2nd AFIR Colloquium 1991, 2: 415-452

Option Pricing Formulae

A. D. Smith

Bacon & Woodrow, St Olaf House, London Bridge City, London SE1 2PE, United Kingdom

Summary

The author develops formulae for the pricing of options, starting from the elementary binomial model, then progressing on to the Black-Scholes formula. The link between the hedging construction and the actuarial present value derivation is explained. Techniques are developed to allow for the possibility of early exercise. The paper is divided into the following sections:

Introduction

1. Background
2. The Binomial Model
3. Present Value Option Pricing
4. Optimal Exercise
5. Hedging and Arbitrage
6. Basic of Continuous Time
7. Hedging and Arbitrage

Bibliography

Résumé

Formules de Tarification d'Options

L'auteur développe des formules pour la tarification des options, en partant du modèle binomial élémentaire, puis en progressant jusqu'à la formule Black-Scholes. Le rapport entre la construction "hedging" et la dérivation actuarielle de la valeur actuelle est expliqué. Des techniques sont mises au point pour permettre la possibilité d'un exercice précoce. L'article est constitué des sections suivantes:

Introduction

1. Arrière-plan.
2. Le modèle binomial
3. Tarification d'option de valeur actuelle
4. Exercice optimal
5. Hedging et arbitrage
6. Bases du temps continu
7. Hedging et arbitrage

Bibliographie

INTRODUCTION

In 1973, Black & Scholes published a paper which gave a pricing model for options. It is generally agreed to be a good model, and their price formula has had considerable influence on the traded options market. In addition, applications have been found for the formula which do not involve options explicitly, for example in the valuation of unit linked life policies with guaranteed sum assured. As a result, many actuaries find themselves applying a formula which they don't understand properly, and which they have never seen demonstrated. This paper aims to correct this situation.

The principal obstacle to the Black-Scholes formula being understood was the level of mathematics involved. Two more recent developments have eased the situation:

(i) In 1979 Cox, Ross and Rubenstein introduced a discrete-time analogue of the model originally used by Black & Scholes. Stochastic integrals were replaced by finite sums, and the theory could be developed from an elementary perspective. The continuous-time version can be recovered by letting the step size tend to zero.

(ii) Also in 1979, Harrison and Kreps published a paper in which they established the Black-Scholes formula using Girsanov's theorem on 'changes of measure.' This method is at first sight even less accessible mathematically than the original method used by Black and Scholes. Conceptually however, it is in principle similar to the process of choosing an appropriate actuarial basis, and then carrying out actuarial valuations. This is the approach I shall adopt here, with appropriate modifications to make the maths more tractable.

This paper derives option pricing formulas in three distinct stages, as follows:

(i) Manipulation of present values, which I hope actuaries will find easy.

(ii) Dynamic Programming arguments. This means repeated optimisation in time. These are used to handle exercise strategies. They will be unfamiliar to many actuaries, but are intuitively plausible and not technically difficult.

(iii) Mathematical Justification. These sections are hard, and mathematically demanding. Since the conclusions are intuitively right, many practitioners may want to take these sections on trust.

The layout of the paper is as follows:

	Discrete time	Continuous time
Present Value manipulations	Chapters 2,3	Chapter 6
Dynamic Programming	Chapter 4	
Basis Justification	Chapter 5	Chapter 7

BACKGROUND

1.1 Contracts Available

A call option gives the bearer of the option the right (if he wishes to exercise it) to purchase from the writer of the option a unit of some underlying security (e.g. an ordinary share) at a fixed price (the exercise price) at a fixed date (the expiry date) in exchange for a fee (or premium) paid by the bearer to the writer.

A put option has a similar definition, except that the bearer has a right to sell a unit of the underlying security to the writer, instead of purchasing it.

A European option restricts the exercise of the option to a fixed date, as above. An American option allows the bearer the right to exercise it on or before the expiry date. This terminology is historical in origin. Both types can now be brought or sold either side of the Atlantic.

1.2 General Assumptions

All option-pricing formulae make assumptions about the market. The principal ones are as follows:

(i) Bid and offer prices on securities are the same
(ii) Frictionless trading - any transaction may be made at no cost at current market prices
(iii) There is no commission, expenses, stamp duty or tax
(iv) There are no dividends payable (or the

security does not go ex-dividend) before the expiry date.

1.3 The Results

Taking the assumptions in 1.2 and an appropriate model for the fluctuation in security prices, it turns out that the premiums on all types of option are uniquely determined, and do not depend on the nature of the participants in the market, provided that they are operating rationally. For example, the formula for the premium on a European call option is

$$P_t = x\Phi\left(\frac{\log(x/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\mathrm{Log}(x/K) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where
x = security price
t = time now
K = exercise price
T = expiry date
r = risk-free force of interest.
σ = security volatility
Φ = cumulative normal distribution function.

1.4 Value of Options at Expiry

Consider a European call option on the expiry date. Suppose K is the exercise price and x is the security price.

If $x > K$ it is in the bearer's interest to exercise the option. Indeed, since it costs him K to buy a security worth x, the contract is worth x-K to him.

If on the other hand, $x \leq K$, the bearer would be better off to buy a security in the market if he wanted it at all. In this case, the option expires worthless.

To summarise, if we define z^+ by

$$z^+ = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases}$$

then the value of the option on expiry is call value = $(x - K)^+$.

Similarly, we can argue that a put option will only be exercised if $x < K$, in which case the

value is $K - x$ to the bearer, so put value = $(K - x)^+$

Note that $(x - K)^+ + K = \max(X, K) = (K - x)^+ + x$,
so
call + cash = put + security
on expiry.

1.5 Revision of Bases and Present Value

I mentioned previously that the approach I was going to adopt involved an approach akin to the present value calculations familiar to actuaries. In its simplest form, an actuarial basis consists of the following elements:

(a) an assumed rate of interest i

(b) an assumed probability model, Q, for any relevant random effects.

Neither (a) nor (b) need to correspond to a best estimate of the real world. Indeed it is often the job of the actuary to make what might be considered prudent assumptions about these factors. We will use P to denote the 'true' probability and Q for the probability used in the actuarial basis.

The present value of a random payment X at time t from now is defined to be

$$PV = (1+i)^{-t} E^Q(X)$$

where E^Q denotes the expected value assuming the probability model Q.

The present value of a series of payments is defined to be the sum of the individual present values. A payment at a random time can be viewed as a sum of random payments, each taking the value 0 unless the payment is made at that time.

The property of present value that makes it so useful is its consistency over time. For example, suppose $s < t < u$. Consider a payment of a (random) sum X at time u. We can calculate the present value of this at time t. Taking present values at time s of (i) the payment X at time u and (ii) the equivalent payment at time t, we find that both give the same answer, irrespective of when t is.

This consistency is a form of what is known as a "no-arbitrage" condition. The process of taking present values is essentially the most

general pricing procedure with this consistency property.

2. The Binomial Model

This model, devised by Cox, Ross and Rubenstein, is a simple discrete model which contains all the important features and none of the tedious technicalities of the continuous models used in practice. The continuous models can be obtained as the limit of this model as the time interval tends to be zero, but in this paper I have derived the continuous models independently.

2.1 Model Specification

The model makes the following assumptions:

(a) There is a constant interest rate i payable on cash deposits and loans.

(b) The security price is S_t on day t, and satisfies

$$S_t = \begin{cases} (1+g)S_{t-1} & \text{either} \\ (1+b)S_{t-1} & \text{or} \end{cases}$$

each with positive probability, and $b < i < g$. We say that day t is "good" in the first case, "bad" in the second.

2.2 The Basis

First of all we choose a basis for valuation of the options as follows:

(a) the assumed rate of interest is the true rate, i.

(b) the probability of good and bad days are

$$Q\ (\text{good day}) = \frac{i-g}{g-b}$$

$$Q\ (\text{bad day}) = \frac{g-i}{g-b}$$

all days being independent of each other.

There is of course no guarantee at the outset that Q is a true representation of the real world, but we shall demonstrate that the Q chosen above has certain characteristics which help to define how the real world will value the option.

2.3 Example

Suppose $g = 30\%$

$$i = 10\%$$
$$b = -10\%$$

Then

$$Q\ (\text{good day}) = 0.5$$
$$\text{and} \quad Q\ (\text{bad day}) = 0.5.$$

Notice that we have said nothing about P except that all possible transitions have positive probability under P. This observation will recur, and we will see that a large amount of probabilistic information is actually irrelevant when setting option prices.

2.4 Consistency

Consider a contract which will pay a sum S_T at a fixed time T, and let t be the time now.

The way to arrange such a contract is to buy a share now, and sell it at time T. The market value of this contract is the current share price S_t.

We calculate the actuarial present value as follows:

The number of good days between now and time T will be distributed binomially under Q with parameters T-t and (i-b)/(g-b)
Thus, for $0 \leq n \leq T-t$

$$Q(n\ \text{gooddays}) = \binom{T-t}{n}\left(\frac{i-b}{g-b}\right)^n\left(\frac{g-i}{g-b}\right)^{T-t-n}$$

In this case the share price is

$$S_T = (1+g)^n(1+b)^{T-t-n}S_t$$

So actuarial present value =

$$(1+i)^{-(T-t)}\sum_{n=0}^{T-t}\binom{T-t}{n}\left(\frac{i-b}{g-b}\right)^{n}\left(\frac{g-i}{g-b}\right)^{T-t-n}(i+g)^{n}(1+b)^{T-t-n}S_t$$

$$=(1+i)^{-(T-t)}\sum_{n=0}^{T-t}\binom{T-t}{n}\left(\frac{(i-b)(1+g)}{g-b}\right)^{n}\left(\frac{(g-i)(1+b)}{g-b}\right)^{T-t-n}S_t$$

$$=(1+i)^{-(T-t)}\left(\frac{(i-b)(1+g)}{g-b}+\frac{(g-i)(1+b)}{g-b}\right)^{T-t}S_t$$

$$=(1+i)^{-(T-t)}\left(\frac{(1+i)(g-b)}{g-b}\right)^{T-t}S_t$$

$$=S_t$$

So the market value is equal to the present value. In this case, we say that S_t is <u>priced consistently</u> with the basis. Of course, the basis was constructed so that this would work.

2.5 Risk Aversion and Prudence

In the literature the market is <u>risk-averse</u> if the market value of the share is less than the discounted expected value (under the 'true' probability model) to reflect the risk involved,

i.e. $S_0 < (1+i)^{-1}E^P(S_1)$ (*)

The basis is <u>prudent</u> if it is pessimistic about the probability of a good day, i.e.

Q (day 1 is good) < P (day 1 is good) (**)

Note that (**) is equivalent to

$E^Q(S_1) < E^P(S_1)$

and since S is priced consistently with Q,

$S_0 = (1+i)^{-1}E^Q(S_1)$

(**) is equivalent to

$S_0 < (1+i)^{-1}E^P(S_1)$

which is (*).

Thus, in this context, actuarial prudence can be seen to be equivalent to risk aversion in the market, and vice versa.

3. Present Value Option Pricing

If a security is priced consistently with a basis, it seems reasonable that options should also be priced according to the basis. In Chapter 5, we will see that this assertion is equivalent to a no-arbitrage condition, but for the moment we will proceed taking present value pricing as an obvious

'sensible' method.

3.1 European Call Valuation

Suppose expiry date = T
current date = t
exercise date = K.

Then on expiry, the value of the option is $(S_T-K)^+$
So the present value is
PV = $(1+i)^{-(T-t)}E^Q(S_T-K)^+$

Recall that under Q

$$Q(n \text{ gooddays}) = \binom{T-t}{n}\left(\frac{i-b}{g-b}\right)^n\left(\frac{g-i}{g-b}\right)^{T-t-n}$$

In this case, the share price is
$S_T = (1+g)^n\ (1+b)^{T-t-n}$
so

$$PV = (1+i)^{-(T-t)}\sum_{n=0}^{T-t}\binom{T-t}{n}\left(\frac{i-b}{g-b}\right)^n\left(\frac{g-i}{g-b}\right)^{T-t-n}\left((1+g)^n(1+b)^{T-t-n}S_t-K\right)^+$$

3.2 European Put Valuation

Reasoning as above, we have

$$PV = (1+i)^{-(T-t)}\sum_{n=0}^{T-t}\binom{T-t}{n}\left(\frac{i-b}{g-b}\right)^n\left(\frac{g-i}{g-b}\right)^{T-t-n}\left(K-(1+g)^n(1+b)^{T-t-n}S_t\right)^+$$

Alternatively, we noted in 1.4 that at time T,

call + cash = put + security.

Taking present values,
call price + $K(1+i)^{-(T-t)}$ = put price + S_t
so put price = $K(1+i)^{-(T-t)} - S_t$ + call price.

3.3 Example

Suppose g = 30%
i = 10%

$b = -10\%$
$T = 2$
$t = 0$
$S_0 = 0.95$
$K = 1.$

So we are considering an option with exercise date 2, exercise price 1 and current price 0.95. Let n = number of good days.

If $n = 0$, then $S_T = 0.95 \times 0.9 \times 0.9$, so call value = 0
$n = 1$, then $S_T = 0.95 \times 0.9 \times 1.3$, so call value = 0.1115.
$n = 2$, then $S_T = 0.95 \times 1.3 \times 1.3$, so call value = 0.6055.

Under Q, we find
$Q(n = 0) = \frac{1}{4}$
$Q(n = 1) = \frac{1}{2}$
$Q(n = 2) = \frac{1}{4}$

So
present value of call $= (1.1)^{-2} (\frac{1}{2} \times 0.1115 + \frac{1}{4} \times 0.6055)$
$= 0.171$ to 3DP.

Similarly, we could calculate the present value of the put option in this way. But we will use the relation

put price $= K(1+i)^{-(T-t)} - S_t$ + call price
$= 0.826 - 0.95 + 0.171$
$= 0.047$ to 3 decimal places

3.4 American Option Valuations

This is much more complicated than the European option contract, because we don't know when the option will be exercised.

We must assume that the bearer works by a rule where at time t he looks at the behaviour of the security so far, and on the basis of that information decides whether or not to exercise the option. If we know the rule which the bearer is using, we can calculate the probability under Q that it will be exercised at any particular time, and hence calculate the present value.

3.5 Example

Taking the same situation as 3.3, but this time with an American call option, we assume (arbitrarily) that the bearer uses the rule:

"Exercise the option the first time the share price exceeds the exercise price".
Examining the four possible outcomes:

Case gg. $S_0 = 0.95$, $S_1 = 1.235$, $S_2 = 1.6055$
So exercise at time 1. Value = 0.235.

Case gb. $S_0 = 0.95$, $S_1 = 1.235$, $S_2 = 1.1115$.
So exercise at time 1. Value = 0.235

Case bg. $S_0 = 0.95$, $S_1 = 0.855$, $S_2 = 1.1115$
So exercise at time 2. Value = 0.1115.

Case bb. $S_0 = 0.95$, $S_1 = 0.855$, $S_2 = 0.7695$.
So never exercise, and value = 0.

(Case gg is when both days have a 'g' outcome, etc) Since under Q all these events have probability ¼, we calculate

$$\text{Present value} = \tfrac{1}{4}\{(1.1)^{-1} \times 0.235 + (1.1)^{-1} \times 0.235 + (1.1)^{-2} \times 0.1115\}$$
$$= 0.130 \text{ to 3 decimal places}$$

4. Optimal Exercise

4.1 In order to value the American option contracts, we need to answer the question "When will the bearer exercise the option?"

Since the value of the option is the actuarial present value, the bearer will try to maximise this.

There are only finitely many possible exercise times, and only finitely many possible paths for the process, so there must be only finitely many decision rules for the time to exercise. We could in theory make a list of all possible rules, and pick out the one which gives the highest present value.

We will need to be very careful about the decision rules we will allow. We can't base our decision to exercise or not at time t on whether day t+1 is good or bad. In technical language, the time of exercise must be a stopping time. This means that you know whether you need to exercise now based only on information currently available.

4.2 **Example**

Suppose that i = 10%, g = 30% and b = -10%. Then, if T = 2, t = 0, S_0 = 0.95 and K = 1, we can list all possible exercise rules, and the present value of call and put options using those rules. You should be able to convince yourself that the list of 26 rules below is exhaustive.
Labelling rules A - Z, we find the following table of exercise times:

Rule	gg	gb	bg	bb	Call	Put
a	0	0	0	0	-0.050	0.050
b	1	1	1	1	0.041	-0.041
c	1	1	2	2	0.082	-0.082
d	1	1	2	never	0.130	-0.130
e	1	1	never	2	0.059	-0.059
f	1	1	never	never	0.107	-0.107
g	2	2	1	1	0.082	-0.082
h	2	never	1	1	0.059	-0.059
i	never	2	1	1	-0.043	0.043
j	never	never	1	1	-0.066	0.066
k	2	2	2	2	0.124	-0.124
l	2	2	2	never	0.171	-0.171
m	2	2	never	2	0.101	-0.101
n	2	2	never	never	0.148	-0.148
o	2	never	2	2	0.101	-0.101
p	2	never	2	never	0.148	-0.148
q	2	never	never	2	0.077	-0.077

r	2	never	never	never	0.125	-0.125
s	never	2	2	2	-0.002	0.002
t	never	2	2	never	0.046	-0.046
u	never	2	never	2	-0.025	0.025
v	never	2	never	never	0.023	-0.023
w	never	never	2	2	-0.025	0.025
x	never	never	2	never	0.023	-0.023
y	never	never	never	2	-0.048	0.048
z	never	never	never	never	0.000	0.000

For example, rule i says

'if both days are good don't exercise; if the first is good and the second is bad exercise at time 2, and if the first day is bad exercise at time 1, whatever happens on day 2'

i.e. 'exercise on the first bad day'.

We can recognise some old friends in this table.

The European call can only be exercised at time 2. It is exercised in the cases gg, gb and bg, so corresponds to l, with present value = 0.171.

The European put can only be exercised at time 2. It is exercised in the case bb, so corresponds to y, with present value = 0.048.

We analysed both of these in section 3.3.

The call example in section 3.5 correspond to d, with value 0.130.

The maximum values are attained as follows:

Call option : maximum is rule l with value 0.171. This illustrates a principle which is more generally true : American call options don't get exercised early.

Put option : maximum is rule j with value 0.066. In this case, the optimal rule is "exercise on day 1 if it is a bad day, otherwise don't exercise at all".

There is another way of evaluating the optimal exercise time, involving working backwards from the expiry date. We will carry this out for the American put option above.

At time 2, there are 3 possible share prices:

	gg	:	S_2	=	1.6055
bg or	bg	:	S_2	=	1.1115
	bb	:	S_2	=	0.7695

If the put option has not yet been exercised, then we would exercise it in the case bb, yielding a profit of 0.2305, and otherwise the option would expire worthless. So we have the following values:

	gg	:	option value	=	0
gb or	bg	:	option value	=	0
	bb	:	option value	=	0.2305

We now calculate the present value of these at time 1, together with share price and the option value if we exercise at time 1:

g	:	S_1	=	1.235, exercise value=-0.235
				present value = 0
b	:	S_1	=	0.855, exercise value=0.145
				present value = 0.105

where 0.105 is calculated as $(1.1)^{-1}.\frac{1}{2}\{0 + 0.2305\}$

We see that in the first case we actually lose money by exercising, while in the second case it is better to exercise now rather than to wait another day.

Finally, taking present values at time 0,

S_0 = 0.95, exercise value = 0.05, present value = 0.066

So at time 0 we are better off waiting a day, getting a present value of 0.066, rather than exercising immediately.

Summarising this, we find that the optimal strategy is to exercise at time 1 if this is a bad day, otherwise to let the option expire worthless.

4.3 The General Case

The intrinsic value of an American option means the value the option would have if it expired now. Thus, all American call options in force, whatever their duration, have the same intrinsic value. If x is the underlying security price, and K is the exercise price, we can see that:

An American call contract has intrinsic value $(x-K)^+$.

An American put contract has intrinsic value $(K-x)^+$.

The price of an American option can never be less than its intrinsic value, for the extension of the duration into the future will never be a disadvantage to the bearer. The difference between the option price and the intrinsic value is called the time value, so

option price = intrinsic value + time value.

The bearer of an American option has to make a choice between

(i) exercising the option now
(ii) deferring exercise for one more day.

If the option has reached expiry, the second choice is to allow the option to expire.

If the bearer takes the first choice, the value realised is the intrinsic value. If he takes the second choice, the price will be the present value of its price one step ahead. A rational bearer will act to maximise this price, so

American option price

$$= \max \begin{cases} \text{intrinsic value} \\ \text{present value of future price} \end{cases}$$

And on expiry,

American option price = intrinsic value.

Thus, if the P_t is the option price at time t, and I_t is the intrinsic value

$$P_t = \max\{I_t, \text{Present value}(P_{t+1})\} \qquad (t<T)$$
$$P_T = I_T.$$

In our model, let

x	=	security price
K	=	exercise price
$P(x,t)$	=	option price at time t when security price is x
I	=	intrinsic value
		$(x-K)^+$ call contract
	=	
		$(K-x)^+$ put contract.

and we find

$$P(x,t) = \max\left[I, \frac{i-b}{(1+i)(g-b)} P((1+g)x, t+1) + \frac{g-i}{(1+i)(g-b)} P((1+b)x, t+1)\right]$$

and $P(x,T) = I$.

4.4 Optimum for Call Options

It turns out that (in the absence of dividends) American call options are never worth exercising early. We will argue this in three steps.

Step 1.

An American call is not worth less a European call with the same expiry date. As far as the bearer is concerned, he has more freedom of choice with the American option. This could only be an advantage, as he is never forced to exercise early.

Step 2.

A European call is not worth less than the intrinsic value of an American call. For on expiry, the European call is never worth less than $S_T - K$. Thus the present value will never be less than the present value of $S_T - K$, which is $S_t - (1+i)^{-(T-t)}K$.

European call price $\geq S_t - (1+i)^{-(T-t)}K$
$\geq S_t - K$

but also, European call price ≥ 0, so
European call price $\geq (S_t - K)^+$ = American put intrinsic value.

Step 3

From Steps 1 and 2, we see that American call price $\geq$ intrinsic value.
So the bearer will be better off reselling the option than exercising it.

Since the American and European call options hold the same benefits to the intelligent bearer, we can conclude that

American call price = European call price
and
American call contracts never get exercised early.

4.5 Optimum for Put Options

In this case it is much harder to give an explicit formula for the optimal strategy. However, we can prove the general form of the best time to exercise. It is as follows:

There exist constants A_0, A_1, A_2, ... A_T with $0 \leq A_t \leq K$ such that an optimal strategy is to exercise the first time $S_t \leq A_t$.

Proof. Let P(x,t) be the option price. We must show that there exists a constant A_t such that

$$P(x,t) = \begin{cases} K - x & x \leq A_t \\ \text{present value } P(., t+1) & x > A_t \end{cases} \quad (*)$$

I claim that
(i) P(x,t) is continuous in x
(ii) P(x,t) $\geq$ 0 for all x
(iii) P(x,t) has gradient $\geq$-1 with respect to x
(iv) P(0,t) = K
(v) A_t exists satisfying *.

Note that $P(x,T) = (K-x)^+$, so (i)->(v) are all satisfied when t=T.
In particular, $A_T = K$.

Suppose that for some value of t, one or more of (i) -> (v) fails. Choose the largest t for which such a failure occurs. We know t < T, since they all hold when t = T.

Let H(x,t) = present value one step ahead.

$$= \frac{(i-b)P((1+g)x,t+1) + (g-i)P((1+b)x,t+1)}{(1+i)(g-b)}$$

Since $t+1 > t$, (i) and (ii) hold for $P(x,t+1)$, and hence for $H(x,t)$.

Also,

$$\text{gradient of } H(x,t) \geq \frac{(i-b)P((1+g)x,t+1)(-1)+(g-i)P((1+b)x,t+1)(-1)}{(1+i)(g-b)}$$

$$= -1$$

putting $x = 0$, and applying (iv) at time $t+1$,

$$H(0,t) = \frac{K}{(1+i)}$$

But, from 4.3,

$$P(x,t) = \max\{ K - x, H(x,t)\}$$

So $P(x,t)$ satisfies (i), (ii) and (iii).

Putting $x = 0$, we see that (iv) holds.

Now, $H(0,t) = \frac{K}{1+i} < K - 0$

$H(K,t) \geq 0 = K - K$

so since $H(x,t)$ and $K-x$ are continuous functions of x, there must be some crossover value A_t with

$H(A_t,t) = K-A_t$

and $0 < A_t < K$

For $x \leq A_t$, we have

$$\frac{H(A_t,t)-H(x,t)}{A_t-x} \geq -1 \quad \text{from (iii),}$$

$$\Rightarrow \quad H(x,t) \leq K-x$$

Similarly, if $x \geq A_t$, we have

$H(x,t) \geq K-x$

So, since by definition,

$P(x,t) = \max\{ K-x, H(x,t) \}$

we have $P(x,t) = \begin{cases} K-x & (x \leq A_t) \\ H(x,t) & (x > A_t) \end{cases}$

and so (v) holds at time t.

Thus all of (i) - (v) hold at time t,

contradicting our choice of t, so in fact (i) - (v) must be true for all t.

4.6 **Example**

We shall assume

$$g = 30\%$$
$$i = 10\%$$
$$b = -10\%$$
$$T = 2$$
$$K = 1$$

and examine the optimal strategy for an American put.

At T = 2, option price is given by

$$\begin{array}{ll} 1-x & \text{if} \quad x < 1 \\ 0 & \text{if} \quad x > 1 \end{array}$$

So $A_2 = 1$, and we exercise if $x < A_2$.

At t = 1, there are three cases when calculating the present value:

Case (i) $0.9x < 1.3x < 1$

Case (ii) $0.9x < 1 < 1.3x$

Case (iii) $1 < 0.9x < 1.3x$

Calculating present values at time 1,

Case (i) $x < 0.769$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{(1-1.3x) + (1-0.9x)\}$$
$$= 0.909-x$$

Case (ii) $0.769 < x < 1.111$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{(1-1.3x) + (1-0.9x)\}$$
$$= 0.455 - 0.409x$$

Case (iii) $1.111 < x$

$$PV = 0.$$

The intrinsic value is $(1-x)^+$, which intersects the present value in case (ii), at $x = 0.922$.

Taking the greater of the intrinsic and present values, we find that the option price at time 1 is

$1-x$	$x < 0.922$	
$0.455 - 0.409x$	$0.922 < x$	< 1.111
0	$1.111 < x$	

So $A_1 = 0.922$.

At $t = 0$, there are five cases when calculating the present value:

Case	(iv)	$1.3x < 0.922$
Case	(v)	$0.9x < 0.922 < 1.3x < 1.111$
Case	(vi)	$0.9x < 0.922 < 1.111 < 1.3x$
Case	(vii)	$0.922 < 0.9x < 1.111 < 1.3x$
Case	(viii)	$0.9x > 1.111$

Calculating present values at time 0,

Case (iv) $x < 0.709$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{(1-1.3x) + (1-0.9x)\}$$
$$= 0.909-x$$

Case (v) $0.709 < x < 0.855$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{(1-0.9x) + (0.455 - 0.409 \times 1.3x)\}$$
$$= 0.661 - 0.651x$$

Case (vi) $0.855 < x < 1.024$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{(1-0.9x) + 0)$$
$$= 0.454 - 0.409x$$

Case (vii) $1.024 < x < 1.234$

$$PV = (1.1)^{-1} \cdot \tfrac{1}{2} \cdot \{0.455 - 0.409 \times 0.92 + 0\}$$
$$= 0.207 - 0.167x$$

Case (vii) $1.234 < x$

$$PV = 0.$$

The intrinsic value is $1-x$, which intersects the present value in case (vi) at $x = 0.924$.

Taking the greater of the intrinsic and present values, we find that the option price at time 1 is

$1-x$	$x < 0.924$

0.454-0.409x	0.924 < x 1.024
0.207-0.167x	1.024 < x < 1.234
0	1.234 < x

and $A_0 = 0.924$.

5. Hedging and Arbitrage

In this chapter, I shall attempt to show why we have used the present value concept to price options. It turns out that the following three statements are equivalent:

(i) All contracts relating to the security are priced in a way that is consistent with the basis chosen previously.

(ii) There is no opportunity for arbitrage, i.e. risk-free profit with zero investment.

(iii) The value of any contract through time can be tracked exactly by a suitable portfolio containing just the share and cash.

(ii), which is probably the most plausible of the three, is the basic assumption of the so called "arbitrage pricing theory" (APT). The logic is argued as follows:

5.2 ;(iii) => (i)
5.3 ;(i) => (iii)
5.4 ;(i) <=> (ii)

In continuous time analogous assertions hold subject to extra regularity conditions.

5.1 Self-Financing Portfolios

Suppose an investor has a portfolio consisting of cash and a share. During the day, the prices are constant, and the investor trades at these prices. Overnight, the trading stops, and prices fluctuate according to the model in 2.1, so the cash earns interest at rate i, while the share either increases by a factor g, or by a factor b.

Suppose that overnight, between days t-1 and t, the investor holds X_t in cash and Y_t units of security.

At the end of day t-1, the portfolio value is

$$\text{cash} + Y_t \times \text{share price}$$

$= \quad X_t + Y_t S_{t-1}.$

At the beginning of day t, the value is
cash + interest + Y_t x new share price

$= \quad (1+i)X_t + Y_t S_t$

So, writing t-1 for t, the value at the beginning of day t-1 is

$(1+i)X_{t-1} + Y_{t-1}S_{t-1}$

We assume the portfolio is self-financing, which means no gains or losses are made through trade. Thus,

value at beginning of day t-1
= value at end of day t-1

We denote this common value by V_{t-1}, so

$V_{t-1} = (1+i)\ X_{t-1} + Y_{t-1}\ S_{t-1} = X_t + Y_t S_{t-1}$

5.2 **Self-Financing => Consistent**

In 2.4 we defined a concept of consistency with a basis to mean that the market prices could be calculated by taking present values. Let V_t be the value of a self-financing portfolio at time t. Then it turns out that the portfolio is priced according to the basis.

To prove this, it suffices to show that

$(1+i)V_{t-1} = E^Q V_t$

for the consistency extends over longer intervals by the consistency property §1.5.

Now $V_t = (1+i)X_t + Y_t S_t.$

We consider the expectation at the end of day t-1. X_t and Y_t are then known, so can be treated as constants, and S_t is consistent with the basis, so

$E^Q(S_t) = (1+i)S_{t-1}$

Thus,

$$\begin{aligned} E^Q(V_t) &= (1+i)X_t + Y_t E^Q(S_t) \\ &= (1+i)\ (X_t + Y_t S_{t-1}) \\ &= (1+i)V_{t-1} \end{aligned}$$

and so V is consistent.

5.3 **Short Sales and Hedging**

A short sale means holding a negative quantity of a security. The argument for allowing this is one of fairness - if the market value of a security is fair, it will pay you the value of the security to take on the obligations attached to the security. A particular case of this is the facility to borrow at the risk-free rate of interest.

Hedging refers to the construction of a portfolio which will track a given index. If we allow short sales, there is a neat characterisation of the indices which can be tracked :

An index with value V_t on day t can be tracked if and only if the index is consistent with the basis.

We know from 5.2 that if an index can be tracked, it is consistent. We now show that consistent indices can be tracked by calculating the necessary portfolio.

Let V be the index value today.
Let V_g be the index tomorrow, if tomorrow is a good day
Let V_b be the index tomorrow if tomorrow is a bad day.

Then, since the index is consistent with the basis,

$$(1+i)V = \frac{i-b}{g-b} V_g + \frac{g-i}{g-b} V_b$$

If S is today's share price, let

$$X = V - \frac{V_g - V_b}{g-b}$$

$$Y = \frac{V_g - V_b}{(g-b)S}$$

The idea is to hold X in cash and Y shares tonight. This will cost us
$X + YS = V$ to buy today.

If tomorrow is good, it will be worth
$(1+i)X + Y(1+g)S$

$$=(1+i)V-(1+i)\frac{V_g-V_b}{g-b}+(1+g)\frac{V_g-V_b}{g-b}$$

$$=\frac{i-b}{g-b}V_g+\frac{g-i}{g-b}V_b-(1+i)\frac{V_g-V_b}{g-b}+(1+g)\frac{V_g-V_b}{g-b}$$

since V is consistent with the basis

$$= V_g.$$

Similarly, if tomorrow is bad the portfolio is worth V_b. We have therefore successfully tracked the index V for one day. We could clearly continue this process indefinitely.

5.4 Arbitrage and Consistency

An arbitrage opportunity is a self-financing portfolio with zero value at time 0 and strictly positive value at some finite fixed time T. If all contracts are priced according to the basis, then this couldn't happen, for the present value of a positive sum is positive, whenever it is paid.

The useful fact is that the converse is true, i.e. if any contract is not consistent with the basis, an arbitrage opportunity is created. Suppose a contract with value U_t at time t is not consistent with the basis. Then for some times $T_1 < T_2$,

U_{T1} is not the present value of U_{T2}

We can construct a portfolio with value V_t at time t, where

V_t = present value of U_{T2} at time t

by Section 5.3, since V_t is consistent with the basis. We now make a composite portfolio, as follows:

Start with no capital at all. Wait until time T_1. Buy one unit of whichever of U_{T1} or V_{T1} is cheaper, and shortsell the more expensive one, investing the difference in cash.

At time T_2, U_{T2} and V_{T2} cancel each other, and we are left with the cash. This is an arbitrage opportunity.

So we have demonstrated that inconsistencies in the market lead to arbitrage opportunities. These are banned by APT, so all products in the market, including options, must be priced consistently with the basis.

6. Basics of Continuous Time

6.1 Brownian Motion

Brownian motion is a time dependant random variable B_t $(t \geq 0)$ which satisfies

(i) $B_0 = 0$

(ii) If $s<t<u$ then $B_u - B_t \sim N(0,u-t)$ independent of B_s

(iii) B_t is a continuous function of time.

It is best thought of as a continuous-time random walk. Especially important are the following characteristics:

(iv) Stationary increments. If $h>0$ then $B_{t+h}-B_t \sim N(0,h)$, so the distribution does not depend on t.

(v) Independent increments. if $r<s<t<u$ then B_s-B_r is independent of B_u-B_t.

Conversely, Levy's theorem, which is a powerful result in stochastic process theory states that if a process X_t is continuous, and has stationary independent increments, then it is of the form

$$X_t = X_0 + \mu t + \sigma B_t$$

where B_t is a Brownian motion and X_0, μ, σ are constants.

6.2 Security Prices

Let S_t be the price of a security at time t. If $s<t$, the return on the security over the time interval (s,t) is defined by the formula

$$\text{return} = \frac{S_t}{S_s} - 1$$

It is widely believed that

(i) S_t should be a continuous function of t

(ii) returns are stationary and independent.

An increment in log S = log S_t - log S_s = log(1 + return). This means that ln S_t must be continuous, with stationary independent increments, ie

$$\log S_t = \log S_0 + \mu t + \sigma B_t$$

for suitable μ,σ. We rewrite this as

$$S_t = S_0 e^{\mu t+\sigma Bt}$$

6.3 The Black-Scholes Model

In their paper, which first presented modern option pricing techniques, Black and Scholes assumed that

(i) there is a risk-free force of interest,r

(ii) the security price S_t satisfies

$$S_t = S_0 e^{\mu t+\sigma Bt}$$

From these assumptions, they derived their now famous option pricing formula.

We can confirm that the formula is sensible using present value techniques. Having got the formula, we will show how an options contract can be tracked by a suitable hedge portfolio.

6.4 The Basis

Again, the idea is to devise a suitable basis to give us an option pricing formula. The basis used will ultimately by justified by hedging considerations, (chapter 7) but we will be able to give a good 'rule of thumb'.

The case when the answer is easiest is when the security is priced consistently with the true probability model, that is to say the current price is the discounted expectation of the future price. In this case, we take a realistic basis, and the market is said to be "risk neutral" since it prices everything by discounted expectation, irrespective of risk.

If the risk free rate is r, and the security price satisfies

$$S_t = S_0 e^{\mu t+\sigma Bt}, \text{ under the "true" model.}$$

Taking T>t, the expected value of S_T at time t is

$$\begin{aligned} E(S_T) &= ES_t e^{\mu(T-t)+\sigma(BT-Bt)} \\ &= S_t e^{\mu(T-t)} E e^{N(0,\sigma^2(T-t))} \\ &= S_t e^{\mu(T-t)} e^{\frac{1}{2}\sigma^2(T-t)} \end{aligned}$$

$$= S_t e^{(\mu+\frac{1}{2}\sigma^2)(T-t)}$$

S is priced consistently if and only if $E(S_T) = S_t e^{r(T-t)}$

i.e $\quad r = \mu+\frac{1}{2}\sigma^2$

The rule for producing the basis is as follows: "Assume the true values of r and σ^2, and assume a new μ given by $r-\frac{1}{2}\sigma^2$".

6.5 The European Call Option

We will now derive the option pricing formula for the European Call, in a risk-neutral market.

We see that $S_T = S_t e^{\mu(T-t)+\sigma(BT-Bt)}$

So S_T has a lognormal distribution, with

$E(\log S_T) = \log S_t + \mu(T-t)$

$Var(\log S_T) = \sigma^2 (T-t)$

The option price will be the present value at time t of $(S_T-K)^+$, i.e.

$$\text{price} = e^{-r(T-t)}\int_K^{+\infty}(w-K)\frac{1}{w\sqrt{2\pi\sigma^2(T-t)}}\exp\left(-\frac{[\log w-\log S_t-\mu(T-t)]^2}{2\sigma^2(T-t)}\right)dw$$

Under the valuation basis, we have $\mu = r - \frac{1}{2}\sigma^2$, and if we observe that

$$\frac{[\log w-\log S_t-(r-\frac{1}{2}\sigma^2)(T-t)]^2}{2\sigma^2(T-t)} = \frac{[\log w-\log S_t-(r+\frac{1}{2}\sigma^2)(T-t)]^2}{2\sigma^2(T-t)}+\log w-\log S_t-r$$

It is not difficult to split the integral, and integrate directly, to give

European call price =

$$x\Phi\left(\frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)-Ke^{-r(T-t)}\Phi\left(\frac{\log(x/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $x = S_t$. This is the Black-Scholes formula.

6.6 The European Put Option

We could value this in the same way as above, by manipulating integrals. Much easier is to realise that from 1.4,

call + cash = put + security.

Taking present values,

European put price = $e^{-r(T-t)}K - x$ + call price

$$Ke^{-r(T-t)}\left(1-\Phi\left(\frac{\log(x/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right)-x\left(1-\Phi\left(\frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\right)$$

Recalling that $1-\Phi(z) = \Phi(-z)$, we can rewrite this as

$$Ke^{-r(T-t)}\Phi\left(\frac{\log(K/x)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)-x\Phi\left(\frac{\log(K/x)-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

6.7 The American Call Option

In continuous time, it is still the case that American call will not be exercised early. If you turn back to 4.4, you will see that the same reasons still hold. It will thus be worth the same as a European call and so
American call price =

$$=x\Phi\left(\frac{\log(x/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)-Ke^{-r(t-t)}\Phi\left(\frac{\log(x/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

6.8 Currency Options

We can test the reasonableness of our proposed valuation basis as follows :
Let $r_£$,$r_\$$ be the risk-free forces of return in the UK and US respectively, and let R_t be the number of dollars to the pound at time t.We suppose

$$R_t = R_0e^{\mu t+\sigma Bt}$$

Consider a european option to buy \$G for £H on exercise date T. The Englishman argues as follows: "I can construct a security whose price is $\$Ge^{-r\$(T-t)}$, by investing in \$ at the risk-free rate. At time T it consists of \$G. Its value in £ is therefore $£Ge^{-r\$(T-t)}/R_t = £S_t$, say. The contract is a european call option on S_t ,with exercise price £H, so its price at time 0 is, using $r = r_£$,

$$S_0\Phi\left(\frac{\log(S_0/H)+(r_£+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)-He^{-r_£T}\Phi\left(\frac{\log(S_0/H)+(r_£-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

but $S_0 = G\ e^{-r\$T}/R_0$, so

$$\text{price}=\frac{Ge^{-r_\$T}}{R_0}\Phi\left(\frac{\log(\frac{G}{R_0H})+(r_£-r_\$+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)-He^{-r_£T}\Phi\left(\frac{\log(\frac{G}{R_0H})+(r_£-r_\$-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

This is the premium I am prepared to pay, in £."

The American argues as follows :
"I can construct a security whose value in £ at time t is $£He^{-r£(T-t)}$, by investing in £ at the risk-free rate. At time T it consists of £H. Its value in $ is therefore $\$Ee^{-r£(T-t)}R_t = \$S_t$,say. (This is a different S_t to the Englishman's). The contract is a european put option on S_t, with exercise price $G, so its price at time 0 is, using $r = r_\$$,

$$Ge^{-r_\$T}\Phi\left(\frac{\log(\frac{G}{S_0})-(r_\$-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)-S_0\Phi\left(\frac{\log(\frac{G}{S_0})-(r_\$+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

but $S_0 = He^{-r£T}R_0$, so

$$\text{price}=Ge^{-r_\$T}\Phi\left(\frac{\log(\frac{G}{R_0H})+(r_£-r_\$+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)-He^{-r_£T}R_0\Phi\left(\frac{\log(\frac{G}{R_0H})+(r_£-r_\$-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

This is the premium I am prepared to pay, in $."

Comparing the two option prices, they differ by a factor of R_0 , which is the current exchange rate between the two currencies. This is exactly what we would hope, and it confirms the reasonableness of our pricing formulae.

This analysis of course only applies to european options, where early exercise is not allowed. American currency options are much more complicated, because in the terms above, the number of units of security covered by the option varies with time. These options may in principle be worth exercising early either way.

7. Hedging and Arbitrage

In this chapter, we derive continuous time analogues for the results in chapter 5, including a justification for the choice of basis. The theory is not as neat, because there are more technicalities, and many results do not carry across at all. However, instead of working with difference equations we find that differential equations arise, so it is generally easier to

solve problems in practice.

7.1 Consistent => No Arbitrage

This is the easier part. If a portfolio has value V_t at time t, and it is priced consistently with the basis, then it is certainly not an arbitrage opportunity - for the present value of a positive sum is positive.

The converse is not true in general without extra boundedness conditions. These conditions are automatically satisfied in the discrete situation, since there are only finitely many possible outcomes in a finite time.

7.2 Quadratic Variation and Ito's Formula

Let B_t be Brownian motion, and let f be twice differentiable. Ito's formula says that

$$df(B_t) = f'(B_t)dB_t + \tfrac{1}{2}f''(B_t)dt.$$

This is basically Taylor's theorem, taking up to second order terms, with the rule that $dB_t^2 = dt$. The reason this works is that if we consider the sum

$$(B_{t1} - B_{to})^2 + (B_{t2} - B_{t1})^2 + \ldots + (B_{tk} - B_{tk-1})^2$$

with $0 = t_o < t_1 \leqslant \ldots\ldots < t_k = t$ as $k \to \infty$ and the differences between the t's -> o, the sum tends (in probability) to t.

More generally, if X_t is a suitable continuous process, and t denotes time, then we have

$$df(X,t) = \frac{\partial f}{\partial X}dX + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}d[X]$$

where [X] is a suitable limit sum of squared increments of X. If $X_t = B_t$, then $[X]_t = t$, as above. [X] is called the quadratic variation process of X.

The Black-Scholes model states that

$$S_t = S_0 e^{\mu t + \sigma B t}$$

so, by Ito's formula,

$$
\begin{aligned}
dS_t &= S_0 e^{\mu t+\sigma B_t}(\mu dt + \sigma dB_t) \\
&\quad + \tfrac{1}{2} S_0 e^{\mu t+\sigma B_t}(\mu dt+\sigma dB_t)^2 \\
&= S_t((\mu+\tfrac{1}{2}\sigma^2)dt + \sigma dB)
\end{aligned}
$$

so

$$(dS_t)^2 = d[S]_t = S_t^2\sigma^2 dt.$$

7.3 A Necessary Hedging Condition

Suppose $V(x,t)$ is twice differentiable. A necessary condition for $V(S_t,t)$ to be the value of a hedged portfolio under the Black-Scholes model is

$$\frac{1}{s}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = rV$$

The original derivation of the Black-Scholes formula involves solving this equation with appropriate boundary conditions.

<u>Proof</u> Let

$$\delta_t = \frac{\partial V(S_t,t)}{\partial S_t}$$

so δ_t is the sensitivity of the portfolio value to changes in S. In order to track $V(S_t,t)$ with a portfolio, δ_t must therefore be the number of units of security to be held in the hedge. Since the total value of the portfolio must be $V(S_t,t)$, the amount of cash held must be

$$\text{cash held} = V - S_t\delta_t.$$

The differential gain in the hedged portfolio will therefore be

δ_t x change in security price
+ $(V - S_t\delta_t)$ x interest,

i.e.

change in portfolio

$$= \delta_t\, dS_t + (V - S_t\delta_t)r dt.$$

On the other hand, Ito's formula tells us that the change in V is given by

$$dV = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

Equating these two, and noting that

$$\delta = \frac{\partial V}{\partial S}$$

$$\frac{\partial V}{\partial S} + (V - S\frac{\partial V}{\partial S})r\,dt = \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

We saw in 7.2 that $dS^2 = S^2\sigma^2 dt$, so this is

$$r(V - S\frac{\partial V}{\partial S}) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

or, equivalently, putting $x = S_t$

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = rV$$

as required.

7.4 A Pathological Portfolio

Take $S_0 = 1$, $r = 0$, $\sigma^2 = 1$, $\mu = -\frac{1}{2}$ in the Black-Scholes model, and let

$$X_t = \int_0^t \frac{dB_s}{1-s} \quad \text{for } t<1$$

so $$dX = \frac{dB}{1-t}$$

As $t \to 1$, X_t will oscillate increasingly wildly, and will hit 1 with certainty before time 1.
If we let
$T = \min\{ t: X_t = 1\}$
then T is therefore less than 1.

Now let $V_t = X_{t\wedge T}$ where $t\wedge T = \min\{t,T\}$

Construct a portfolio with holdings in S_t given by

$$\text{no. of units} = \begin{cases} 1/((1-t)S_t) & t<T \\ =0 & t\geq T. \end{cases}$$

If this portfolio has initial value zero, and all holdings are financed by borrowing at the zero risk free rate,

change in portfolio $=dS_t/((1-t)S_t)$

but $dS_t = S_t dB_t$, by Ito's formula, so

change in portfolio $= dB/(1-t)$ $\quad t<T$
$\quad 0 \quad t\geq T$

and hence the portfolio tracks V_t exactly.

But $V_1 = X_{1\wedge T} = X_T = 1$, since $T \leq 1$.

So this is an arbitrage opportunity. To avoid this kind of strategy, we state the following criterion for investors:

The number of units of security held, and, the amount of cash held must both be <u>bounded</u> processes.

A process X_t is bounded if for each time t there is some fixed upper limit C_t which $|X|$ is not allowed to exceed up to and including time t. We will see in § 7.6 that this restriction suffices to eliminate arbitrage. It can be justified on the grounds that unbounded holdings in shares will distort the price structure, so the simple model here is no longer appropriate.

7.5 Bounded Portfolio Condition

Suppose V(x,t) is twice differentiable. A necessary and sufficient condition for V(S,t) to be the value of a bounded hedged portfolio under the Black-Scholes model is

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} = rV$$

and
V(x,t) and x V(1/x,t) are Lipschitz in x for each t. (In this context, Lipschitz means having a bounded derivative)

<u>Proof</u> V(x,t) Lispchitz <=>

$$\frac{\partial V}{\partial x} \text{ bounded}$$

and xV(1/x,t) Lipschitz <=>
putting y=1/x,xV(1/x,t) Lipschitz <=>

$$\frac{\partial}{\partial x}V(\tfrac{1}{x},t)=V(\tfrac{1}{x},t)-\frac{1}{x}V'(\tfrac{1}{x},t) \text{ bounded}$$

$$V(y,t)-y\frac{\partial}{\partial y}V(y,t) \text{ bounded}$$

But from 7.3, the these quantities are precisely the holdings in security and cash for the hedge portfolio. They are bounded if and only if the Lipschitz conditions hold.

7.6 Change of Basis

We saw in 7.5 that the possible hedging strategies will depend on the values of r and σ^2 in the Black-Scholes model, but not on μ. This is the reason for the rule of thumb stated in 6.4, which was

> "The constants r and σ^2 in the basis must correspond to reality, but μ in the basis can be chosen to be anything convenient".

As we saw, the value $\mu = r + \frac{1}{2}\sigma^2$ was particularly convenient, so we took that.

For those who want the technical jargon, we want to choose a basis by an absolutely continuous change of measure. This means that events with zero probability in reality must also have zero probability in the basis. We actually saw this principle at work in the discrete case. We stipulated in 1.2 that all transitions must have positive (i.e. non-zero) probability. This is equivalent to saying that the basis we stated in 2.2 is absolutely continuous with respect to the truth. In the same way, the σ^2 parameters in continuous time can be expressed as a (probability 1) limit of sums of squares of log prices. If σ^2 in the basis was not equal to the true value, then the probability of this limit being the basis value will be 1 under the basis, but 0 in real life, contradicting absolute continuity.

It is not obvious that a change in μ constitutes an absolutely continuous change of measure, indeed since

$$\lim_{t\to\infty}\frac{\log S_t}{t}=\lim_{t\to\infty}\frac{\log S_0+\mu t+\sigma B_t}{t}=\mu \text{ (almost surely)}$$

we might apply the above argument to deduce that we can't change μ either. There is a technical trick we can use to get around this, using the following theorem :

<u>Girsanov's Theorem</u> :Let B_t be a Brownian Motion under P, and let c_t be a previsible process with

$$\int_0^{\infty} c_s^2 ds < K \text{ for some fixed constant } K$$

then there is a probability law Q, absolutely continuous with respect to P under which

$$\tilde{B}_t = B_t - \int_0^t c_s ds \text{ is a Brownian motion}$$

We then take c to be the difference between the real and the basis values of μ for $t \leq T$, and zero thereafter. Girsanov's theorem then tells us that our basis change is okay, provided we don't try to price anything beyond time T.

It is essential (and true) for our hedging constructions that absolutely continuous changes will preserve stochastic integrals. This will ensure that a basis change will not change the self-financing criterion.

7.7 <u>Self-Financing => Consistent</u>

Suppose that at time t, we hold x_t in cash, and y_t units of security with price S_t. The value of the portfolio at time t is then

$$V_t = x_t + y_t S_t.$$

The value at time 0 is

$$V_0 = x_0 + y_0 S_0.$$

The interest gained up to time t will be

$$r\int_0^t x_u du$$

While the accumulated changes in the price S for us will be

$$\int_0^t y_u dS_u$$

If there are no new cash injections, and no consumption, so the portfolio is self financing, we will have

$$V_t = V_o + \text{interest} + \text{price changes}$$

i.e.

$$x_t+y_tS_t=x_0+y_0S_0+r\int_0^t x_u du+\int_0^t y_u dS_u$$

I shall show that if

X_t, Y_t are bounded
S_t is consistent with the basis
the portfolio is self-financing,

then

V_t, the portfolio value, is consistent with the basis.

The key to proving this is to note that V_t is consistent with the basis if and only if $e^{-rt}V_t$ is a <u>martingale</u> under Q for if $e^{-er}V_t$ is a martingale, then for $t < T$, taking expectations at time t,

$$e^{-rt}V_t = E^Q(e^{-rT}V_T)$$

i.e

$$V_t = e^{-r(T-t)}E^QV_T$$

which says that V_t is consistent with the basis.

$$\begin{aligned}
\text{Since} \quad V_t &= x_t + y_tS_t \\
\text{and} \quad dV_t &= rx_tdt + y_tdS_t \\
\text{so} \quad d(e^{-rt}V_t) &= V_tde^{-rt} + e^{-rt}dV_t \\
&= -rV_te^{-rt}dt + e^{-rt}dV_t \\
&= -re^{-rt}(x_t - re^{-rt}y_tS_t)dt + \\
&\quad e^{-rt}(rx_tdt + e^{-rt}y_tdS_t) \\
&= y_t(e^{-rt}dS_t + re^{-rt}S_tdt) \\
&= y_td(e^{-rt}S^t).
\end{aligned}$$

But under the basis, $e^{-rt}S_t$ is a martingale, indeed it is a square-integrable martingale. Since y_t is bounded, it follows from the construction of the stochastic integral that $e^{-rt}V_t$ is also a square-integrable martingale, so V_t is consistent with the basis.

Bibliography

Beenstock, M & **Brasse**,V : Using Options to Price Maturity Guarantees. J.Inst. Act. 113 (1986), pp 151-165

Black, F. & **Scholes**,M : The Pricing of Options and Corporate Liabilities. J.Political Economy. Vol 81 (1973) pp 637-654

Brenner, M (ed) : Option Pricing, Theory and Applications. Lexington Books,1983

Cox, J., **Ross**, S. & **Rubenstein**, M. : Option Pricing: A simplified approach. J.Financial Economics vol 7 (1979) pp 229-263

Duffie, D. : Security Markets (Stochastic Models). Academic Press 1988.

Harrison, J.M. & **Kreps**, D. : Martingales and Arbitrage in Multiperiod Securities Markets. J. Economic Theory. Vol 20 (1979) pp 381-408

Jarrow, R.A. & **Rudd**, A : Option Pricing. Richard D. Irwin, Homewood, Illinois (1983)

Rogers, L.C.G. & **Williams**, D. : Diffusions, Markov Processes and Martingales. Vol 2. Ito Calculus. Wiley 1987

Wilkie, A.D. : An Option Pricing Approach to Bonus Policy. J. Inst. Act. 114(1987) pp21-77