage-free derivative pricing model and should be of fundamental importance for fields such as portfolio management. For example intriguing and challenging questions like "why are there securities and strategies that have expected returns above the riskless rate ?" suggested by Elton and Gruber(1991), note to the fourth edition, makes sense in this framework and could eventually be tackled for some class of perfectly hedged investment strategies.

2. Characterization of perfectly hedged equity-linked contracts.

Following the program sketched at the end of Section 1, let us adapt (1.4) to the finance context. Our presentation follows Hürlimann (1994c).

Let the stochastic process \(S_t, t \geq 0\) represent the market value at time \(t\) of a risky investment made at time 0. Consider the set \(\text{Com}(S_t)\) of comonotonic stochastic processes \(U_t, V_t, t \geq 0\), such that \((U_t, V_t) \in \text{Com}(S_t)\) if there exists non-decreasing functions \(u(s), v(s)\) such that \(U_t = u(S_t), V_t = v(S_t)\) and \(u(s) + v(s) = s\). The processes \(U_t, V_t\) represent market values at time \(t\) of derivative instruments with underlying asset \(S_t\). The comonotonic property means that only those derivative instruments are considered for which neither the buyer nor the writer will benefit in case the market value of the underlying asset increases. Since \(S_t\) is non-negative, one has furthermore \(0 \leq u(s), v(s) \leq s\). A feasible pair of derivative instruments \((U_t, V_t) \in \text{Com}(S_t)\) is said to guarantee the asset value \(g\) if the following real number exists and is finite:

\[(2.1) \quad g = \sup_{s \in \mathbb{R}} \{u(s)\} < \infty.\]
The set of feasible pairs of comonotonic derivative instruments with guaranteed asset value \( g \) is denoted by

\[
V_g = \{ (U_t, V_t) \in \text{Com}(S_t) \text{ such that } (2.1) \text{ holds } \}.
\]

One sees that for \((U_t, V_t) \in V_g\) the function

\[
w(s) = g - u(s) - g + v(s) - s
\]

is always non-negative and defines a transformed random variable \( W_t = w(S_t) \) such that with probability one

\[
(2.4) \quad s_t + w_t = g + v_t.
\]

With \( s_g = \inf_{u(s) = d} \{ s \} \), one has for all \( s \geq s_g \)

\[
(2.5) \quad w(s) = 0, \quad v(s) = s - g.
\]

It is possible to interpret the pair \((g, V_t)\) as a perfectly hedged equity-linked contract with guaranteed asset value \( g \) and dividend \( V_t = v(S_t) \geq 0 \). Indeed an investor, who holds long the asset \( S_t \) and buys the derivative instrument \( W_t \), can always meet the liability \( g + V_t \) according to (2.4). A stronger mathematical characterization is obtained as follows. Suppose there is given an equity-linked contract \((g, V_t)\) with guaranteed asset value \( g \) and dividend \( V_t = v(S_t) \geq 0 \), where one assumes that \((U_t = S_t - V_t) \in \text{Com}(S_t)\). Let further \( W_t = w(S_t) \) be some derivative instrument with underlying asset \( S_t \), where a priori \( W_t \) is not defined by (2.4). Given is an investor, who holds long the asset \( S_t \) and short the equity-linked contract \((g, V_t)\). After payment of the dividend, the value of the investor's investment is \( U_t = S_t - V_t \). To reduce the financial risk of a loss \( U_t < g \), which may be quite important, the investor chooses to buy the derivative instrument \( W_t \). At time \( t \) the investment value \( U_t + W_t \) has to meet as closely as possible the guaranteed liability \( g \). Under these conditions, which choice of \( W_t \) is appropriate? As decision criterion suppose the investor applies the minimum square loss principle, which consists to minimize the expected square difference between assets and liabilities. At time \( t \) one has to minimize the risk quantity

\[
(2.6) \quad R(t) = \mathbb{E}[(U_t + W_t - g)^2] = \min.
\]
From $R(t) = (E[U_t + W_t] - g)^2 + \text{Var}[U_t] + \text{Var}[W_t] + 2\text{Cov}[U_t, W_t]$, one sees that a minimum is attained provided $g = E[U_t + W_t]$, and $\text{Cov}[U_t, W_t] = -\text{Var}[W_t]$ or $\text{Cov}[U_t, W_t] = -\text{Var}[U_t]$. In the first situation one has $R_{\min}(t) = \text{Var}[U_t] - \{1 - \rho(U_t, W_t)^2\}$, where $\rho(U_t, W_t)$ is the correlation coefficient between $U_t$ and $W_t$. The second situation is similar. In particular the risk of the investor can be completely eliminated (perfect hedge), that is $R_{\min}(t) = 0$, provided there exists $U_t, W_t$ such that $\text{Cov}[U_t, W_t] = -\text{Var}[W_t] = -\text{Var}[U_t]$. The result is summarized as follows (see also Hürlimann(1994c/94d)).

**Proposition 2.1.** Given is an equity-linked contract $(g, V_t)$ with guaranteed asset value $g$ and dividend $V_t = v(S_t)$, where $(U_t = S_t - V_t, V_t) \in \text{Com}(S_t)$. Let further $W_t = w(S_t)$ be some derivative instrument with underlying asset $S_t$. Assume that the set $\{s \in \mathbb{R} : w(s) = 0\}$ is non-empty. Then the following conditions are equivalent:

(C1) $R_{\min}(t) = E[(U_t + W_t - g)^2] = 0$

(C2) $\text{Cov}[U_t, W_t] = -\text{Var}[W_t] = -\text{Var}[U_t]$

(C3) One has $W_t = g - U_t$, $(U_t, V_t) \in \text{Com}$, and $(g, V_t)$ defines a perfectly hedged equity-linked contract with guaranteed asset value $g$ and dividend $V_t$.

### 3. CAPM derivative pricing.

Our aim is to derive pricing formulas for a perfectly hedged equity-linked contract with guaranteed asset value characterized by Proposition 2.1. The needed hedging derivative instrument $W_t$ satisfies the relation

\begin{equation}
S_t + W_t = g + V_t.
\end{equation}

Let us make use of the CAPM pricing method suggested by Borch(1982), formula (4). Assume a random variable $X_t$ is split into two components $Y_t, Z_t$ such that $X_t = Y_t + Z_t$. In order to avoid arbitrage opportunities, the problem is to design a pricing principle $H[-]$, representing prices at initial time 0, which satisfies the additive property $H[X_t] = H[Y_t] + H[Z_t]$. Indeed suppose on the contrary that for example $H[X_t] > H[Y_t] + H[Z_t]$. Then a market agent could choose to sell short $X_t$ and hold long the components $Y_t, Z_t$. The price at time 0 of this investment equals $H[X_t] - (H[Y_t] + H[Z_t]) > 0$ while at time $t$ the investment value is $-X_t + (Y_t + Z_t) = 0$. It follows that this agent has made a riskless profit, which is inconsistent with an economic equilibrium. Let $D_t$ be a (possibly stochastic) discount function. Then Borch(1982) suggests to
price the components $Y_t$, $Z_t$ of $X_t$ using the following CAPM like relationships:

\begin{align}
(3.2) \quad H[Y_t] &= E[D_t Y_t] + \frac{(H[X_t] - E[D_t X_t]) - \text{Cov}[D_t X_t, D_t Y_t]}{\text{Var}[D_t X_t]} \\
H[Z_t] &= E[D_t Z_t] + \frac{(H[X_t] - E[D_t X_t]) - \text{Cov}[D_t X_t, D_t Z_t]}{\text{Var}[D_t X_t]}
\end{align}

Alternative justifications of this pricing method will be presented in Hürlimann (1994d), Section 1.

Applied to our situation let $X_t = g + V_t$, $Y_t = S_t$, $Z_t = W_t$, and assume for simplicity that $D_t = e^{rt}$, with $r$ the risk-free compounded rate of interest. Then clearly $H[Y_t] = S_0$ is the known market value of the underlying asset at time 0. Furthermore one has the parity relations:

\begin{align}
(3.3) \quad H[X_t] &= g e^{rt} + H[V_t] = S_0 + H[W_t], \\
E[D_t X_t] &= g e^{rt} + e^{rt} E[V_t] = e^{rt} (S_0 + E[W_t]).
\end{align}

Assume that $E[S_t] = e^{\delta_S} S_0$, where $\delta$ is the mean expected compounded yield rate over the interval $[0,t]$. Using (3.2) one gets the CAPM derivative price

\begin{align}
(3.4) \quad H[V_t] &= E[e^{\gamma V_t}] + \delta_S (1 - e^{\delta_S r t}) - \text{Var}[V_t]/\text{Cov}[V_t, S_t].
\end{align}

The price of the derivative instrument $W_t$ is determined by (3.4) and the first parity relation in (3.3).

It is important to observe that "martingale" derivative prices, of use in complete markets, are obtained through specialization. The "fundamental theorem of asset pricing" states that a stochastic process $S_t$ satisfies the "no-arbitrage condition" if and only if there exists an "equivalent martingale measure", denoted by *, such that $E'[S_t] = e^{\delta_S S_0} = e^{\delta S_0}$. A mathematical proof of this result in discrete time is given by Schachemayer (1992). Background material is found in Duffie (1992). Taking prices with respect to the measure *, the relation (3.4) reduces to the "no-arbitrage price"

\begin{align}
(3.5) \quad H[V_t] &= E'[e^{\gamma V_t}],
\end{align}

which equals the expected discounted value of the derivative process at time $t$ with respect to the equivalent martingale measure.

An important advantage of the CAPM method is its generality and its straightforward application from a statistical point of view. For this reason it is a valuable candidate for pricing derivatives in incomplete
markets (major unsolved problem). For the needed empirical work in this area, let us mention the interesting introduction to the statistical modelling of financial price processes by Taylor (1992), where also a lot of empirical data work is presented.

**Example 3.1** (portfolio insurance) Example 1.1, equation (1.2), yields the most popular equity-linked call-option contract with guaranteed asset value $g=K$, $V_t=(S_t-K)_+$, $W_t=(K-S_t)_+$. A straightforward calculation shows that

\[(3.6) \quad \text{Cov}[V_t, S_t] = \text{Var}[V_t] + E[V_t] - E[W_t], \text{ with } E[W_t] = K - S_0e^{\delta t} + E[V_t].\]

The **CAPM based call-option price** is thus given by

\[(3.7) \quad H[V_t] = E[e^{\delta t}V_t] - S_0(e^{\delta t}e^{-\delta t} - 1) - \text{Var}[V_t]/(\text{Var}[V_t] + E[V_t] - E[W_t]).\]

From a qualitative point of view, three cases must be distinguished:

\[(3.8) \quad H[V_t] = \begin{cases} E[e^{\delta t}V_t], & \text{if } \delta > r \text{ (expected yield over risk-free rate)} \\
E[e^{\delta t}V_t], & \text{if } \delta = r \text{ (risk-neutral valuation)} \\
> E[e^{\delta t}V_t], & \text{if } \delta < r \text{ (expected yield under risk-free rate)} \end{cases}\]

Observe these pricing relations are related to the problem suggested by Elton and Gruber (1991) in their note to the fourth edition.

4. **OPT. CAPM and guaranteed rates of return.**

In this Section a generalization of the notion of equity-linked contract with guaranteed asset value, called **experience rated finance strategy**, is considered. This strategy is a triple $(S, F, D)$, where $S$ represents the future unknown value of a financial asset (e.g. stock price), $F$ is its future (expected) price (e.g. forward price of stock), and $D$ is a dividend (possible gain of the investment strategy), contingent on the value of $S$, made by the investor on this strategy in a financial market with derivative environment. For simplicity let us assume that the period of investment is one-year, that is $T=1$. Suppose the investor wants to guarantee some future price, say $F=S_0r$, where $S_0$ is the value of $S$ at time of investment and $r$ is a **guaranteed accumulated rate of return**. For the special dividend $D=(S-F-B)_+$, $B$ a constant, it is shown in Hürlimann (1991) that application of option
pricing theory implies the restriction \( r \leq r_f \), where \( r_f \) is the risk-free accumulated rate of return used in option pricing formulas. Furthermore there exist capital protection options strategies, which guarantee the rate \( r \) with certainty, that is with probability one, and the market rate of return is recovered in the mean.

In the present Section these results are generalized to arbitrary dividend formulas \( D \) satisfying \( 0 \leq D \leq (S-F) \), and we analyze when the applied option pricing method is consistent with capital asset pricing models of the types introduced by Sharpe-Lintner-Mossin and Black.

In derivative applications the dividend \( D \) will often be the payoff at time \( T=1 \) of a linear combination of call- and put-options of the form

\[
D = \sum a_i (S-\alpha_i) + \sum b_i (\beta_i - S).
\]

The accumulated market rate of return on the financial asset \( S \) is denoted by \( r_s \). Let \( NO = S-F-D \) be the net outcome after dividend in a long position of the experience rated finance strategy. We assume that the net equivalence principle holds, that is \( E[NO]=0 \) or equivalently

\[
E[S] = S_0 r_s = F + E[D].
\]

4.1. An option pricing method.

In option pricing theory the price of the dividend \( D \) at the time of investment is equal to the present value

\[
PV(D) = \sum a_i C(\alpha_i) + \sum b_i P(\beta_i),
\]

where \( C(\cdot) \) and \( D(\cdot) \) are call- and put-prices evaluated according to some option pricing model, e.g. Black and Scholes (1973) model. Option pricing theory is based on the no-arbitrage condition. On the other side the assumption \( E[NO]=0 \) means that the present value of NO vanishes, that is \( PV(NO) = S_0 r_s F - PV(D) = 0 \). Equivalently one has

\[
PV(D) = r_f, S_0 (r_f - r(D)) \geq 0,
\]

where the guaranteed rate of return \( r = r(D) \) depends upon the dividend formula chosen. Since by assumption \( 0 \leq D \leq G_s \), hence \( 0 \leq PV(D) \leq C(F) \), it follows from (4.4) that
Furthermore the following capital protection options strategy

invest $S_0 (1 - r(D) r_f^{-1}) = PV(D)$ in a linear combination of call- and put-options of the form (1.1),
invest $S_0 r(D) r_f^{-1}$ in the risk-free investment

guarantees the rate of return $r(D)$, or the future price $F$, and in the mean the invested total capital $S_0$ is recovered accumulated at the market rate $r_s$. Indeed the value at time $T=1$ of this investment strategy is equal to

$$S_0 r(D) + D = F + D \geq F.$$  

The mean value of this investment is, according to (4.2), equal to

$$F + E[D] = E[S] = S_0 r_s,$$

which shows that the market rate of the financial asset is recovered in the mean. Furthermore it follows from (4.5) that the maximum experience rated dividend, which can be recovered in the mean, is attained if and only if there exists a feasible dividend formula $D$ satisfying the conditions

$$PV(D) = C(S_0 r) = r_f^{-1} S_0 (r_f - r).$$

For example the dividend formula $D=(S-F)$, such that

$$PV(D) = C(F) = S_0 - r_f^{-1} F$$
satisfies this property. This may be of interest in practice in case it is likely to be expected that $r_s \geq r_f$.

**Remark 4.1.** Observe that only the case $D=0$ guarantees with certainty the risk-free rate by investing the total capital $S_0$ in the risk-free investment. The advantage of an options strategy of the above type with $D \neq 0$ lies in the fact that the investor can participate to the up and down movements of the market price $S$, and thus realize in the long run a rate of return, which lies eventually above the risk-free rate. A more detailed analysis of capital protection options strategies follows in Section 5.
4.2. Risk adjusted capital asset pricing models.

Our construction is based on the idea presented in Hürlimann (1991), Section 6. Given are n experience rated finance strategies \( \{S_i, F_i, D_i\} \), \( i = 1, \ldots, n \). The market portfolio is represented by an experience rated finance strategy \( \{S_M, F_M, D_M\} \). Denote by \( r_i, i = 1, \ldots, n \), and \( r_m \) the accumulated rates of return on the considered financial assets and assume that (dividend dependent) rates of return \( r_i = r_i(D_i), i = 1, \ldots, n \), \( r_m = r_m(D_M) \) have to be guaranteed. Then one has necessarily the relations

\[
(4.11) \quad F_i = S_i \rho r_i, \quad i = 1, \ldots, n, \quad F_M = S_M \rho r_M,
\]

and from assumption (4.2)

\[
(4.12) \quad r_{sj} = r_i + E[D_i]/S_{i0}, \quad i = 1, \ldots, n, \quad r_{sM} = r_m + E[D_M]/S_{M0}.
\]

The observed accumulated rates of return on the financial market are described by the random variables \( R_{s_j} = S_j / S_{i0} \), \( i = 1, \ldots, n \), and \( R_{sM} = S_M / S_{M0} \), such that \( r_{sj} = E[R_{sj}] \) and \( r_{sM} = E[R_{sM}] \). In general the opinion or expectations about \( r_i, r_m \) will depend upon the investors risk preferences. We assume that the return distributions of the investors can be represented by random variables \( R_i, R_m \) such that \( r_i = E[R_i] \), \( r_m = E[R_m] \). Under these assumptions the classical CAPM yields the linear relation

\[
(4.13) \quad r_i - r_f = (r_m - r_f) \beta_i, \quad i = 1, \ldots, n,
\]

\[ \beta_i = \text{Cov}[R_i, R_m] / \text{Var}[R_m] \]

an ex ante defined beta-factor.

Inserting the relation (4.12) into the CAPM (4.13) one obtains a "risk adjusted capital asset pricing model" of the form

\[
(4.14) \quad r_{sj} = r_f + E[D_i]/S_{i0} + (r_{sM} - r_f - E[D_M]/S_{M0}) \beta_i, \quad i = 1, \ldots, n.
\]

It is an important question to analyze when a relation of the form (4.14) can hold. We study first budget equations and then derive a condition for consistency with the option pricing method. For this consider the total market supply described by the market portfolio. Let \( S_{f0} = \sum S_{i0} \) be the capital invested in the risk-free asset with rate \( r_f \), \( S_{i0} \) the amount contributed by investor number \( i \), and let \( S_{i0} \) be the capital invested in the risky asset number \( i, i = 1, \ldots, n \). The total capital invested is thus \( S_{M0} = S_{f0} + \sum S_{i0} \). On the
other side the market portfolio can be represented by the vector of weights

\[ w^M_t = (w_1^M, w_2^M, ..., w_n^M), \quad w_i^M = \frac{S_{i0}}{S_{M0}}, \quad i = 1, ..., n. \]

Then the following budget equations must hold:

\[ \sum w_i^M + w_t^M = 1, \quad r_M = \sum w_i^M r_i + w_t^M r_t, \quad r_{SM} = \sum w_i^M r_{Sj} + w_t^M r_t. \]

By subtraction one obtains the necessary condition

\[ r_{SM} - r_M = \sum w_i^M (r_{Sj} - r_i). \]

Inserting (4.12) into (4.17) it follows that

\[ E[D_M] = \sum E[D_i]. \]

Therefore the expected dividend of the market portfolio is equal to the sum of the expected dividends of the individual investors.

**Example 4.1.** If one sets \( D_M = S_{M0} (R_{SM} - r_{SM}), \quad D_i = S_{i0} (R_{SM} - r_{SM}), \quad i = 1, ..., n, \)

which means that all investors buy call-options on the market index with exercise price \( r_{SM}, \)

which is known to be uniquely defined in a risk-averse economy by Hürlimann (1994e), then (4.18) is valid only if

\[ \sum w_i^M = 1, \]

which corresponds to the assumption: the market supply of the risk-free asset vanishes, that is

\[ w_t^M = 0 \quad \text{or} \quad S_{t0} = \sum S_{i0,t0} = 0, \]

also made in Hürlimann (1991), Section 6. Observe that some of the \( S_{i0,t0} \)'s may be positive, others negative, since investors may lend or borrow capital at the risk-free rate. Thus the assumption \( S_{t0} = 0 \) may be fulfilled in a non-trivial way. The corresponding risk-adjusted CAPM is of the form

\[ r_{Sj} = (1 - \beta_j) (r_t + E[(R_{SM} - r_{SM}),]) + \beta_j r_M, \quad i = 1, ..., n, \]

as derived first in Hürlimann (1991), formula (6.9).

A risk adjusted CAPM of the form (4.13) or (4.14) will be consistent
with the option pricing method provided the guaranteed rates $r_i$ and $r_M$ are evaluated following (4.5):

$$r_i = r_t - r_t PV(D)/S_{t,0}, \quad r_M = r_t - r_t PV(D)/S_{M,0}.$$  

Inserted in (4.13) one obtains the condition

$$PV(D_i) = \beta_i w_i^M PV(D_M), \quad i=1, \ldots, n.$$  

Add these equations and assume that $PV(D_M) = \sum PV(D_i)$. It follows that

$$\sum \beta_i w_i^M = 1.$$  

Using the first equation in (4.16) one must have

$$w_i^M = \sum w_i^M (\beta_i - 1) = \sum w_i^M (\beta_i - \beta_M).$$

Since $\beta_i$ may take values greater or less than 1, the assumption $w_i^M = 0$ (market supply of the risk-free asset is zero), obtained in Example 4.1, can be fulfilled in a non-trivial way.

4.3. Original CAPM and option pricing method.

At the present stage of our investigation, it is not known if a risk adjusted CAPM of the form (4.14) may be a valuable empirical tool. The simpler original CAPM, though much criticized, still finds wide application in the real world (see Harrington (1987)). Thus it seems important to analyze under which conditions the original CAPM may be consistent with the option pricing method. The CAPM postulates a linear relation of the form

$$r_sj = r_t + (r_{SM} - r_t) \beta_{sj}, \quad i=1, \ldots, n,$$

with $\beta_{sj} = \text{Cov}[R_{sj}, R_{SM}]/\text{Var}[R_{sj}]$ an *ex post* defined beta-factor.

Consider the net outcome $NO_i=S_i F_i - D_i$ associated to the experience rated finance strategy $\{S_i, F_i, D_i\}$, where $F_i = S_{i,0} r_i$ is calculated according to (4.12). Taking expected values one gets

$$E[NO_i] = S_{i,0} (r_{sj} - r_t) - E[D_i] = S_{i,0} (r_{sj} - r_t) + r_t PV(D_i) - E[D_i].$$
Under the assumption $E[N_O]=0$ this means that

\begin{equation}
S_{io}(r_s - r_f) = E[D_i] - r_f PV(D_i), \quad i=1,...,n.
\end{equation}

These equations provide a link between the mean excess return (expected return above the risk-free rate) and capital protection options strategies. The same relation holds for the market portfolio, that is

\begin{equation}
S_{M0}(r_{SM} - r_f) = E[D_M] - r_f PV(D_M).
\end{equation}

Now assume that

\begin{equation}
E[D_M] = \sum E[D_i], \quad PV(D_M) = \sum PV(D_i).
\end{equation}

Then adding the $n$ equations (4.27) and comparing with (4.28) yields the relation between mean excess returns

\begin{equation}
r_{SM} - r_f = \sum w_i^M (r_s - r_f).
\end{equation}

This equilibrium relation, based on the option pricing method, shows that, independently of the sign of the mean excess return of the market portfolio, there may exist individual experience rated finance strategies with positive as well as with negative mean excess returns. This is a property which is observed in the present-day financial markets. One may also relate this result to the suggestion made by Elton and Gruber (1991) in the note to their revised book. Observe that the same property holds for the original CAPM (4.25) if and only if there exist negative beta-factors, which is likely impossible in perfect markets (exclusion through diversification).

Now the CAPM will be consistent with the option pricing method if the equations (4.27) are replaced by the more restrictive conditions

\begin{equation}
S_{io}(r_s - r_f) = S_{io}(r_{SM} - r_f) \beta_{is} = E[D_i] - r_f PV(D_i), \quad i=1,...,n.
\end{equation}

Since the original CAPM has confirmed some predictive power (see Harrington (1987)), these conditions may be useful for predicting expected returns of capital protection option strategies. Indeed from (4.8) and (4.31) one deduces that the mean expected return is equal to

\begin{equation}
E[S_i] = F + E[D_i] = F + r_f PV(D_i) + S_{io}(r_{SM} - r_f) \beta_{is}.
\end{equation}
Since $E[S_i] \geq F$ for a capital protection option strategy, the relation (4.32) suggests the following option pricing based test of the CAPM. The original CAPM holds if and only if the following inequality is true

$$r_{S,M} \geq r_f - (r_f / \beta_{S,j})(PV(D_i)/S_{i,0}).$$

Using (4.5) one gets

$$r_{S,M} \geq r_f - (r_f - r_i)/\beta_{S,j}.$$

Since the trivial capital protection option strategy $D_i=0$ yields $r_i=r_f$ the risk-free rate, one must necessarily have $r_{S,M} \geq r_f$. This means that the original CAPM can only be valid if the mean market rate of return, that is the long run average market rate of return, is greater than the risk-free rate of return. This strong implication must be fulfilled in order to validate an efficient market theory. As consequence from the inequality $r_{S,M} \geq r_f$ and (4.32), one sees that in an efficient market a capital protection option strategy has necessarily a mean expected return of amount

$$E[S_i] \geq F + r_f PV(D_i)=S_{i,0}r_f.$$

Moreover investing in options whose underlying securities have high beta factors offer the highest mean expected return.

Another consistency condition, which has to be fulfilled, follows by inserting (4.25) into the option pricing relation (4.30). One obtains

$$\sum \beta_{S,j} w_i^M = 1.$$

Using (4.16), that is $w_i^M=1 - \sum w_i^M$, one gets the compatibility condition

$$w_i^M = \sum w_i^M (\beta_{S,j} - 1) = \sum w_i^M (\beta_{S,j} - \beta_{S,M}).$$

This is of the form (4.24) with the ex post $\beta_{S,j}$ instead of the ex ante $\beta_i$.

5. **On the return of capital protection option strategies.**

In Hürlimann(1991) some special cases of capital protection option strategies were mentioned. Using option pricing theory a method has been introduced for the evaluation of guaranteed rates of return, which were
shown to be necessarily less than the risk-free rate. In this Section we consider this method within the framework of capital protection option strategies. The idea is to look for a stability criterion, which helps find "good" option strategies. The same notations as in Section 4 are used.

Under an option strategy we understand any linear combination of put- and call-options whose payoff is of the form

\[ D = D(S) = \sum a_i(S - \alpha_i) + \sum b_i(\beta_i - S). \]  

Its present value over a time period \( \tau \) is given by

\[ PV(D) = r^\tau \left( \sum a_i C(\alpha_i) + \sum b_i P(\beta_i) \right), \]

where \( C( \cdot ) \), \( P( \cdot ) \) are call- and put-prices as given in options markets.

A capital protection option strategy with dividend \( D \) and investment capital \( K \) is defined as follows. Invest the amount \( K - PV(D) \) in cash at the risk-free accumulated rate \( r^\tau \) and pay \( PV(D) \) to buy the dividend. The outcome of such a strategy after the period of time \( \tau \), is described by the payoff function

\[ P(S) = (K - PV(D)) r^\tau + D. \]

Imagine one wants to guarantee a one-year accumulated rate of return \( r \leq r^\tau \). Set \( F = S_0 r^\tau \) for the guaranteed future price, \( F_r = S_0 r^\tau \) for the theoretical (=arbitrage-free) forward price, where \( S_0 \) is the initial value of the contingent claim \( S \). Using a long position in some experience rated finance strategy of the form \( \{ S, F, D \} \) such that \( 0 \leq D \leq F_S = (S - F) \), it is possible to construct capital protection option strategies which guarantee the capital \( F \), that is such that \( P(S) = F + D \). In Hürlimann(1991), p. 227, formula (3.1), the following special case and variants of it was derived:

\[ D = (S - F_d) \alpha, \quad F = F_d - C(F_d). \]

It is important to remark that this choice may not necessarily "optimize" the dividend outcome \( D \). Consider more generally an arbitrary dividend call \( D = (S - d) \), and assume that \( D \leq (S - F) \), that is \( d \geq F \). Set \( K = \alpha S_0 \) for the invested capital and invest \( \alpha \) shares in the experience rated finance strategy \( \{ S, F, D/\alpha \} \). The total net outcome of this investment is equal to
(5.5) \[ NO = \alpha G - D = \alpha(S - F) - D. \]

If one assumes that \( E[NO]=0 \), one has also that the present value vanishes, that is \( PV(NO)=0 \). Doing the calculation it follows that

(5.6) \[ \alpha = r_I^\ast C(d)/(F_t - F), \ d\geq F. \]

The choice \( d=F \) maximizes the probability of a dividend payment, but the capital needed for the investment is also greatest. The capital protection option strategy corresponding to (5.6) has the payoff

(5.7) \[ P(S)=(\alpha S_0 - C(d))r_I^\ast + (S-d)_+ = \alpha F_t - r_I^\ast C(d) + (S-d)_+ = \alpha F + (S-d)_+. \]

Hence the existence of capital protection option strategies with guaranteed rate of return \( r \) has been settled. Moreover, and this must be emphasized, the experience rated strategy associated to the \( \alpha \) shares in the finance strategy \( \{S,F,D/\alpha\} \), allows via the risk reserve \( R=\alpha(S - F)_+ - D \) (see H"urlimann(1991), Section 1), to guarantee in the long run the mean market return value of \( D=(S-d)_+ \), \( d\geq F \), such that \( E[P(S)]=\alpha F+E[(S-d)_+] \). This point is also implicitly contained in Section 4.

If one wants to guarantee the rate \( r \) using an arbitrary option strategy \( D \), one proceeds as follows. One needs a capital of \( K=\beta S_0 \), where

(5.8) \[ \beta = r_I^\ast PV(D)/(F_t - F). \]

Indeed the payoff of the corresponding capital protection option strategy is

(5.9) \[ P(S) = (\beta S_0 - PV(D))r_I^\ast + D = \beta F_t - r_I^\ast PV(D) + D = \beta F + D. \]

However the mean market return \( E[D] \) of the dividend can be guaranteed in the long run only if \( 0\leq D\leq \beta(S-F)_+ \). If this condition is not fulfilled, the reserve \( R=\beta(S-F)_+ \) of the corresponding finance strategy is negative, that is a credit position, which must be financed by borrowing capital. But then the guaranteed rate will be obviously less than the desired one. How is it possible to find "good" option strategies? One way is to proceed as follows. It is intuitively clear that a desirable condition is \( P(S)=\beta F+D\geq \beta F_0 \), which means that the payoff should at least be the payoff of the risk-free investment. This is equivalent to the condition
As a first "naive" stability criterion, one can try to maximize the probability of this event over the period of length $\tau$:

\begin{equation}
(5.11) \quad \Pr( D \geq r_\tau PV(D) ) = \max !.
\end{equation}

In some sense this criterion is dual to the one-period probability of loss criterion widely used in insurance rate-making. It states that if the pair $(X,P)$, $X$ the insurance claims, $P$ the risk premium, represents the insurance contract, then one requires that

\begin{equation}
(5.12) \quad \Pr( X \geq P ) \leq \varepsilon,
\end{equation}

where $\varepsilon>0$ is a sufficiently small probability level of non-solvability. More generally it would be interesting to consider also the "dual" criterion of the more sophisticated "ruin" stability criterion of classical risk theory.

Using computer programs, it is possible to evaluate the conditional probability given $S=S_\tau$ at the end of the period $\tau$, namely

\begin{equation}
(5.13) \quad \Pr( D \geq r_\tau PV(D) | S=S_\tau ).
\end{equation}

To illustrate we have computed the cases of a call $D=(S-d)$, and of a straddle $D=(S-d)_+ + (d-S)_- = S-dl$. To value the price of the call-option, let us use Black and Scholes(1973) model. In particular the unknown accumulated rate of return $S/S_0$ at the end of the period $\tau$ is lognormally distributed with parameters $\mu=\ln(r_\tau)-\frac{1}{2}\sigma^2$, $\sigma$ the volatility. The results are displayed in the next tables.
Table 1: conditional probabilities for the excess return, call option strategy $r_f=1.07$, $S_0=100$, $\tau=3/12$ (3 months)

<table>
<thead>
<tr>
<th>strike</th>
<th>conditional probabilities given $S=S_0$ in %</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>80</td>
</tr>
<tr>
<td>70</td>
<td>0.7</td>
</tr>
<tr>
<td>80</td>
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<tr>
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<td>0.3</td>
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<tr>
<td>110</td>
<td>0.0</td>
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<tr>
<td>120</td>
<td>0.0</td>
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<tr>
<td>130</td>
<td>0.0</td>
</tr>
<tr>
<td>140</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table 2: conditional probabilities for the excess return, straddle option
\( r_f = 1.07, S_0 = 100, \tau = 3/12 \) (3 months)

<table>
<thead>
<tr>
<th>strike</th>
<th>conditional probabilities given ( S = S_t ) in %</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>0.7 10.2 23.2 41.3 60.6 76.9 94.6</td>
</tr>
<tr>
<td>80</td>
<td>0.8 10.1 23.1 41.1 60.4 76.7 94.5</td>
</tr>
<tr>
<td>90</td>
<td>38.9 15.5 22.5 38.0 56.8 73.7 93.4</td>
</tr>
<tr>
<td>100</td>
<td>92.5 62.9 47.6 41.4 46.3 58.7 84.3</td>
</tr>
<tr>
<td>110</td>
<td>98.6 84.6 68.8 45.4 37.0 32.0 48.1</td>
</tr>
<tr>
<td>120</td>
<td>99.2 89.0 75.4 50.3 37.9 22.7 11.6</td>
</tr>
<tr>
<td>130</td>
<td>99.3 89.8 76.6 51.8 39.2 23.0 5.6</td>
</tr>
<tr>
<td>140</td>
<td>99.3 89.8 76.8 52.0 39.4 23.1 5.4</td>
</tr>
</tbody>
</table>


The rest of this paper is an abridged version of Hürlimann (1993).

In an economic environment subject to risk and uncertainty a mismatch between future values of assets and liabilities is almost always observed. For an effective financial management and/or performance analysis of investment portfolios, it may be useful to compare this mismatch with the corresponding difference on some reference investment used as a benchmark, e.g. a "Treasury bill", or a market index, or an "almost" efficient portfolio, etc. Our aim is to consider a quantity which permits to measure the deviation between future positions of assets and liabilities. A first study of this quantity is undertaken using various hedging schemes.

The following notations are needed:

\( A_t \): random value of assets at time \( t \)
\( L_t \): random value of liabilities at time \( t \)
\( R_t \): random accumulated rate of return at time \( t \) of a reference investment
\( S \): amount invested in a reference investment at time \( 0 \)
$r_L$: one-year accumulated rate of return to discount liabilities

$P[L_t]$: pricing principle such that $P[L_t]$ is the price of the liabilities at time 0

We shall assume that $r_L \leq r_f$ the risk-free rate, a condition which can be justified by option pricing theory for example.

The quantity of interest is defined by a coefficient of risk exposure between assets and liabilities relative to a reference investment at time $t$:

$$\xi_t = \frac{(E[A_t] - r_L P[L_t])}{S(E[R_t] - r_L)}.$$

**Example 1.1.** What is meant is best illustrated by the well-known CAPM. Suppose that $A_t = S_t$ is the price of a security at time $t$, $r_t = E[S_t]/S_0$ the one-year accumulated rate of return on this security, $r_L = r_f$, $S = S_0$, $L_t = S r_f$, $P[L_t] = S_0$, and $E[R_t] = r_M$ is the one-year accumulated rate of return on a market index. Then the coefficient of risk exposure at time $t=1$, namely $\xi_t = (r - r_f)/(r_M - r_f)$, identifies with the beta factor expressed by $\beta = \frac{\text{Cov}[S_t, SR_t]}{\text{Var}[SR_t]}$. The beta factor is also called systematic risk of the security relative to the market index.

Mathematically the beta factor is an illustration for a simple linear regression. The slightly more general method of hedging by sequential regression has been applied successfully for pricing options by Föllmer and Schweizer (1988). In the pricing theory of derivative securities other hedging schemes are widely used, e.g. hedging based on coefficients such as $\Delta$ (delta), $\Theta$ (theta), $\Gamma$ (gamma), $\Lambda$ (lambda) and $\rho$ (see e.g. Hull (1989), chap. 8). Our aim is to consider some hedging schemes, compare and combine them, and analyze their effect on the coefficient of risk exposure defined in (6.1). The obtained results provide information on the unknown expected value $E[A_t]$, or equivalently on the rate of return on financial assets, and are part of a relative asset pricing theory.

Only the important special case of an option of European type with strike price $E$ on an underlying asset with random price $S_t$ and exercise date $T$ is discussed. Of main interest is a call-option such that $A_T = L_T = (S_T - E) +$, $S = S_0$, $E[R_T] = r_T$. The price of the call-option at time 0 is denoted by $C(E) = P[L_T]$ and the corresponding put-price is denoted by $P(E)$. We denote by $F = S r_T^T$ the expected future price of the reference investment and by $F_L = S r_L^T$ its guaranteed future price, both prices valid at time $t=0$. The natural assumption $F = E[S_T]$ is made throughout.
Hedging schemes are classified by some characterizing property. For a first analysis we consider the following four hedging types.

**P-hedging scheme.** A "P-hedge" or perfect hedge is obtained if the financial risk encountered in handling an option can be completely eliminated. This approach is represented by the major breakthrough in option pricing theory summarized by Black and Scholes (1973). Main examples are Black-Scholes formula in a continuous time setting and the binomial option pricing model by Cox, Ross and Rubinstein (1979).

**Δ-hedging scheme.** The notion of "Δ-hedge", speak delta hedge, is derived from Black-Scholes analysis. Holding a short position in a derivative security with price \( P=P(S) \) and holding long \( \Delta=\partial P/\partial S \) shares of the security defines an instantaneously riskless portfolio.

**R-hedging scheme.** In general and because a continuous trading strategy cannot be realized at 100%, the risk involved in handling an option cannot vanish completely. Applying a sequential regression scheme, Föllmer and Schweizer (1988) have shown how the a priori risk (as measured by the variance) can be reduced to an irreducible intrinsic part. For a single period model, the method to be applied is equivalent to a simple linear regression. For example a "simple R-hedge", speak simple regression hedge, for a call-option is characterized by its coefficient of risk exposure, namely

\[
\xi^R_T = (E[(S_T - E)_+]) - r^T C(E))/(F - F_L) = \text{Cov}[(S_T - E)_+,S_T]/\text{Var}[S_T].
\]

This coefficient represents the systematic risk of the contingent "stop-loss" claim \((S_T - E)_+\) relative to the security price \(S_T\). Details follow in Section 7.

**S-hedging scheme.** The sequential regression scheme, as well as the more general method developed by Föllmer and Sonnemann (1986), are closely related to the notion of mean self-financing trading strategy. Another mean self-financing model is found in Hürlimann (1991). The author considers "financial risk protection models", which guarantee the future price \(F_L\) with probability one, while the expected future price \(F\) is only guaranteed in the mean. Among all feasible models a special stable model can be summarized by a relationship of the form

\[
F - F_L = E[(S_T - F_L - B)_+],
\]
where $B$ plays the role of an amount to be reserved at time $T$ in order to be able to cover in the mean the financial risk involved in the guarantee. Setting $B = E - F_L$ with $E > F_L$ for a positive reserve, one sees that actually $F - F_L = E[(S_T - E)_{+}]$. For the coefficient of risk exposure, denoted by $\xi^s_T$, this means that

$$1 - \xi^s_T = \frac{\tau_L^T C(E)}{(F - F_L)}.$$  

Due to a stable property of the model (see H"{u}rlimann(1991), Section 1), the relation (6.4) defines a so-called "$S$-hedge", speak stable hedge.

7. R-hedging and the fair price of options in a single period model.

It is assumed that financial assets do not pay dividends. To determine the price of a call-option, one applies the R-hedging scheme as described in F"{o}llmer and Schweizer(1988). In a single period model it suffices to construct a dynamical arbitrage portfolio, which realizes the call at expiration. Consider the following portfolio $\phi$:

- buy $\xi$ units of the risky asset with market price $S$ at time $t=0$
- borrow $\eta$ units of money at the accumulated rate $r_L$

Denoting by $V_t(\phi)$ the value of the portfolio $\phi$ at time $t$, one has

$$V_0(\phi) = \xi S + \eta, \quad V_T(\phi) = \xi S_T + \eta_T = H,$$

where $H=(S_T-E)_{+}$ is the value of the call-option at expiration date $T$ given the market price $S_T$, and $\eta_T$ is the new value of the loan needed to match the call exactly. Consider the costs of this dynamical trading strategy. Denote by $C_t(\phi)$ the accumulated costs at the rate $r_L$. Clearly one has $C_0(\phi) = V_0(\phi)$. At time $T$ the supplementary costs for adjusting the borrowed money are given by $C_T(\phi)-r_L^T C_0(\phi)=\eta_T-r_L^T \eta$. With (7.1) one gets

$$C_T(\phi) - r_L^T C_0(\phi) = r_L^T [r_L^T H - V_0(\phi) - \xi \delta S],$$

where one uses the notation $\delta S=r_L^T S_T - S$. By definition of the R-hedging scheme one chooses now the optimal trading strategy $(\xi,\eta)$, which minimizes the remaining risk at initial time, as measured by the expected supplementary quadratic costs of the trading strategy. This is equivalent to solving the following optimization problem:
(7.3) \[ R = E[(C_t(\phi) - r_L^T C_0(\phi))^2] = r_L^{-2T} E[(r_L^{-T} H - V_0(\phi) - \xi \delta S)^2] = \min. \]

Applying a "simple" linear regression, which consists to find the best linear estimate of \( r_L^{-T} H \) based on \( \delta S \), one finds the solution

(7.4) \[ \xi = \frac{\text{Cov}[H, S_T]}{\text{Var}[S_T]}, \quad V_0(\phi) = r_L^{-T} E[H] - \xi \delta S. \]

This optimal initial value of the portfolio is denoted by \( C(E) \) and is called the fair price of the call-option. Rewrite (7.4) to get

(7.5) \[ r_L^{-T} C(E) = S_L(E) - \xi (F - F_T), \]

where \( S_L(E) = E[(S_T - E)_+] \) is the so-called stop-loss premium of the risky asset \( S_T \) at time \( T \) to the priority \( E \). An immediate check shows that

(7.6) \[ r_L^{-T} E[C_T(\phi)] = C_0(\phi), \]

which means that the random variable representing the discounted accumulated costs is a martingale. One says that the optimal trading strategy is mean self-financing. Furthermore the minimum remaining risk involved in handling a call-option following the above R-hedging scheme equals

(7.7) \[ R_{min} = E[(C_T(\phi) - r_L^T C_0(\phi))^2] = (1 - \rho^2) \text{Var}(E) = \frac{(1 - \rho^2)}{\rho^2} \xi^2 \text{Var}[S_T], \]

where \( \text{Var}(E) = \text{Var}[(S_T - E)_+] \) denotes the variance of the stop-loss claim \( (S_T - E)_+ \), and \( \rho \) is the correlation coefficient between \( (S_T - E)_+ \) and \( S_T \).

**Remark 7.1.** Among all possible choices for the random variable \( S_T \), the R-hedging scheme with minimum remaining risk \( R_{min} = 0 \) characterizes the diatomic random variable models. First it is an exercise to check that \( R_{min} = 0 \) if \( S_T \) is a diatomic random variable. Second if \( R_{min} = 0 \), then \( \rho^2 = 1 \), which implies (see e.g. Fisz(1973), Satz 3.6.5, p. 112) that \( \text{Pr}((S_T - E)_+ = a S_T + b) = 1 \), \( a, b \) constants. But then \( S_T \) must be a diatomic random variable. In other words a R-hedge is a P-hedge if and only if \( S_T \) is a diatomic random variable. In multiple period models this leads to the binomial option pricing model of Cox, Ross and Rubinstein(1979).
8. **Options and synthetic securities.**

For the Δ-hedging method and the R-hedging method we show how to construct *costless hedges* for a given long position of a risky asset by combining linearly call- and put-options with the same underlying asset but possibly different exercise prices. In case of the R-hedging method these hedges reduce the a priori risk of the underlying risky asset to its intrinsic minimum. The results obtained constitute the fundamental instrument used in our further analysis of hedging schemes in this paper.

8.1. **The Δ-hedging scheme.**

In general a portfolio of risky assets is said to be *Δ-hedged* by a portfolio of hedging instruments if the Δ-coefficients of both portfolios are identical. Since the Δ-operator is linear, it follows that the stated problem is solved by a system of two equations in two unknowns. Given is a long position of one unit of the risky asset with market price S at time 0. Suppose that a Δ-hedged portfolio contains a short position of α units of a call-option with exercise price E₁ and a short position of β units of a put-option with exercise price E₂. A *costless Δ-hedge* satisfies the linear system of equations ((−)ₘ denotes a partial derivative with respect to S)

\[
\begin{align*}
\alpha C(E₁) + \beta P(E₂) &= 0, \\
\alpha C(E₁)ₘ + \beta P(E₂)ₘ &= 1,
\end{align*}
\]

with solution

\[
\begin{align*}
\alpha &= \frac{P(E₂)C(E₁)ₘ - C(E₁)P(E₂)ₘ}{[C(E₁)ₘP(E₂)ₘ - P(E₂)C(E₁)ₘ]}, \\
\beta &= \frac{C(E₁)ₘP(E₂)ₘ - P(E₂)C(E₁)ₘ}{[C(E₁)ₘP(E₂)ₘ - P(E₂)C(E₁)ₘ]}.
\end{align*}
\]

8.2. **The R-hedging method.**

One proceeds as in Section 7 and the same notations are used. The R-hedged portfolio for the given risky asset is assumed to contain the same hedging instruments as the Δ hedged portfolio in section 8.1. Let φ denote the portfolio mix containing the long position of the risky asset and the R-hedged portfolio. We use the simpler notations C₁ = C(E₁), P₂ = P(E₂). To obtain a *costless R-hedge*, the value of the portfolio φ should be as follow:

\[
\begin{align*}
\text{at time } t=0 : \quad & V₀(φ) = S₀ - αC₁ - βP₂ = S₀, \text{ or equivalently} \\
\text{at time } t=T : \quad & αC₁ + βP₂ = 0 \text{ (no-cost relation),}
\end{align*}
\]
In general it will not be possible to satisfy these equations for given $E_1 \neq E_2$ (an exception is the case of a diatomic random variable for $S_T$). For a given exercise price $E$ assume the following relations hold (see Section 7):

\begin{align}
(8.5) & \quad SL(E) = r_L T C(E) + \xi(F - F_L), \quad \xi = \text{Cov}[(S_T - E)_+, S_T] / \text{Var}[S_T], \\
(8.6) & \quad SL'(E) = E[(E - S_T)_+] = E - F + SL(E) = r_L T P(E) + \eta(F - F_L), \\
& \quad \eta = \text{Cov}[(E - S_T)_+, S_T] / \text{Var}[S_T], \\
(8.7) & \quad \xi - \eta = 1, \\
(8.8) & \quad r_L T (P(E) - C(E)) = E - F_L. \quad \text{(put-call parity relation)}
\end{align}

To find values of $\alpha$, $\beta$ matching (8.4) in an optimal way, consider as in Section 7 the accumulated costs $C_t(\phi)$ at the rate $r_L$. At time $T$ the supplementary costs needed to hedge the portfolio $\phi$ are given by

\begin{align}
(8.9) & \quad C_T(\phi) - r_L T C_0(\phi) = S_T - \alpha(S_T - E)_+ - \beta(E_2 - S_T)_+ - F_L.
\end{align}

The optimal hedging strategy $(\alpha, \beta)$ which minimizes the expected supplementary quadratic costs is given by the solution to the following constrained optimization problem:

\begin{align}
(8.10) & \quad R = E[(S_T - F_L - \alpha(S_T - E)_+ - \beta(E_2 - S_T)_+)^2] = \min.
\end{align}

under the constraint $\alpha C_1 + \beta P_2 = 0$. Since $\beta = -(C_1/P_2)\alpha$, this problem is equivalent to the one-parametric problem

\begin{align}
(8.11) & \quad R = E[(S_T - F_L - \alpha X)^2] = \min, \quad \text{where} \\
(8.12) & \quad X = (S_T - E_1)_+ - (C_1/P_2)(E_2 - S_T)_+.
\end{align}

The solution follows again by linear regression. The best linear estimate of $S_T$ based on $X$ yields the optimal values:

\begin{align}
(8.13) & \quad \alpha = \text{Cov}[S_T, X] / \text{Var}[X], \quad F_L = F - \alpha E[X].
\end{align}

The last equation represents actually a best estimate of $r_L$:

\begin{align}
(8.14) & \quad \ln(r_L) = (1/T) \ln\{E[S_T - \alpha X] - \ln\{S_0\}\}.
\end{align}
The remaining risk of the optimization procedure is given by

\[ R_{\text{min}} = \text{Var}[S_T - \alpha X] = \text{Var}[S_T] - (1 - \rho(S_T, X)^2), \]

where \( \rho(S_T, X) \) is the correlation coefficient between \( S_T \) and \( X \).

In the special case \( E_i = F_i \), one gets from relation (8.8) that \( C_i = P_i \), hence \( \beta = -\alpha \). It follows that \( X = S_T - F_i, \ \alpha = 1 \) and \( R_{\text{min}} = 0 \), which is a well-known result (e.g. Sharpe(1985), chap. 17).

### 8.3. The costless R-hedge with equal exercise prices.

For the purpose of this paper it will suffice to consider the special case \( E = E_i = F_i \) (\( E = F_i \) is trivial as seen at the end of Section 8.2). To simplify notation set \( C = C(E) \), \( P = P(E) \), and assume (8.5) to (8.8) hold. To construct a costless R-hedge, one has to evaluate \( \alpha \) in (8.13), where \( X = (S_T - E)_+ - Q - (E - S_T)_+ \), \( Q = (E - F_i) / P \). Rewritten as \( X = (1 - Q)(S_T - E) + Q(F_i - E) \), one gets \( E[X] = (1 - Q)S(T) - Q(F_i) \). Inserted in (8.13) one gets

\[ F - F_i = \alpha \{ (1 - Q)S(T) - Q(F_i) \}. \]

The relation (8.8) can be rewritten as \( r^T_C = \{ Q/(1 - Q) \} - (E - F_i) \). Inserted in (8.5) one gets after straightforward calculation

\[ (F - F_i)(Q + \xi(1 - Q)) = (1 - Q)S(T) + Q(F_i - E). \]

Through comparison with (8.16) it follows that

\[ \alpha = 1/[Q + \xi(1 - Q)], \]

which is the solution to our costless R-hedging problem. It depends on the systematic risk of the call-option and on the ratio call- to put-price.


Notations are as in section 8. Imagine that a long position of one unit of a risky asset should be hedged against a portfolio containing short positions of \( \alpha \) units of a call-option and \( \beta \) units of a put-option with the same exercise prices \( E \) and the given risky asset as underlying asset. In this situation we analyze under which condition the hedging portfolio constitutes
a costless \((\Delta, R)\)-hedge, that is a simultaneous costless \(\Delta\)-hedge and \(R\)-hedge. The condition for a costless \(\Delta\)-hedge has been derived in section 8.1. With \(Q = C/P, P_s = C_s - 1\), one gets from (8.2) that

\[
\alpha = P/(PC_s - CP_s) = 1/(Q + C_s(1 - Q)).
\]

This is to be compared with the solution (8.18) for a costless \(R\)-hedge. A simultaneous costless \((\Delta, R)\)-hedge is only possible if the systematic risk of the call-option equals the hedge ratio of the call, that is

\[
\xi = C_s.
\]

**Remark 9.1.** In case this condition is fulfilled, the rate of return on the financial risky asset satisfies the relative asset pricing relationship

\[
\xi(F - F_d) = SL(E) - r_t^T C(E), \quad \xi = C(E)_s.
\]

Observe that the net stop-loss premium \(SL(E)\) usually depends on the unknown rate of return \(r\). Therefore the relation (9.3) is an implicit equation for \(r\). In applications \(C(E), C(E)_s\) can be evaluated using market information and/or option pricing models, and \(SL(E)\) can be evaluated from a mathematical model for financial asset prices.

In view of the outlined perspective, it is important to analyze under which model assumptions the condition (9.2) is effectively or approximately satisfied. Only the simplest fundamental cases are considered. In this Section we discuss the binomial and the lognormal case.

**9.1. The single-period binomial model.**

Suppose that the asset price \(S_T\) at the end of the period will be either \(uS\) or \(dS\) with probability \(q = \text{Pr}(S_T = uS)\). Assume that \(T = 1\) and \(dS < F, F_d = Sr_T, E < uS, r_f\) the risk-free rate. Let \(\sigma\) be the standard deviation of the return such that \(\text{Var}[S_T] = S^2\sigma^2\). From the properties of a diatomic random variable one has necessarily the relations

\[
q = (r - d)/(u - d), \quad \sigma^2 = (u - r)(r - d).
\]

To value the call-option the objective probability measure \(q\) has to be
replaced by the risk-neutral probability measure

\[(9.5) \quad p = (r_d)/(u-d).\]

Under these assumptions one has

\[(9.6) \quad C(E) = r_t^{-1}p(uS-E), \quad C(E)_S = r_t^{-1}pu.\]

On the other hand straightforward calculation shows that

\[(9.7) \quad SL(E) = q(uS - E), \quad SL(E)SL^c(E) = q(1-q)(uS - E)(E - dS), \quad Var(E) = q(1-q)(uS - E)^2.\]

It follows that

\[(9.8) \quad \xi = \text{Cov}[(S_t-E)_+, S_t]/\text{Var}[S_t] = (\text{Var}(E) + SL(E)SL^c(E))/\sigma^2S^2 = q(1-q)(uS-E)(u-d)/\sigma^2S^2 = (u - E/S)/(u - d).\]

Under our assumptions one checks that a R-hedge is possible only if \(r_e = r_t\)
(or \(d=0\)). In particular the R-hedging relationship does not provide any additional information about the rate of return \(r\) of a financial asset. A comparison shows that \(\xi = C(E)_S\) if and only if

\[(9.9) \quad E/S = r_t^{-1}du.\]

This means that for a simultaneous costless (\(\Delta R\))-hedge, the exercise price per unit of asset price should be equal to the discounted product of the up and down relative asset price movements at the end of the period.

9.2. The lognormal case.

Assume that the asset price \(S_t\) at time \(T\) is lognormally distributed with parameters \(\mu, \sigma\), with probability density function

\[(9.10) \quad f(x) = \text{Pr}(S_t=x) = \{1/(2\pi)^\frac{1}{2}\sigma x\}^{-\exp[-(\ln x - \mu)/2\sigma^2]}\].

The mean and the variance of the asset price are given by

\[(9.11) \quad F = E[S_t] = \exp(\mu+\frac{1}{2}\sigma^2), \quad \text{Var}[S_t] = F^2(e^{\sigma^2}-1).\]
Denote by $N(x)$ the cumulative distribution function of the standard normal variate and set $\Phi(x)=N'(x)$. The following formulas hold:

\begin{align}
(9.12) \quad SL(E) &= FN(x) - EN(x-\sigma), \quad x=\frac{\sigma}{2} + \ln(F/E)/\sigma, \\
(9.13) \quad \text{Cov}[(S_t-E)_+, S_t] &= F \cdot \{-e^{\sigma^2}SL(\Phi(\sigma)) - SL(E)\}.
\end{align}

On the other hand the Black-Scholes call-option price at time 0 is given by

\begin{equation}
(9.14) \quad C(E) = SN(y) - rTEN(y-\sigma), \quad y=\frac{\sigma}{2} + \ln(F/E)/\sigma, \quad F_S rT.
\end{equation}

It follows that the partial derivative with respect to $S$ equals

\begin{equation}
(9.15) \quad C(E)_S = N(y) + \{\Phi(y) - (E/F_S)\Phi(y-\sigma)\}/\sigma.
\end{equation}

Using the definition of $y$ in (9.14) one sees without difficulty that $F_S \Phi(y)=E \Phi(y-\sigma)$. It follows that the delta hedge ratio of a call option is

\begin{equation}
(9.16) \quad \Delta=C(E)_S=N(y).
\end{equation}

The condition (9.2) for a costless $(\Delta, R)$-hedge $\text{Cov}[(S_t-E)_+, S_t]=\Delta - \text{Var}[S_t]$ can be rewritten using the above formulas (9.11) to (9.13) as

\begin{equation}
(9.17) \quad Fe^\sigma[N(x+\sigma)-N(y)] = F[N(x)-N(y)] + E[N(x)-N(x-\sigma)].
\end{equation}

To solve this equation, use the approximation $N(\xi) = \frac{1}{2} + \xi \Phi(0)$ (valid for small $\xi$) to get the following unique approximate exercise price:

\begin{equation}
(9.18) \quad E = F \cdot \{-e^{\sigma^2} + (1/\sigma^2) \cdot \ln(F/F_S) - (e^{\sigma^2}-1)\}.
\end{equation}

If $\sigma$ is small, then $e^{\sigma^2}-1 \approx \sigma^2$, hence $E = F \cdot \{-e^{\sigma^2} + \ln(F/F_S)\}$.

**Example 2.1.** In the limiting important practical case $F=F_S$ (*no-arbitrage* condition of Financial Economics), one has $E=Fe^\sigma$ and the relative asset pricing relationship (9.3) yields the formula

\begin{equation}
(9.19) \quad F_L = Fe^\sigma \cdot \{[1-N(3\sigma/2)]/[1-N(\sigma/2)]\}.
\end{equation}

If $\sigma$ is small, then one has as above approximately $N(3\sigma/2)=3N(\sigma/2)-1$ and $N(\sigma/2)=\frac{1}{2}(1 + \sigma/(2\pi)^{1/2})=\frac{1}{2}(1 + 0.4\sigma)$. With $T=1$ it follows that
Using a simple S-hedge with exercise price \(E = F = F_t\) (defining equation \(F_t - SL(F_t)\)), one obtains the following alternative formula for a rate of return to discount liabilities (see Hürlimann (1991), formula (3.9)):

\[
(9.21) \quad r_L^* = r_t - 2\left(1 - N(\sigma/2)\right) = r_t(1 - 0.4\sigma).
\]

A direct comparison shows that whatsoever \(\sigma\), one has always \(r_L^* \geq r_L\). Numerically one has for example \(r_t = 0.86\) \(r_t, r_L^* = 0.92\) \(r_t\) if \(\sigma = 0.2\) and \(r_t, r_L^* = 0.93\) \(r_t, r_L^* = 0.96\) \(r_t\) if \(\sigma = 0.1\). A similar comparison can be made for a S-hedge with exercise price \(E = F, e^{\sigma}\). The solution of the S-hedging problem \(F_t - SL(F_t)\) yields:

\[
(10.1) \quad F_t - SL(F_t) = r_t - \frac{1}{2} \{N(\sigma(2) + e^{\sigma}(2 - 3N(\sigma(2)))\} = r_t - \frac{1}{2} \{1 + 0.4\sigma + e^{\sigma}(1 - 1.2\sigma)\}.
\]

If \(\sigma = 0.2\), then \(r_t = 0.94\) \(r_t\) and if \(\sigma = 0.1\), then \(r_L^* = 0.96\) \(r_t\).

### 10. A simple binomial \((R,S)\)-hedging scheme.

The conditions for a \(R\)-hedge in a single-period binomial model with \(T=1\) have been derived in section 9.1 (formulas (9.4) to (9.8)). In particular one has necessarily \(r_L = r_t\). For a simultaneous \((R,S)\)-hedge one must satisfy the additional constraint

\[
(10.1) \quad F_t - SL(E), B = E - F_t > 0.
\]

In the usual notations and with \(e = E/S\), this is equivalent to the condition

\[
(10.2) \quad r(e - d) = r(u - d) + d(e - u).
\]

To illustrate what implications such a model has, let us specialize to the simple case \(e = r\). In this situation \(B = (r - r_t)S\) can be interpreted as the price for bearing the financial risk involved in holding the risky asset. Against payment of the "risk premium" \(B\), the investor expects a rate of return which lies \(B/S = r-u\) above the risk-free rate. Observe that if one imposes this "risk premium" interpretation to \(B\), then one obtains necessarily the unique simple case \(e = r\).

Let us fix the probability \(q\) and try to express \(d, u\) in terms of \(q\) and \(r\). By definition of \(q\) one has \(r = d + q(u-d)\). From the variance constraint \(\sigma^2 = (u-r)(r-d)\), one gets \(r = u - \sigma^2/q(u-d)\). It follows that
Now insert (10.3) and (10.4) into (10.2) for $e=r$ to get after simplification

\[ r_q = r_r - d(1-q). \]

Since $r = uq + d(1-q)$ one has further

\[ d + q^2(u - d) = r_r. \]

Taking into account (10.3) the following binomial model follows:

\[ d = r_r - \sigma \left( \frac{q}{(1-q)^{1/2}} \right), \quad u = r_r + \sigma \left( \frac{1+q}{(1-q)^{1/2}} \right), \quad r = r_r + \sigma \left( \frac{q}{(1-q)^{1/2}} \right). \]

**Example 10.1.** The maximum rate of return on a financial asset is attained when $q = \frac{1}{2}$. Assume this might be possible for some market index. In this situation the corresponding simple binomial $(R,S)$-hedge is specified by

\[
d = r_r - \sigma \left( \frac{q}{(1-q)^{1/2}} \right), \quad u = r_r + \sigma \left( \frac{1+q}{(1-q)^{1/2}} \right), \quad r = r_r + \sigma \left( \frac{q}{(1-q)^{1/2}} \right).
\]

**Remark 10.1.** In this situation the R-hedge relation $SL(F) = r_r C(F) + \xi(F - F_r)$ takes the form $\frac{1}{2} \sigma_M + \frac{1}{2} \sigma_M + \frac{1}{2} (r_M - r_r) S$. The expected value of the stop-loss contingent claim separates half and half between the accumulated option price and the component for bearing the systematic risk of the market.

Consider any other investment with rate of return $r$. If all investors adopt simple binomial $(R,S)$-hedging schemes and if the market supply for the risk-free investment vanishes, then the rates $r$ and $r_M$ are linked by the "risk-adjusted" CAPM (see Example 4.1 and H"urlimann(1991), Section 6):

\[ r = r_r + \frac{1}{2} \sigma_M + (r_M - r_r - \frac{1}{2} \sigma_M) \beta, \]

with $\beta$ some ex ante defined beta factor. In the model (10.8) one gets

\[ r = r_M = r_r + \frac{1}{2} \sigma_M. \]

Therefore the rate of return on an arbitrary financial asset equals the
maximum market rate of return. How does one specify the corresponding simple binomial \((R,S)\)-hedging scheme? \((10.10)\) is only possible if

\[
(10.11) \quad r = r_t + \sigma [q(1-q)]^{1/2} = r_t + \frac{1}{2} \sigma_M.
\]

It follows that

\[
(10.12) \quad q = \frac{1}{2} \left(1 + \left[1 - (\sigma_M / \sigma)^2\right]^{1/2}\right).
\]

From \((9.8)\) one sees that the systematic risk of a call-option with exercise price \(E = Sr\) equals \(\xi = (u-r)/(u-d) = 1-q\). Suppose \(\xi \leq \frac{1}{2}\), hence \(q \geq \frac{1}{2}\). In this case the simple binomial \((R,S)\)-hedging scheme of an arbitrary asset is characterized by the formulas \((10.7)\) with the probability \(q\) equal to

\[
(10.13) \quad q = \frac{1}{2} \left(1 + \left[1 - (\sigma_M / \sigma)^2\right]^{1/2}\right).
\]

**Remark 10.1.** In the same way it is possible to define other simultaneous \((R,S)\)-hedging schemes and \((\Delta,S)\)-hedging schemes. The implications of a simultaneous \((\Delta,R,S)\)-hedging scheme, that is a costless \((\Delta,R)\)-hedge combined with a \(S\)-hedge have been discussed in Hürlimann(1993).

**References.**


