A Mean Reverting Process for Pricing Treasury Bills and Futures Contracts

Ieuan G. Morgan & Edwin H. Neave

School of Business, Queen's University, Kingston, Ontario K7L 3N6, Canada

Summary

This paper develops a discrete time single factor model for consistently valuing treasury bills or similar instruments, and futures contracts written against them. A multiplicative binomial process with reversion to the mean describes spot rate evolution. Interest rates can become neither negative nor arbitrarily large. The model uses an explicit process for forward rates to generate consistent formulae for bond prices, future prices, their conditional variances and risk premia. Thus it permits a study of term structure evolution which is more clearly related to both model and data than in other similar works.

The model's predictions are tested with data for futures contracts written against US treasury bills and eurodollar deposits. Some of the predictions about the conditional risk premia and conditional variances of the futures prices are supported by the test results.

Résumé

Un Procédé de Retour al la Moyenne pour Fixer le Prix des Bons du Trésor et des Contrats à Terme

Cet article expose un modèle à facteur unique en temps discret pour évaluer régulièrement les bons du trésor ou instruments semblables et les contrats à terme garantis par eux. Un procédé de binôme multiplicatif avec retour à la moyenne décrit l'évolution du cours du disponible. Les taux d'intérêts ne peuvent devenir ni négatifs ni arbitrairement grand. Le modèle utilise un procédé explicite pour les taux des opérations à terme afin de générer des formules régulières pour les prix des obligations, les prix des transactions à terme, leurs écarts conditionnels et les primes de risque. Il permet ainsi une étude de l'évolution de la structure d'échéance qui est plus clairement liée au modèle et aux données que dans d'autres travaux similaires.

Les prédictions du modèle sont testées avec des données concernant les contrats à terme garantis par des bons du trésor américains et des dépôts d'eurodollars. Certaines prévisions sur les primes de risque conditionnelles et les écarts conditionnels des prix des transactions à terme sont confirmées par les résultats des tests.
1. Introduction

This paper uses a discrete time multiplicative binomial model of spot interest rate evolution, and its implied evolution of the term structure, to derive consistent pricing formulae for treasury bills and futures contracts written against them. All results are developed under assumptions of zero arbitrage profits. In comparison to other treatments, the model has the advantages of beginning with forward rates as a building block (cf. Heath, Jarrow, Morton [1990]), can formulate mean reverting interest rate processes, and provides means of explicitly evaluating martingale probabilities from bond price data.

Although the model treats initially estimated forward rates parametrically, the parameters appear in the analytic expressions, permitting ready assessment of their influence. Contract delivery dates and the maturity dates of underlying instruments also figure in the formulae. In addition, expressions for the conditional variance and conditional risk premia of both bill values and futures prices are developed for each point in time. Finally, we develop functional relations between conditional variances and risk premia, providing new theoretical support for relations used in empirical work such as Engle, Lilien, and Robins [1987].

1.1 Organization of paper

The paper is organized as follows. The rest of this section reviews relevant literature on futures prices. The model and its underlying assumptions are described in Section 2, which specifies how both the spot rate and the term structure evolve. Section 3 develops mutually consistent formulae for treasury bill prices, contingent claim prices and conditional martingale probabilities and relates them to the restrictions imposed by initial assumptions regarding the term structure. Section 4 develops formulae for futures prices, their conditional variances, and conditional risk premia. The formulae lead to predictions which are evaluated as part of the test results reported in section 5. Section 6 concludes.

1.2 Review of theory of futures prices

Groundwork for the theory of commodities futures pricing is developed in Black [1976]. Important properties of forward and futures contracts and prices are developed in Cox, Ingersoll, and Ross [1981], Jarrow and Oldfield [1981], and Richard and Sundaresan [1981]. The pricing theories developed in these papers, as well as ours, do not recognize delivery options. Discussions of how delivery options reduce futures prices appear in Gay and Manaster [1984], Kane and Marcus [1986]
and Boyle [1989]. Our tests will use data for contracts that do not involve any important delivery options. An early empirical paper on treasury bill futures prices is Jacobs and Jones [1980].

2. Analytical model

Our discrete time model is based on an approach to options pricing developed in Cox, Ross and Rubinstein [1979] and Cox and Rubinstein [1985].

2.1 Useful model properties

The model uses spot interest rates as a state variable and allows the term structure to evolve with fewer constraints than do related papers such as Ho and Lee [1986]. A model that is in some ways similar to ours but designed to reflect both interest rate and asset price uncertainty in contingent claims is presented by Kishimoto [1989]. However, Kishimoto does not develop explicit formulae, as does this paper. Bliss and Ronn [1989] offer a trinomial version of the Ho and Lee model applied to interest rate claims, but again do not develop explicit formulae.

Our model is closely related, but not identical, to the continuous time model of Heath, Jarrow, and Morton [1990], hereafter HJM. While we start with forward rates as do HJM, we use a mean reverting process and examine the martingale probabilities more explicitly than do HJM. In addition, our estimation aims and techniques differ from those of HJM.

In sum, our model examines consequences of zero arbitrage profits explicitly while using palatable assumptions regarding spot rate evolution and its implications for the term structure. Moreover, the results are achieved without analytic complexity. The test results are consistent with some of the model’s predictions regarding the behavior of futures prices’ conditional risk premia and conditional variances, and particularly the relation between these two measures.

2.2 Evolution of interest rates

Let \( R_t \) be unity plus a one-period interest rate. The term \( R_0 \) is taken to be the initial spot rate, and the terms \( \{R_t\}, \ t \in \{1, 2, ..., M - 1\} \), will be interpreted below as a series of one period forward rates on bonds, as inferred from data at time zero for a horizon of \( M \).

The realized spot rates are assumed to evolve stochastically about the \( \{R_t\} \) according to a function of a multiplicative factor \( u > 1 \). Letting \( S(t) \) be the sum of
a series whose details are defined below, one plus the time 1 spot rate evolves as in the following diagram:

\[ \begin{align*}
R_0 & \rightarrow u^{S(1)}R_1 \\
& \rightarrow u^{S(2)}R_2 \\
& \rightarrow u^{S(3)}R_3 \\
& \rightarrow u^{S(3)/3}R_3 \\
& \rightarrow u^{S(3)/3}R_3 \\
& \rightarrow u^{S(3)/3}R_3 \\
& \rightarrow u^{S(3)}R_3
\end{align*} \]

In this binomial process, the probability of an upward move is \( p \), that of a downward move \( q \). Thus the probabilities of attaining the outcomes in the last column, which refers to time 3, are \( p^3, 3p^2q, 3pq^2 \), and \( q^3 \) respectively.

To specify the evolution more fully, note that if \( S(t) = t \) the process diverges, while if \( S(t) = 0 \) for all \( t \), the process is deterministic. Except where otherwise specified explicitly, we shall use convergent series because of their desirable properties. Formally, for any series \( S(t) \) we define the maximal spot rate as \( u^{S(t)}R_t \) and the minimal value as \( u^{-S(t)}R_t \), and define

\[
\begin{align*}
u_{d(t, t)} &= u^{S(t) - J^{S(t)} / t}, \quad j \in J_t = \{0, 2, \ldots, 2t\}, \quad (2.2.1)
\end{align*}
\]

to indicate the values the spot rate is permitted to assume. Notice that if \( t \) is even, \( d(t, t) = 0 \), so that \( u^{d(t, t)} = 1 \) and the multiplicative factor has returned to its starting point.

A process using any series \( S(t) \) can be characterized as follows. Ignoring the influence of the \( \{R_t\} \), let \( U_t \) be the random variable described by (2.2.1); the values are distributed binomially as described above. The process' mean is

\[
E(U_t) = \{pu^{S(t)/t} + qu^{-S(t)/t}\}^t,
\]

\(^1\text{This observation gives a useful check on the formulae to be derived later; cf. Heath, Jarrow, Morton [1990].}\)
its variance

\[ V(U_t) = \{pu^{2S(t)/t} + qu^{S(t)/t}\} - \{pu^{S(t)/t} + qu^{-S(t)/t}\}^{2t}, \]

and its conditional variance as a proportion of the interest rate level is

\[ V^*(t) = pq(u^{2S(t)/t} - 1)^2. \]

Note the latter is independent of the rate level, and decreasing in \( t \) for sufficiently large values of \( t \).

To ensure the absence of arbitrage opportunities, the conditions

\[ u^{d(i+1)R_{t+1}} < u^{d(i, i)R_{t}} < u^{d(i+2, i+1)R_{t+1}} \]  

(2.2.2)

must be satisfied for all \( t \leq T; j \in \{0, 2, ..., 2t\} \). Thus in computational work any value chosen for \( u \) must satisfy (2.2.2) for the values \( \{R_d\} \) implied by the term structure.

For example, if we take \( S(0) = 0 \) and define a convergent series like

\[ S(t) = \sum_{k=1}^{t} \left( \frac{1}{2} \right)^k, \]

(2.2.3)

the spot rate process has a finite upper bound and a zero lower bound for any finite\(^3\) time horizon \( T \). In this case \( S(3) = 7/8, \) and, viewed from the perspective of time 0, the time 3 forward rate is distributed as the product of \( R_3 \) and one of the multiplicative factors \( u^{21/24}, u^{7/24}, u^{7/24}, u^{21/24} \). Calculations show that, ignoring the influence of the \( \{R_d\}, \) the series (2.2.3) implies reversion to the mean of both the spot rate and the underlying term structure.

3. Securities and claim prices

\(^2\)If \( S(t) \) is a convergent series, the spot rates are mean reverting, as can readily be verified by noting that in this case, \( E(U_t) \) converges to unity. Moreover, \( V(U_t) \), while positive for any finite value of \( t \), converges to zero in the limit.

\(^3\)Or, for that matter, an infinite time horizon. For convenience, we work exclusively with finite time horizons.
This section relates bond prices, contingent claim prices and interest rates for a process ending at time $M \leq T$. Given a set of forward rates, these relations define a particular set of claim values and related conditional probabilities. The same contingent claim values are used in section 4 to derive futures prices consistent with zero arbitrage profits.

3.1 Relations between interest rates, claim prices and bond prices

We first consider how the martingale probabilities can be consistent with both data and the model. Given the initially estimated forward rates $\{R_t\}, t \in \{1, 2, ..., T\}$, all discount bonds should have an initial value determined by the geometric mean of the current spot and inferred forward rates.\(^4\) Evaluation quickly shows that only certain assumptions regarding the martingale probabilities are consistent with this assumption, as discussed further in note 5 below. In particular the martingale probabilities cannot be constant if a) the model is assumed to be true; and b) the values of bonds are given by the geometric means of the spot rate $R_0$ and the forward rates $\{R_t\}$. Thus, while our principal task is to derive formulae for securities prices which will be true for any values of the martingale probabilities, we wish simultaneously to recognize that initially observed values restrict the choice of martingale consistent both with the model and the data.\(^5\)

Bonds are defined to have a value of unity at maturity. Let $B_t(j, M)$ represent the market price at time $t$, when the spot rate is $u^{d(j, t)}R_0$ of a bond with maturity $M$, $t \in \{0, 1, 2, ..., M-1\}; j \in J_t$. Then

$$B_{M-1}(j, M) = 1 / u^{d(j, M-1)}R_{M-1}. \quad (3.1.1)$$

Let $c_t(j, j)$ be the time $t$ value of a (one-period) unit claim to be paid at time $t+1$ if the spot rate rises (from $u^{d(j, t)}R_t$ to $u^{d(j, t+1)}R_{t+1}$), and let $c_t(j, j + 2)$ be the time

\(^4\) Most other authors do not explicitly discuss the need for consistency of these values. In most continuous time papers, for example, the existence of the martingale is established, but, with the exception of HJM, its exact relation to the model and the data is not discussed. Moreover, HJM are concerned with methods that eliminate the need to calculate or to estimate the martingale probabilities. In discrete time papers the martingale probabilities are assumed constant without comment.

\(^5\) Dybvig [1989] notes that several authors in effect force the term structure to fit the model.
the value of a (one-period) unit claim to be paid if the spot rate falls to \( u^{d(j+1)}R_{t+1} \).

Since only the two outcomes are possible, we have:

\[
    c(j, 0) + c(j, j + 2) = 1 / u^{d(j)}R_t .
\]

(3.1.2)

Under the assumption of no arbitrage profits a martingale exists and the conditional (pseudo) probabilities defining the equivalent martingale measure are:

\[
p(j) = c(j, j) / [c(j, j) + c(j, j + 2)];
\]

(3.1.3)

\[
q(j) = c(j, j + 2) / [c(j, j) + c(j, j + 2)],
\]

(3.1.4)

cf., for example, Huang and Litzenberger [1988, pp. 223-234]. Hence

\[
c(j, j) = p(j) / u^{d(j)}R_t , \text{ and }
\]

(3.1.5)

\[
c(j, j + 2) = q(j) / u^{d(j)}R_t .
\]

(3.1.6)

To examine the implied nature of martingales consistent with the data and the model, we first analyze a system of equations for prices of bonds with maturities up to \( M = 3 \), then discuss the general result.

### 3.2 Three period bond example

In this example we assume the convergent series is given by (2.2.2). The one period contingent claim values at \( t = 0 \) are

\[
c(0, 0) = p_0 / R_0 , \text{ and } c(0, 2) = q_0 / R_0 .
\]

(3.2.1)

The time \( t = 0 \) prices of bonds with maturities of \( M \in \{1, 2, 3\} \) are given by

\[
B(0, M) = 1 / \prod_{i=0}^{M-1} R_i ;
\]

(3.2.2)

\[
B(0, 1) = 1 / R_0 = [p_0 + q_0] / R_0 ;
\]

(3.2.3)

\[
B(0, 2) = 1 / R_0 R_1 = [p_0 / (u^{1/2}R_0R_1)] + [q_0 / (u^{1/2}R_0R_1)].
\]

(3.2.4)

The values satisfying (3.2.3) and (3.2.4) are
\[ p_0 = u^{1/2} / [u^{1/2} + 1], \quad q_0 = 1 / [u^{1/2} + 1]. \] (3.2.5)

Since the time 1 price of a bond with maturity \( M = 3 \), conditional on the current interest rate, is either \( B_1(0, 3) \) or \( B_1(2, 3) \) and \( B_3(j, 3) = 1 \) regardless of the outcome,

\[
B_1(0,3) = \frac{p_1(0)}{u^{1/2}R_1} B_2(0,3) + \frac{q_1(0)}{u^{1/2}R_1} B_2(2,3) \]
\[
= \frac{p_1(0) + u^{3/4}q_1(0)}{u^{5/4}R_1R_2}, \] (3.2.6)

\[
B_1(2,3) = \frac{p_1(2)}{u^{-1/2}R_1} B_2(2,3) + \frac{q_1(2)}{u^{-1/2}R_1} B_2(4,3) \]
\[
= u^{1/2} \frac{p_1(2) + u^{3/4}q_1(2)}{R_1R_2}. \] (3.2.7)

Also, at time 0, to find martingale probabilities consistent with both the data and the model,

\[ B_0(0,3) = [p_0 / R_0]B_1(0,3) + [q_0 / R_0]B_1(2,3) = 1 / R_0R_1R_2. \] (3.2.8)

Substitution from (3.2.5) and elimination of \( R_0, R_1, R_2 \) gives

\[ 1 = \frac{p_1(0) + u^{3/4}q_1(0) + u^{5/4}[p_1(2) + u^{3/4}q_1(2)]}{u^{3/4}(u^{1/2} + 1)} \]

which simplifies to \(^6\)

\[ p_1(0) = u^{5/4}q_1(2). \] (3.2.9)

\(^6\)The analytic form of martingale probability for periods \( t > 2 \) is more complex. However, the values of \( p_t \) still increase in \( t \) as indicated by the example.
Given $u$, $R_0$, $R_1$, $R_2$, the system consists of five equations, two being (3.2.4) and (3.2.8) and three being of the type $p_i(j) + q_i(j) = 1$. Two were used in solving for $p_0$ and $q_0$. Since there are four unknown martingale probabilities at time 1, we need one more equation to resolve the indeterminacy illustrated by (3.2.9). Any one of the following three conditions implies both of the others and solves the system,

\begin{align*}
p_t(0) &= p_t(2), \\
B_t(2,3) / B_t(0,3) &= u^{7/4}, \\
p_t(0) &= u^{5/4} / \{1 + u^{5/4}\}. 
\end{align*}

The market will be dynamically complete if bonds with longer lives and linearly independent payoffs can be traded at each point on the lattice (cf. Huang and Litzenberger [1988, 196-203; 242-244]). In the binomial model the minimum number of such bonds is two, and the conditions for dynamic completeness are satisfied for a horizon of $t$ if, at time $t = 0$, bonds with maturities of $T + 1$ and $T + 2$ are available.\(^7\)

3.3 General results for claim and bond prices

As well as bonds with maturities $T + 1$ and $T + 2$ we also assume, of course, bonds with maturities $1, 2, \ldots, T$. Our problem is to price all bonds and derivative securities consistently under the martingale. This is achieved in the following way. One of the following conditions implies the other and solves the system corresponding to a bond with maturity $M$,

\begin{align*}
p_i(j) &= p_i, \\
\frac{B(j+2,M)}{B(j,M)} &= v_{t,M} = \prod_{k=1}^{M-1} u^{25(k)/k} 
\end{align*}

Whenever it creates no ambiguity, we write $v_{t,M}$ as $v_t$. For convenience in writing formulae like (3.3.8) below, we define $v_{M,M} = 1$.

\(^7\)Alternatively, for the existing set of securities, Appendix B shows that an assumption of constant relative risk aversion is sufficient to establish (3.2.11).
Equation (3.3.2) says that at time \( t \) the ratio of bond prices at two adjacent points on the lattice is the same for all such pairs of adjacent points. Let

\[
w(t) = 2S(t) / t; \quad t = 1, 2, \ldots, M.
\]  

(3.3.3)

Bond prices at time \( t \) are related to those at \( t + 1 \) by the possible rate outcomes and the values of the contingent claims associated with them. We now simplify the notation for the bond price by fixing the maturity, \( M \), and drop this argument from the price except where it is needed for clarity. Then the bond prices can be written recursively as:

\[
B_t(j) = c_t(j, j)B_{t+1}(j) + c_t(j, j + 2)B_{t+1}(j + 2).
\]  

(3.3.4)

Substituting for the claim prices, as in (3.1.6),

\[
B_t(j) = [p_tB_{t+1}(j) + q_tB_{t+1}(j + 2)] / u^{d(j,t)}R_t. 
\]  

(3.3.5)

Then starting at time \( M - 1 \) (using the form of prices given in (3.1.1) and proceeding by backward induction), we can derive the bond price at any point on the lattice,\(^8\)

\[
B_t(j) = [p_t + v_{t+1}q_t]B_{t+1}(j) / u^{d(j,t)}R_t, 
\]  

(3.3.6)

for \( t \in \{1, 2, \ldots, M - 1\} \). The analysis uses

\[
B_{t+1}(j + 2) = v_{t+1}B_{t+1}(j)
\]  

(3.3.7)

for all admissible values of \( t \) and \( j \), as in (3.3.2). Further recursive application of (3.3.6) with \( j = t \) gives

\[\text{\footnotesize Rewriting (3.3.6) to express the martingale in terms of bond prices shows how the price evolution would be constrained by assuming the martingale probability is constant. That is, (3.3.6) says that a constant martingale probability } p^* \text{ must satisfy}
\]

\[
p^* = \left[ u^{d(j,t)}R_tB_t(j) - u^{w(t+1)}B_{t+1}(j) \right] / \left[ 1 - u^{w(t+1)}B_{t+1}(j) \right],
\]

for all \( t \) and for all \( j \). We know of no arguments favouring this proposition.
where $t \in \{0, 1, \ldots, M - 1\}$. Here we have chosen to express the bond price in terms of the maximum possible interest rate one period prior to the bond's maturity.

The term structure also evolves in a particular manner, since at time $t$ and in state $j$ it is defined by the bond price formulae (3.3.8) for bonds of differing maturities $M$. For example, (one plus) the first two values of the yield curve defined at time $t$ and state $j$ are $1 / B_t(j, t + 1)$ and $[1 / B_t(j, t + 2)]^{1/2}$. By the same token, (one plus) the one-period forward rate$^9$ between times $t + 1$ and $t + 2$, conditional on reaching time $t$ and state $j$, is

$$B_t(j, t + 1) / B_t(j, t + 2).$$

Calculation shows that for the convergent series (2.2.3) the yield curve following an upward (downward) move of the spot rate is decreasing convex (increasing concave); i.e., mean reverting. On the other hand, for the divergent series $S(t) = t$, the yield curve following an upward (downward) move of the spot rate is increasing concave (decreasing convex).$^{10}$

3.4 Conditional variance of bond prices

Let the process be in state $j$ at time $t$. Then since $B_{t+1}(j)$ occurs with probability $p$ and $B_{t+2}(j + 2)$ with probability $q$, application of the results of Appendix A to (3.3.7), using $y = v_{t+1}B_{t+1}(j)$ and $x = B_{t+1}(j)$, implies that the conditional variance of the bond price is:

$$V_{B_t}(j) = pq(v_{t+1} - 1)^2 [B_{t+1}(j + 1)]^2.$$ (3.4.1)

$^9$Recall from (3.3.2) that $v_t$ is also dependent on $M$. This must be recognized in trying to simplify explicit formulae for the forward rates.

$^{10}$These effects are sometimes characterized by saying the former type of yield curve indicates the effects of informational shocks wear off with time, but that the latter type indicates the effects are permanent.
The conditional variance of the bond return is found by using (3.3.6) to express $B_{t+1}(j+1)$ in terms of $B_t(j)$, and then dividing by $B_t(j)$:

$$V^*_B(j) = pq\left(\frac{v_{t+1} - 1}{p_t + v_{t+1}q_t}\right)^2 \left(u^{d(j+1)}R_t\right)^2.$$  

(3.4.2)

3.5 Conditional risk premia

Let $J$ be a random variable assuming one of the values $\{j, j + 2\}$. We define the conditional risk premium to be the conditionally expected price at time $t + 1$ less the bond price at time $t$ times the current spot rate,

$$P_{B_t}(j) = E[B_{t+1}(J) - B_t(j)u^{d(j+1)}R_t],$$  

(3.5.1)

or as the expected excess rate of return

$$P^*_B(j) = E_t[B_{t+1}(J) / B_t(j)] - u^{d(j+1)}R_t.$$  

(3.5.2)

When all bond prices in (3.5.2) are expressed in terms of $B_t(j)$, and (3.3.5) and (3.3.6) are used, the expected excess rate of return simplifies to

$$P^*_B(j) = \left(\frac{p_t + v_{t+1}q_t}{p_t + v_{t+1}q_t} - 1\right)(u^{d(j+1)}R_t).$$  

(3.5.3)

3.6 Relation between risk premia and conditional variance

The expected excess rate of return on the bond can be expressed in terms of the conditional standard deviation $[V^*_B(j)]^{1/2}$. Together, (3.4.2) and (3.5.3) give

$$P^*_B(j) / [V^*_B(j)]^{1/2} = K^*_{B_t},$$  

(3.6.1)

where

$$K^*_{B_t} = \left[\frac{p_t + v_{t+1}q_t}{p_t + v_{t+1}q_t} - 1\right] \left[\frac{p_t + v_{t+1}q_t}{(pq)^{1/2}(v_{t+1} - 1)}\right].$$

This simplifies to

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For there to be a positive risk premium, the probability $p_t$ under the equivalent martingale must exceed the actual probability $p$ of an interest rate increase.\footnote{We cannot describe the relationship between $p$ and $p_t$ precisely without modelling the underlying economy; i.e., without using a general equilibrium analysis rather than examining relative prices. Nevertheless, if agents are risk averse we still know that $p_t > p$.} Equation (3.6.1) also shows that the conditional risk premium and the conditional standard deviation are strictly proportional at any instant in time, depending on the difference $p_t - p$ and on the standard deviation of a unit binomial random variable.

Equation (3.6.1) offers theoretical support for relations assumed in the literature. Specifically, the ARCH-M model of Engle, Lilien, and Robins [1987], based on Engle [1982], postulates an ex ante relationship between the conditional mean and conditional variance of returns when there is one risky and one riskless asset. They estimate models in which the conditional mean of bond returns is a linear function of the conditional standard deviation and a linear function of its logarithm. Different assumptions about investors' utility functions lead to the different functional forms.

### 3.7 Relation to the Ho-Lee and similar models

It is instructive to relate the current bond price model to the perturbation functions used by Ho and Lee [1986]. Adapted to our model and in our notation, equations (1) and (2) of Bliss and Ronn [1989] summarize the relations between the perturbation functions $h(M - t)$, $h^* (M - t)$ as

\begin{align}
B_{t+1}(i) &= B_t(i)u^{d(i)}R_t h(M - t), \tag{3.7.1} \\
B_{t+1}^*(i) &= B_t(i)u^{d(i)}R_t h^*(M - t). \tag{3.7.2}
\end{align}

Using (3.3.6) to rewrite (3.7.1) gives

$$h(M - t) = 1 / [p_t + v_{t+1}^t q_t],$$

and with further use of (3.4.1), (3.7.2) gives

$$h^*(M - t) = v_{t+1}^t / [p_t + v_{t+1}^t q_t].$$
Reference to equation (3) of Bliss and Ronn then shows that \( p_t \) plays a role analogous to Ho and Lee's \( \pi \) while \( v_{t+1} \) plays a role analogous to their \( \delta^{M-t} \).

Two important differences should be noted. First, our \( p_t \) is time dependent, while \( \pi \) is treated as a constant. Second we show in section 3.2, by reference to the existing term structure, that the assumption of no arbitrage profit implies a functional relationship between \( p_t \) and \( u_t \), while Ho and Lee, Bliss and Ronn, and Kishimoto [1989], in treating \( \pi \) and \( \delta \) as independent, overlook the need for this further restriction. A more general analysis, in which neither the movements of interest rates nor the martingale probabilities are assumed constant is included in Pedersen, Shiu and Thorlacius [1989].

4. Futures prices

This section develops expressions for futures prices, their conditional variances, and risk premia.

4.1 Recursive calculation of futures prices

Let \( F_t(j, T, M) \) be the futures price at time \( t \), when the spot rate is \( u^{\alpha(t)} R_t \), on a contract with delivery date \( T < M \) written against a bond maturing at time \( M \). On the delivery date, the futures price equals the value of the underlying instrument; cf. Cox, Ingersoll, Ross [1981]. Therefore,

\[
F_T(j, T, M) = B_T(j, M). \tag{4.1.1}
\]

In period \( t, t \in \{0, 1, 2, \ldots, T-1\} \), the futures price is defined as \( F_t(j, T, M) \). However, the arguments \( T \) and \( M \) will be suppressed whenever no ambiguity results, and the futures price will usually be written \( F_t(j); t \in \{0, 1, 2, \ldots, T\} \).

Under the assumptions of perfect markets and no arbitrage opportunities and as a result of zero cash outlay at the initiation of a futures position, together with marking to market of the position, futures prices satisfy the condition

\[
[c_t(j, j) + c_{t}(j, j + 2)]F_t(j) = c_t(j, j)F_{t+1}(j) + c_t(j, j + 2)F_{t+1}(j + 2); \tag{4.1.2}
\]

\( t \in \{1, 2, \ldots, T-2\} \), and, for \( t = T - 1 \),

\[
[c_{T-1}(j, j) + c_{T-1}(j, j + 2)]F_{T-1}(j) = c_{T-1}(j, j)B_{T}(j) + c_{T-1}(j, j + 2)B_{T}(j + 2); \tag{4.1.2}
\]
Using (3.3.6) and (3.3.7), conditions (4.1.2) can be rewritten

\[ F_{t}(j) = p_{t}F_{t+1}(j) + q_{t}F_{t+1}(j + 2); \quad t \in \{1, 2, \ldots, T-1\}; \quad \text{and} \]

\[ F_{T}(j) = B_{T}(j) = \nu_{T}B_{T}(0) = \nu_{T}F_{2}(0); \quad j \in J_{T}. \]  

Taking (4.1.3) with \( t = T - 1 \) and using the results of (4.1.4) gives

\[ F_{T-1}(j) = p_{T-1}F_{T}(j) + q_{T-1}F_{T}(j + 2) = [p_{T-1} + \nu_{T}q_{T-1}]F_{T}(j). \]  

Similarly,

\[ F_{T-2}(j) = [p_{T-2} + \nu_{T}q_{T-2}]F_{T-1}(j), \]  

for all admissible \( j \). It follows that

\[ F_{T}(j + 2) = \nu_{T}F_{0}(j), \quad \text{and} \]

\[ F_{t}(j) = [p_{t} + \nu_{t}q_{t}]F_{t+1}(j), \quad j \in J_{t}, \quad t = 0, 1, \ldots, T - 1. \]  

Finally, setting \( j = 0 \) and applying (4.1.8) recursively gives an explicit formula for the futures price at time zero,

\[ F_{0}(0) = B_{T}(0)\prod_{t=0}^{T-1} [p_{t} + \nu_{t}q_{t}]. \]  

The futures price depends on \( M, T, B_{T}(0), \) and \( u \), but not on \( \{R_{t}\}, \quad t \in \{1, 2, \ldots, T - 1\} \).

### 4.2 Conditional variance of futures prices

Conditional on a realization \( F_{t}(j) \), the futures price at time \( t + 1 \) is either \( F_{t+1}(j) \), with probability \( p \), or \( F_{t+1}(j + 2) \), with probability \( q \). From (4.1.7), the second of these outcomes can be written

\[ F_{t+1}(j + 2) = \nu_{T}F_{t+1}(j). \]  

Appendix A shows, by taking \( y = \nu_{T}F_{t+1}(j) \) and \( x = F_{t+1}(j) \), that the conditional variance of this binomial distribution is

\[ V_{F_{t}}(j) = pq\{(\nu_{T} - 1)F_{t+1}(j)\}^{2}. \]  

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Then, using (4.1.8), (4.2.1) can be rewritten as

$$V_{F,t}(j) = \frac{pq[(v_T - 1)F_t(j)]^2}{[p_t + v_Tq_d]^2}.$$  \hspace{1cm} (4.2.2)

Since the conditional variance of the rate of change of the futures price is given by

$$V^*_t(j) = V_{F,t}(j) / [F_t(j)]^2,$$  \hspace{1cm} (4.2.3)

it follows immediately from (4.2.2) that

$$V^*_t(j) = pq\left\{\frac{v_T - 1}{p_t + v_Tq_d}\right\}^2,$$  \hspace{1cm} (4.2.4)

independent of \( j \). Also, \( V^*_t(j) \) increases in \( t \) if and only if \( p_t + v_Tq_t > p_{t+1} + v_Tq_{t+1} \), or, equivalently, \( p_t < p_{t+1} \). The conditional variance for the rate of change of futures price in (4.2.4) has the same form as that for the bond return in (3.4.3) but it is different in two respects. It is, through \( v_T \), a function of the contract expiration date and it lacks the factor equal to the square of one plus the current interest rate.

### 4.3 Conditional risk premia in futures prices

Let \( J \) be a random variable, \( J \in \{j, j + 2\} \). Define the conditional risk premium in the futures price to be the expected price change,

$$P_{F,t}(j) = E_{t+1}(J) - F_t(j).$$  \hspace{1cm} (4.3.1)

Using (4.1.7) and (4.1.8), (4.3.1) can be rewritten

$$P_{F,t}(j) = \{p + v_Tq_t\} - \{p_t + v_Tq_d\} F_{t+1}(j)$$

$$= \{p + v_Tq_t\} - \{p_t + v_Tq_d\} \frac{F_t(j)}{p_t + v_Tq_d}.$$  \hspace{1cm} (4.3.2)

The risk premium can also be expressed in terms of the rate of change of futures price,

$$P^*_{F,t}(j) = P_{F,t}(j) / F_t(j).$$  \hspace{1cm} (4.3.3)
The risk premia are positive in any period \( t \) for which \( p < p_t \), as in section 3.6. The conditional risk premium for the rate of change of the futures price in (4.3.3) differs from that for the bond returns in (3.5.3) by being a function of the contract expiration date and by the absence of the factor equal to the current interest rate.

4.4 Relations between conditional risk premia and variance

From (4.2.4) and (4.3.3),

\[
P_{F_t}(j) / \left[ V_{F_t}(j) \right]^{1/2} = K_{F,t}, \text{ where}
\]

\[
K_{F,t} = \left\{ [p + v_T q] - [p_t + v_T q_t] \right\} / \left\{ (pq)^{1/2}[v_T - 1] \right\}.
\]

After simplifying,

\[
K_{F,t} = (p_t - p) / (pq)^{1/2}.
\]

That is, the proportionality constant \( K_{F,t} \) is the same for futures prices as for bond prices; cf. (3.6.2). For the bond returns, the current interest rate factor appears in both the numerator and the denominator and cancels to give \( K_{b,t} \).

5. Empirical tests

In applying the model to the analysis of daily futures data we are making the implicit assumption that futures markets are open for a single instant once a day. Section 4 describes the evolution of futures prices and their rates of change. The two fundamental model parameters \( u \) and \( p \) are augmented by \( \{R_t\}, T, M \), and the derived conditional martingale probabilities \( p_t \) and \( q_t \). Rather than attempting to estimate these parameters, we concentrate on principal model predictions that can be tested in an amenable, related stochastic process.

5.1 Relevant model properties

For present estimation purposes, the relevant model properties are found in (4.2.4) and (4.4.2). First if, as in the example of section 3.2, the martingale probability \( p_t \) is increasing in \( t \), the conditional variance in (4.2.4) is an increasing convex function of \( t \). Second, the ratio of the conditional risk premium to the conditional standard deviation is an increasing concave function of \( t \); see (4.4.2). However, exploratory computations based on appropriate model data show that the
conditional variance in (4.2.4) increases very slowly, as does $K_{F,t}^\gamma$. For present purposes the latter will be treated as a constant in the estimation.

We also test a stringent form of null hypothesis that no variable other than those in the model should determine either the conditional risk premium or the conditional variance. This hypothesis is less important as a testing device than as a point of departure for developing the model form that will actually be estimated. The alternative hypothesis is not specific but if a more complicated model is necessary to fit the data, additions to the test equations may reveal in what direction the current theory needs to be modified. The model for which the estimates and the results of diagnostic tests are shown is one that was developed after the original null hypothesis was rejected.

Our estimation procedures are designed for time series with persistent heteroscedasticity: we employ univariate ARCH models, as in Engle [1982], Bollerslev [1986], and Engle, Lilien and Robins [1987]. Let $x_{t-1}$ and $g_{t-1}$ be vectors of variables known at $t-1$ and $\gamma$ and $\phi$ be corresponding vectors of parameters. Let $\left(\frac{F_t}{F_{t-1}}\right) - 1$ be the observed rate of change of futures price corresponding to the expectation in (4.3.3) plus an error term $\epsilon_t$ that will be assumed to be conditionally normally distributed with mean zero and variance $h_t$.

The specification of the conditional variance accommodates persistence from period to period but it is less restrictive than that in (4.2.4) by including other determinants of variation. The test equation system and its estimation are similar to those used by McCurdy and Morgan [1988] for foreign currency futures,

$$\left(\frac{F_t}{F_{t-1}}\right) - 1 = \gamma x_{t-1} + \phi h_{t-1}^{1/2} + \psi e_{t-1} + \epsilon_t,$$

$$h_t = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta h_{t-1} + \phi g_{t-1}^\phi.$$  \hspace{1cm} (5.1.1)  \hspace{1cm} (5.1.2)

The theoretical model corresponds to the restrictions $\gamma = 0$, $\psi = 0$ in (5.1.1) and, subject to the appropriateness of the assumption of an ARCH process for the evolution of the conditional variance, $\phi = 0$ in (5.1.2).

The log likelihood is maximized by numerical methods. Since the maintained hypothesis of conditional normality does not hold, we follow a scheme similar to that of Bollerslev and Wooldridge (1988). All standard errors quoted in the paper are computed to be robust, as in White (1982). Let $A$ be the numerical approximation to the matrix of second derivatives with respect to the free variables. Let $B$ be the numerical estimate of the information matrix, formed by taking the average of the
period by period outer products of the gradient. The standard errors are computed from the diagonal elements of the matrix $A^tBA^{-1}$.

5.2 Data

At any time, two futures contracts for US treasury bills are actively traded on the International Monetary Market (IMM) of the Chicago Mercantile Exchange. Several futures contracts against eurodollar deposits also trade actively on the same market. Futures contracts against 90 day treasury bills trade actively in the last six months of the contract life, while the contracts against 90 day eurodollar time deposits trade actively for a considerably longer time. The prices are not generally the same for these two sets of contracts. One difference between the two is that the futures contract promises specifies delivery of a treasury bill maturing 90 days after contract delivery date, while the eurodollar time deposit contract is settled by a cash payment. The final settlement price for the eurodollar time deposit contract is determined from a trimmed average of the London Interbank Offered Rate (LIBOR) quoted by 12 randomly chosen banks of at least 20 banks participating in the London eurodollar market.

We obtained settlement price IMM index data for the two shortest remaining life contracts for the treasury bill futures and for the three shortest remaining life contracts for the eurodollar futures for the years 1982 through 1988. The IMM index values were converted into futures prices. We also collected overnight eurocurrency interest rates for the USA. Limits were tighter on the movement of treasury bill futures prices than on the eurodollar futures. In 1982 there were 9 days on which the limit of 0.60 of the IMM index was reached in one or the other contract for treasury bills; in the eurodollar contracts, there were only 2 days with limit moves of 1.00 of the IMM index in 1982. No limit moves were found in the other years but, for purposes of defining indicator variables, the date 871020 was classified as a limit move day because of the very sharp drop in interest rates the day after the stock market crash. The IMM index in all contracts analyzed jumped by more than 1.00 on 871020. There were 1769 observations in each series.

5.3 Results

The ARCH-M model corresponding to the main hypothesis of a direct link between the conditional mean and the conditional standard deviation survived all preliminary tests and is the basic structure of the estimated models. However, the final models chosen (after the simple model corresponding to the null hypothesis was rejected) include variables additional to those specified in the theory. The purpose
is to see what can be learned from the data about possible directions for adaptations of the theory.

We fit similar models to all five sets of contracts. Occasionally, this means that a parameter which is strongly significant in one series is included in another, even though its explanatory power in the second case is slight. On the other hand, such a parameter is eliminated when its inclusion interferes with convergence in the numerical optimization.

Table 1 shows the parameter estimates and standard errors for the five series. The first two rows show estimates for the treasury bill contracts with the shortest and next shortest remaining lives. The row headed TB1 corresponds to data from contracts with 91 to 1 days of life remaining and the row headed TB2 corresponds to data from contracts with 182 to 92 days remaining. The remaining three rows display estimates for the eurodollar contracts, EU1, EU2 and EU3, for which the remaining lives are 91 to 1 days, 182 to 92, and 273 to 183 days respectively.

Each row of table 1 shows estimates of the conditional mean (5.1.1) and lines 6 through 13 to the conditional variance function (5.1.2). The ARCH-M model specifies the conditional mean to be a function of the conditional standard deviation. In table 1 the estimates of the ARCH-M parameter $\theta$ are positive in all cases and more precisely estimated in the shorter remaining life contracts, TB1 and EU1. The main hypothesis of the model is retained in these analyses.

In other respects, the theoretical model does not fare as well. Column 2 of table 1 reveals that the lagged dependent variable has a significantly positive coefficient estimate in all cases except TB2, for which a moving average error term, (with coefficient estimate in column 4) proved to be more in tune with the data. Significant coefficient estimates for autoregressive or moving average terms suggest there are relevant risk premium variables omitted from the model. Column 3 also shows a significantly negative coefficient estimate for the indicator variable representing the first business day of the week. Lower prices of assets or contracts at the beginning of the week are common for as yet unknown reasons.

Table 1 also presents evidence of additional sources of changing conditional variance. The conditional variance is higher when the short term interest rate is high, (as it is on weekends when it reflects three days' interest). Offsetting this are the negative coefficients for the Monday indicator variable. The conditional variance is high on Friday (from Thursday close to Friday close). It is lower as the end of the contract life approaches and, because the square of the remaining life is the variable used, the relationship between the conditional variance and remaining life is convex.
The last column of Table 1 gives the estimates for the limit move indicator variable. This variable is included because the equilibrium futures price cannot be observed when trading is halted at the limit price. The results also reflect the very large jump in futures price on 871020.

Tables 2 and 3 present the diagnostic tests and tests for omitted variables. Such evidence was used in the process of choosing the form of the empirical models. For example, the need for indicator variables for the first business day of the week and for Friday was identified by tests of the type shown in these two tables. The first column of table 2 is a likelihood ratio test showing that the restricted model with \( \gamma = 0, \psi = 0, \) and \( \phi = 0 \) in (5.1.1) and (5.1.2) is rejected in favour of the chosen model. Table 2 includes tests for autocorrelation in the standardized residuals, a test for remaining heteroscedasticity based on the autocorrelation function for squared residuals, and conditional moment [Newey (1985), Tauchen (1985)] tests for skewness and kurtosis. The models are satisfactory in most respects but in the test for kurtosis there is evidence against the hypothesis of conditional normality.

The last test shown in table 2 is the Pagan-Sabau [1987] consistency test. The eurodollar futures series are satisfactory with respect to this test, in which the difference between the squared residual and its estimated conditional variance is regressed on the estimated conditional variance. For treasury bills futures, the regression slope coefficient test statistics reveal a weak negative relationship, suggesting that some sharp changes in treasury bill futures prices are not adequately predicted by the chosen variance function.

Table 3 tests for variables that may have been inappropriately excluded from the fitted models. Because the models for the longer contracts are similar to those for the shortest remaining life contract series, only the shortest series are analysed. The p-values in the table are not small except for the squared lagged dependent variable and the Tuesday indicator variable in the conditional variance in the treasury bill series. Linear and concave forms of the time to delivery date are not relevant to either the conditional mean or the conditional variance.

The theoretical model specifies no role for the current level of interest rates in the risk premium for futures prices. In contrast, Jacobs and Jones [1980] postulate the risk premium to be a concave function of the remaining time to delivery of the treasury bill futures contract, with a multiplicative factor linear in the current spot interest rate. They conclude the risk premium decreases as the interest rate rises. Table 3 shows that the current spot rate of interest, \( R - I \), is properly excluded from the risk premium (or mean rate of change of price) in our model, even though it does appear in the conditional variance. In other words, the large p-values for the
interest rate in table 3 are consistent with our model but provide no support for the form of risk premium postulated by Jacobs and Jones.

6. Conclusions

We have developed a model for pure discount bond prices and futures prices for contracts written against the bonds and we have tested the model with futures data. The evidence shows that the ARCH-M form predicted by the model, for the link between the conditional risk premium and the conditional standard deviation, is supported in the empirical tests. This evidence occurs in all five series examined. The theoretical model also specifies no role for the current level of interest rates in the risk premium or rate of change of bond futures prices. Our tests are consistent with this theory. In other respects, the empirical models are more complicated, in both the conditional mean and the conditional variance, than the theory specifies. Perhaps the most important addition is the autoregressive term in the conditional mean of four of the series (and the moving average term in the remaining series). These terms suggest that, in the risk premia, the short run dependence is stronger than that specified by the model.

7. References


Dybvig, Philip H., 1989, Bond and bond option pricing based on the current term structure, manuscript.


Table 1: Equation estimates for treasury bill and eurodollar futures

\[
(F_t / F_{t-1}) - 1 = \gamma_0 + \gamma_1 [(F_{t-1} / F_{t-2}) - 1] + \gamma_2 M_t + \theta \eta_t^{1/2} + \psi e_{t-1} + e_t
\]

<table>
<thead>
<tr>
<th></th>
<th>$\gamma_0$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\theta$</th>
<th>$\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TB1</td>
<td>-0.033</td>
<td>0.060</td>
<td>-0.033</td>
<td>0.210</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(0.024)</td>
<td>(0.011)</td>
<td>(0.063)</td>
<td></td>
</tr>
<tr>
<td>B2</td>
<td>-0.026</td>
<td>0.040</td>
<td>-0.040</td>
<td>0.174</td>
<td>0.065</td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.013)</td>
<td>(0.038)</td>
<td>(0.024)</td>
<td></td>
</tr>
<tr>
<td>EU1</td>
<td>-0.022</td>
<td>0.058</td>
<td>-0.031</td>
<td>0.157</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.014)</td>
<td>(0.026)</td>
<td>(0.012)</td>
<td>(0.066)</td>
<td></td>
</tr>
<tr>
<td>EU2</td>
<td>-0.036</td>
<td>0.057</td>
<td>-0.039</td>
<td>0.186</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.024)</td>
<td>(0.015)</td>
<td>(0.074)</td>
<td></td>
</tr>
<tr>
<td>EU3</td>
<td>-0.022</td>
<td>0.058</td>
<td>-0.040</td>
<td>0.143</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.024)</td>
<td>(0.015)</td>
<td>(0.074)</td>
<td></td>
</tr>
</tbody>
</table>
Table 1: Continued

variance

\[ h_t = \alpha_0 + \alpha_1 \hat{x}_{t-1} + \beta h_{t-1} + \phi_1 (R_{t-1} - 1)^2 + \phi_2 (T - t)^2 + \phi_3 M_t + \phi_4 F_t + \phi_5 L_t \]

<table>
<thead>
<tr>
<th></th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \beta )</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
<th>( \phi_4 )</th>
<th>( \phi_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TB1</td>
<td>-0.0007</td>
<td>0.089</td>
<td>0.866</td>
<td>0.040</td>
<td>0.385</td>
<td>-0.0149</td>
<td>0.0156</td>
<td>0.504</td>
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<td></td>
<td>(0.0111)</td>
<td>(0.016)</td>
<td>(0.022)</td>
<td>(0.021)</td>
<td>(0.126)</td>
<td>(0.0039)</td>
<td>(0.0043)</td>
<td>(0.240)</td>
</tr>
<tr>
<td>TB2</td>
<td>0.0003</td>
<td>0.080</td>
<td>0.896</td>
<td></td>
<td></td>
<td>-0.0150</td>
<td>0.0215</td>
<td>0.406</td>
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<tr>
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<td>(0.013)</td>
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<td>(0.023)</td>
<td></td>
<td></td>
<td>(0.0053)</td>
<td>(0.0053)</td>
<td>(0.191)</td>
</tr>
<tr>
<td>EU1</td>
<td>-0.0017</td>
<td>0.090</td>
<td>0.873</td>
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<td>-0.0120</td>
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<td>0.936</td>
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<td></td>
<td>(0.0014)</td>
<td>(0.021)</td>
<td>(0.029)</td>
<td>(0.030)</td>
<td>(0.142)</td>
<td>(0.0038)</td>
<td>(0.0041)</td>
<td>(0.502)</td>
</tr>
<tr>
<td>EU2</td>
<td>-0.0008</td>
<td>0.081</td>
<td>0.889</td>
<td>0.074</td>
<td></td>
<td>-0.0142</td>
<td>0.0149</td>
<td>0.720</td>
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<td></td>
<td>(0.0027)</td>
<td>(0.020)</td>
<td>(0.032)</td>
<td>(0.049)</td>
<td></td>
<td>(0.0078)</td>
<td>(0.0075)</td>
<td>(0.468)</td>
</tr>
<tr>
<td>EU3</td>
<td>-0.0015</td>
<td>0.090</td>
<td>0.879</td>
<td>0.074</td>
<td></td>
<td>-0.0122</td>
<td>0.0168</td>
<td>0.740</td>
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<td></td>
<td>(0.0026)</td>
<td>(0.022)</td>
<td>(0.035)</td>
<td>(0.049)</td>
<td></td>
<td>(0.0076)</td>
<td>(0.0075)</td>
<td>(0.456)</td>
</tr>
</tbody>
</table>

Standard errors shown in parentheses are robust. TB1 and TB2 refer to the treasury bill futures contracts with the shortest and next shortest remaining lives, respectively. Similarly, EU1, EU2 and EU3 refer to the eurodollar contracts with the three shortest remaining lives. M is an indicator variable taking the value 1 on a Monday or a Tuesday following a holiday Monday, F an indicator for Friday, and L an indicator for a limit move.
Table 2: Diagnostic checks on models in table 1

<table>
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<tr>
<th></th>
<th>LR</th>
<th>R</th>
<th>Q(10)</th>
<th>Q²(10)</th>
<th>S</th>
<th>K</th>
<th>P</th>
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<tbody>
<tr>
<td>TB1</td>
<td>146.0</td>
<td>-0.80</td>
<td>7.60</td>
<td>10.20</td>
<td>1.07</td>
<td>35.3</td>
<td>-1.99 (0.05)</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.42)</td>
<td>(0.67)</td>
<td>(0.42)</td>
<td>(0.30)</td>
<td>(0.00)</td>
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</tr>
<tr>
<td>TB2</td>
<td>117.9</td>
<td>-1.57</td>
<td>5.75</td>
<td>17.70</td>
<td>0.71</td>
<td>33.5</td>
<td>-2.09 (0.04)</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.12)</td>
<td>(0.83)</td>
<td>(0.06)</td>
<td>(0.40)</td>
<td>(0.00)</td>
<td></td>
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<tr>
<td>EU1</td>
<td>133.2</td>
<td>1.39</td>
<td>7.84</td>
<td>11.04</td>
<td>0.65</td>
<td>38.2</td>
<td>-0.60 (0.55)</td>
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<tr>
<td></td>
<td>(0.00)</td>
<td>(0.16)</td>
<td>(0.64)</td>
<td>(0.35)</td>
<td>(0.42)</td>
<td>(0.00)</td>
<td></td>
</tr>
<tr>
<td>EU2</td>
<td>119.4</td>
<td>-0.09</td>
<td>8.87</td>
<td>14.56</td>
<td>0.97</td>
<td>38.9</td>
<td>-1.01 (0.31)</td>
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<td></td>
<td>(0.00)</td>
<td>(0.93)</td>
<td>(0.54)</td>
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<td>(0.32)</td>
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<td>EU3</td>
<td>114.6</td>
<td>-0.28</td>
<td>9.18</td>
<td>15.80</td>
<td>2.15</td>
<td>39.6</td>
<td>-1.12 (0.26)</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.78)</td>
<td>(0.52)</td>
<td>(0.11)</td>
<td>(0.14)</td>
<td>(0.00)</td>
<td></td>
</tr>
</tbody>
</table>

LR is the log likelihood ratio test statistic for the restriction $\gamma = 0$, $\psi = 0$, $\phi = 0$. $R$ is the test statistic for runs above the mean. $Q(10)$ the Ljung-Box form of the portmanteau statistic for autocorrelation in the first 10 lags of the standardized residuals, $Q^2(10)$ the same for the squared standardized residuals. $S$ and $K$ the [Newey (1985), Tauchen (1985)] conditional moment test statistics for skewness and kurtosis, respectively, and $P$ is the Pagan-Sabau consistency test statistic computed from robust standard errors. The p-values, shown in parenthesis, are for the chi-square distribution except for $R$ and $P$, where they are for the unit normal distribution.
Table 3: OPG tests for omitted variables for shortest contract equations

<table>
<thead>
<tr>
<th></th>
<th>TB1</th>
<th>EU1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean variance</td>
<td>mean variance</td>
</tr>
<tr>
<td>((F_{t-1}/F_{t-2}) - 1)</td>
<td>5.10 (0.02)</td>
<td>1.36 (0.24)</td>
</tr>
<tr>
<td>(R_{t-1} - 1)</td>
<td>3.01 (0.08)</td>
<td>1.90 (0.17)</td>
</tr>
<tr>
<td>(e_{t-1})</td>
<td>0.79 (0.37)</td>
<td>2.65 (0.10)</td>
</tr>
<tr>
<td>Tu</td>
<td>0.01 (0.92)</td>
<td>8.10 (0.00)</td>
</tr>
<tr>
<td>W</td>
<td>0.00 (0.95)</td>
<td>1.03 (0.31)</td>
</tr>
<tr>
<td>Th</td>
<td>0.79 (0.37)</td>
<td>4.37 (0.04)</td>
</tr>
<tr>
<td>F</td>
<td>1.56 (0.21)</td>
<td>3.84 (0.05)</td>
</tr>
<tr>
<td>N</td>
<td>2.49 (0.11)</td>
<td>0.72 (0.40)</td>
</tr>
<tr>
<td>L</td>
<td>2.49 (0.11)</td>
<td>0.64 (0.42)</td>
</tr>
<tr>
<td>((T - t))</td>
<td>1.34 (0.25)</td>
<td>0.23 (0.63)</td>
</tr>
<tr>
<td>((T - t)^{1/2})</td>
<td>1.80 (0.18)</td>
<td>0.21 (0.65)</td>
</tr>
<tr>
<td>((T - t)^{-1/2})</td>
<td>3.07 (0.08)</td>
<td>0.43 (0.51)</td>
</tr>
</tbody>
</table>

p-values, for the chi-square distribution with 1 degree of freedom are shown in parenthesis. For the lagged dependent variable, \((F_{t-1}/F_{t-2}) - 1\), and the time to contract expiration variables, the square of the variable is tested in the variance. For the indicator variables, including indicators for the day of the week, limit moves, \(L\), and the first day of trading of a new futures contract, \(N\), the same form is, of course, tested in the variance and the mean.
Appendix A. Variance of a binomial distribution

Let $Z$ be a binomial random variable assuming the values $x$ and $y$ with probabilities $p$ and $q = 1 - p$ respectively. Suppose also that $y > x$. Then

$$Z' = (Z - x) / (y - x)$$

is a standardized random variable with outcomes 0 and 1 occurring with probabilities $p$ and $q$ respectively, and hence has variance $pq$. Then since

$$V(Z') = pq = V(Z) / (y - x)^2,$$

it follows immediately that

$$V(Z) = pq(y - x)^2.$$

Appendix B. Constant relative risk aversion and bond prices

This appendix shows that the assumption of constant relative risk aversion is sufficient to resolve the indeterminacy in (3.2.9) by establishing (3.2.11).

Let utility of wealth, $W$, be $U(W) = (1 / \lambda)W^\lambda$. At time 1, conditional on the current interest rate $u^{a(i)}R_1$, let the certainty equivalent for the holder of a bond with two periods remaining to maturity be $C(j)$. The utility of the certainty equivalent is the expected utility of the bond values at time 2. Before discounting back to time 1,

$$U(C(0)) = pU(1 / u^{3/4}R_2) + qU(1 / R_2) \quad (B.1)$$

$$= \frac{1}{\lambda} \left( \frac{1}{u^{3/4}R_2} \right)^\lambda (p + q u^{3/4}) \quad (B.2)$$

At time 1, with discounting, the insurance premium,

$$\pi(0) = B_j(0, 3) - C(0) / u^{1/2}R_1 \quad (B.3)$$

can be expressed as a proportion of current wealth,
\[ \frac{\pi(0)}{B_1(0, 3)} = 1 - \frac{[C(0) / u^{1/2}R_2B_1(0, 3)]}{[C(0) / u^{1/2}B_1(0, 3)]}. \]  \hspace{1cm} (B.4)

Similar steps for a downward movement in the interest rate give

\[ U(C(2)) = pU(1 / R_2) + qU(u^{3/4} / R_2), \]  \hspace{1cm} (B.5)

\[ = \frac{1}{\lambda} \left( \frac{1}{R_2} \right)^{\lambda} \left( \rho + qu^{3/4} \right) \]

\[ = u^{3/4}U(C(0)), \]  \hspace{1cm} (B.6)

which implies

\[ C(2) / C(0) = u^{3/4}. \]  \hspace{1cm} (B.7)

The insurance premium after a downward movement in the interest rate,

\[ \pi(2) - B_2(2, 3) - C(2) / (u^{1/2}R_2), \]  \hspace{1cm} (B.8)

as a proportion of current wealth is

\[ \frac{\pi(2)}{B_2(2, 3)} = 1 - \frac{[u^{1/2}C(2) / R_2B_2(2, 3)]}{C(0)}. \]  \hspace{1cm} (B.9)

Constant relative risk aversion implies that (B.4) and (B.9) are equal, so that

\[ C(0) / u^{1/2}B_2(0, 3) = u^{1/2}C(2) / R_2B_2(2, 3), \]

\[ B_2(2, 3) / B_2(0, 3) = uC(2) / C(0), \]

\[ B_1(2, 3) / B_1(0, 3) = u^{7/4}, \]  \hspace{1cm} (B.10)

which is also equation (3.2.11).