Market Value of Risk Transfer: Catastrophe Reinsurance Case

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Abstract Pricing risk transfer for derivative products in complete markets can rely on specific arbitrage-free probability transforms, but the reinsurance market is not of this nature. The theory of martingale pricing in incomplete markets has produced various optimized probability transforms including the minimum martingale measure and the minimum entropy martingale measure. Previous work has shown how these can be applied to insurance pricing. Some data on pricing of catastrophe reinsurance programs provides a test of how well these measures can explain layer pricing in an actual market, illustrating the valuation of risk transfer in the reinsurance market.

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Pricing in incomplete markets suffers from an overabundance of alternatives. Arbitrage-free pricing requires transforming the value process to a martingale and pricing all transactions by their expected values under the transform. But whereas some complete markets have unique transforms, there are many alternative possible transforms in incomplete markets. Recent work has emphasized two transforms that are appealing in that they optimize something reasonable. These are the minimum martingale transform and the minimum entropy martingale transform. Formulas for applying these to insurance pricing were presented by Møller (2003)[1].

Section 1 reviews this background. Section 2 fits transforms to some actual market reinsurance prices as a test of the methodology. One of the transforms works reasonably well, which provides a theoretically consistent basis for measuring the market value of this risk transfer.

1. Background on martingale pricing of insurance

Numerous authors have addressed pricing of single-term insurance contracts through probability transforms and valuing the entire claims process through martingale measures. The basic result for arbitrage-free pricing is that prices are expected values from a transformed process which is a martingale. This criterion requires that there is no expected drift in the transformed process. For insurance this implies that the combined transformed frequency and severity processes have a mean drift equal to that of the overall loaded premium. Then premium minus transformed losses has no expected drift.

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[1] Møller, T. (2003) "Stochastic orders in dynamic reinsurance markets," *ASTIN Colloquium Papers* http://www.astin2003.de/img/papers/moeller.pdf and to appear *Finance and Stochastics*

Møller shows that much of the earlier work can be put into a system of transforming frequency and severity probabilities in a coordinated fashion. Given a process with a constant Poisson frequencyλ and density $g(y)$ for the loss size variable Y, this system uses a function $\phi(y)$, with the only restriction that $\phi(y) > -1$ for all positive losses y. The transformed frequency parameter is then $\lambda[1+E\phi(Y)]$ and the severity density gets transformed to $g(y)[1+\phi(y)]/[1+E\phi(Y)]$.

Two interesting choices for the $\phi(y)$ function come from the minimum martingale transform and the minimum entropy martingale transform of the theory of pricing in incomplete markets. In a complete market there is often a perfect hedging strategy available which is associated with a single specific martingale transform. When this is not possible, it might be desirable to find the hedging strategy that will minimize the variance of the payoff risk. The martingale that produces this strategy is the minimum martingale transform. Thus it is minimal in the sense of quadratic risk.

The relative entropy between two measures P and Q is $E_P[dQ/dP \log(dQ/dP)]$. This is a distance of a sort, as it is zero if P=Q and is otherwise positive. However it is not symmetric in P and Q. Minimizing the relative entropy is popular and is related to optimizing a fit given the information available, according to principles of information theory. In the insurance pricing case, P is the real-world measure and a martingale Q is sought that will minimize the relative entropy – of course under the constraint that the transformed mean loss is the loaded expected loss in the premium. Q is then the martingale closest to P in the sense of relative entropy.

Møller provides the $\phi(y)$ functions that give the minimum martingale measure (MMM) and minimum entropy measure (MEM) for the surplus process (loaded premium less compound Poisson claims). Using the notation CV for the severity ratio of standard deviation to mean, so that $EY^2/(EY)^2 = 1+CV^2$, the MMM ϕ can be expressed as:

$\phi_M(y) = (y/EY)\theta/[1+CV^2]$, where $1+\theta$ is the ground-up loading factor.

This leads to $1+E\phi_M(y) = 1+\theta/[1+CV^2] = [1+CV^2+\theta]/[1+CV^2]$ so the severity density gets transformed by the factor $[1+CV^2+(y/EY)\theta]/[1+CV^2+\theta]$. This can be simplified by setting $s = \theta/[1+CV^2+\theta]$. Then:

$$\phi_M(y) = (sy/EY)/(1-s)$$
$$1+E\phi_M(Y) = 1/(1-s)$$
$$\lambda_M = \lambda/[1-s]$$
$$g_M(y) = g(y)[1-s + sy/EY]$$

The severity probability increases for losses above the mean and decreases below the mean.

For MEM, the frequency mean and severity probabilities are also multiplied by factors:

$$\phi_E(y) = e^{cy}-1, \text{ so}$$
$$1+E\phi_E(Y) = Ee^{cY}$$
$$\lambda_E = \lambda Ee^{cY}$$
$$g_E(y) = g(y)e^{cy}/Ee^{cY}.$$

It follows that $\lambda_E E_E Y = \lambda E[Ye^{cY}]$, so $1+\theta = E[Ye^{cY}]/EY$. This can be used to define c if θ is known.

Both MMM and MEM avoid a probability-transform pricing inconsistency, originally attributed to Thomas Mack, which we will call Mack's paradox. Probability transforms increase the probability attributed to larger losses so that there will be positive risk loads for higher layers. But the probability for the smaller losses then must decrease to keep the total probability equal to one. This can lead to negative risk loads for lower layers, e.g., deductible buy-backs, such as a contract that pays the loss if it is less than some fairly low limit and nothing if it is above the limit. This problem affects most of the probability transforms that have been suggested to date.

The MMM and MEM transforms circumvent Mack's paradox by increasing the frequency probability more than any severity probability is decreased. Their frequency factors are the reciprocals of the severity factors at zero. Since for these measures the severity transform factor is increasing in y, the severity factor at zero is smaller than the actual factor for any $y>0$. This is not necessarily true for ϕ functions in general and Møller has some examples where it is not.

2. Application to reinsurance pricing

We have a proprietary database of some catastrophe reinsurance contracts for individual perils, so there are separate contracts for hurricane (H), earthquake shake damage (E), and fire following earthquake (FE). Separate catastrophe model loss estimates were also available by peril for each contract, and these were used by reinsurers in the treaty pricing. From this database the s and c parameters of the MMM and MEM transforms were estimated for the three perils.

The loss size y used in the transforms was the total market loss from the cat model, defined as the sum of the separate company cat losses for each event the model simulates. This takes the view that reinsurers will base their pricing on their exposure to the total market loss. The sum of the company results available is used as a proxy for the total loss.

Although some portion of the reinsurance premium is for expenses, this was ignored in these fits. Thus the transformed losses were fit to approximate the premiums for the layers. This was done in part to understand the value of the reinsurance from the buyer's viewpoint – i.e., to model the value of the risk transfer as a transform of the losses ceded.

A convenient notation in this arena is to divide the layer expected losses and premiums by the layer limit to yield loss on line (LOL) and rate on line (ROL). The objective function minimized by the fit was chosen somewhat arbitrarily to be the sum of the squared relative errors. This was done in part to get the best fit for the low LOL layers, where the loadings ROL/LOL tend to be highest and the value of risk transfer less well understood.

Since both measures are intuitive but not compelling, a mixture of them was also attempted. This uses a weight w = k/(k+$y^{½}$) for MMM and 1 – w for MEM. The weights were applied to the ϕ functions so that $\phi_W(y) = w\phi_M(y) + (1-w)\phi_E(y)$. This produces:

$$1+\phi_W(y) = \frac{k}{k+y^{1/2}}\left(1+\frac{sy}{(1-s)EY}\right)+e^{cy}y^{1/2}\Big/k+y^{1/2}$$ and

$$1+E\phi_W(Y) = \frac{ks}{(1-s)EY}E\left(\frac{Y}{k+Y^{1/2}}\right)+E\left(\frac{Y^{1/2}e^{cY}}{k+Y^{1/2}}\right)+E\left(\frac{k}{k+Y^{1/2}}\right).$$

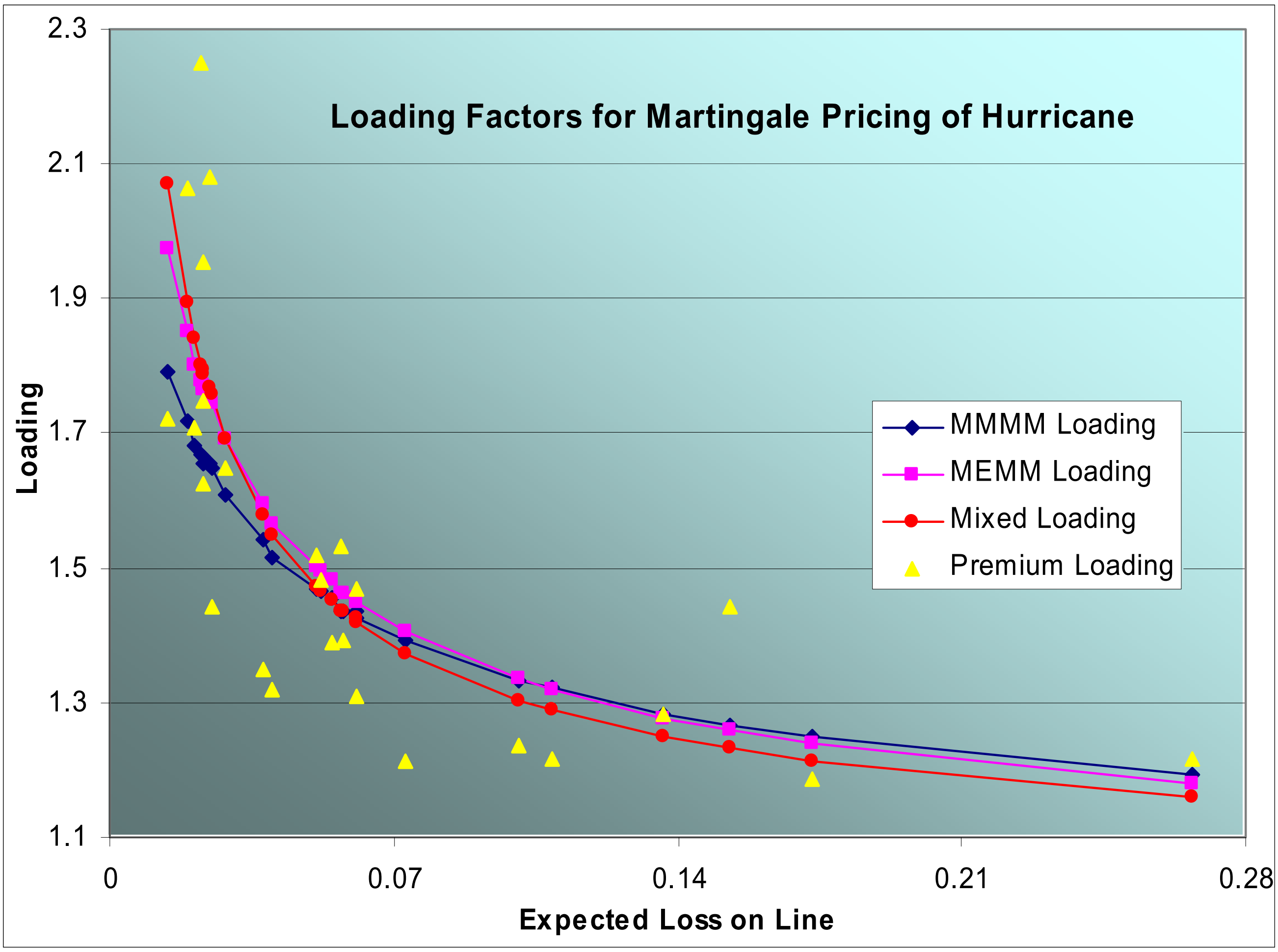
The exponent of ½ on y in the weight function was selected after some experimentation. Letting the exponent itself be an additional parameter improved each fit only slightly. For s=0, ϕ_W is a weighting of no loading with the MEM load, so does not reduce to the MEM load. The parameters that worked best did not give weights that varied between 0 and 1. The maximum weight to the MEM tended to be small, but the c parameter could be enough larger for this to give more loading to the low LOL layers, which is where the errors were often larger.

The fitted parameters and resulting optimized error measures for the three perils are:

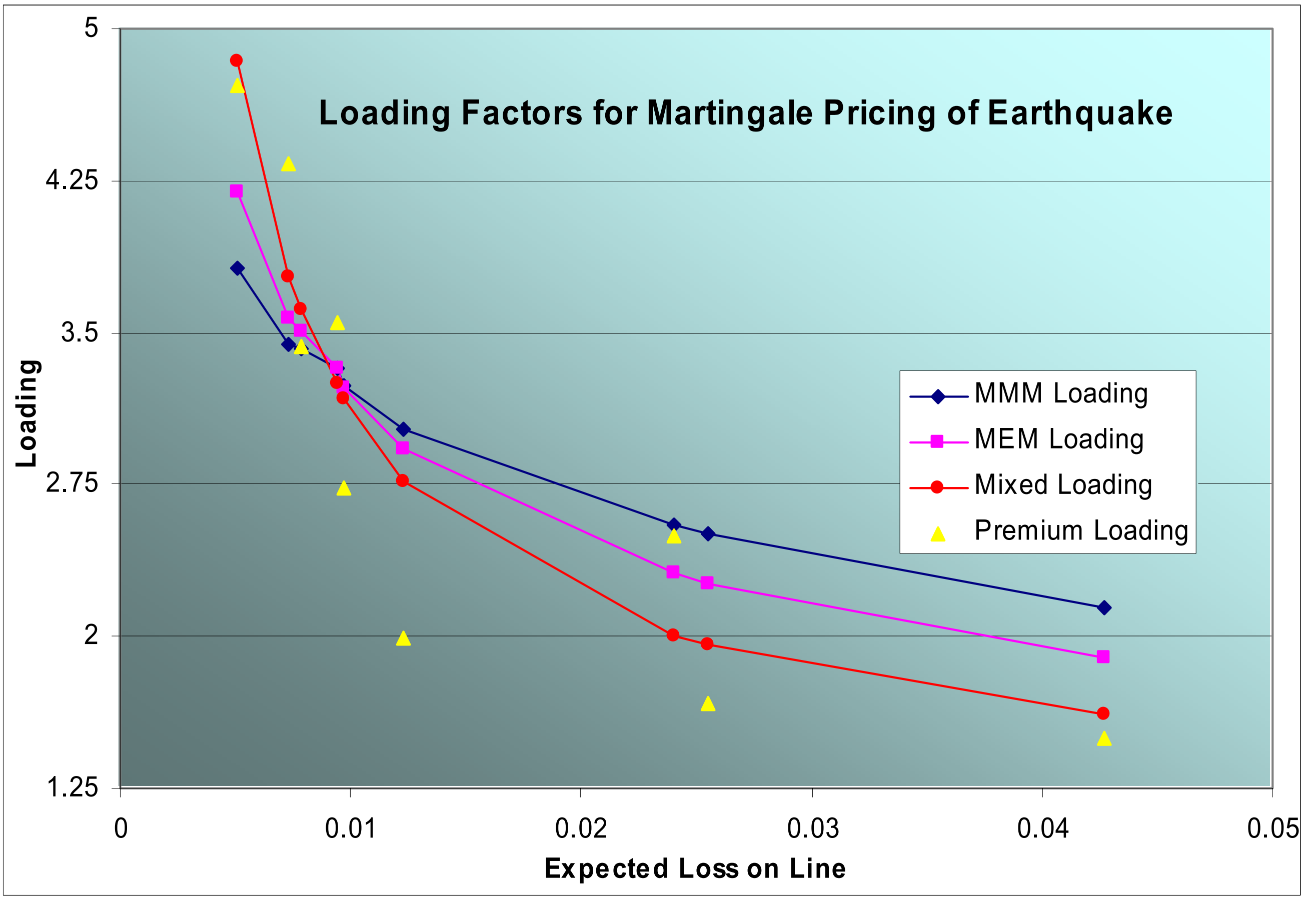
	H	**H Error**	**E**	**E Error**	**FE**	**FE Error**
MMM s	.017	.381	.021	.470	.036	.160
MEM ln c	-28.2	.308	-26.3	.311	-26.6	.082
Mixed	.011, -27.0	.298	0, -25.5	.220	.116, -26.7	.064

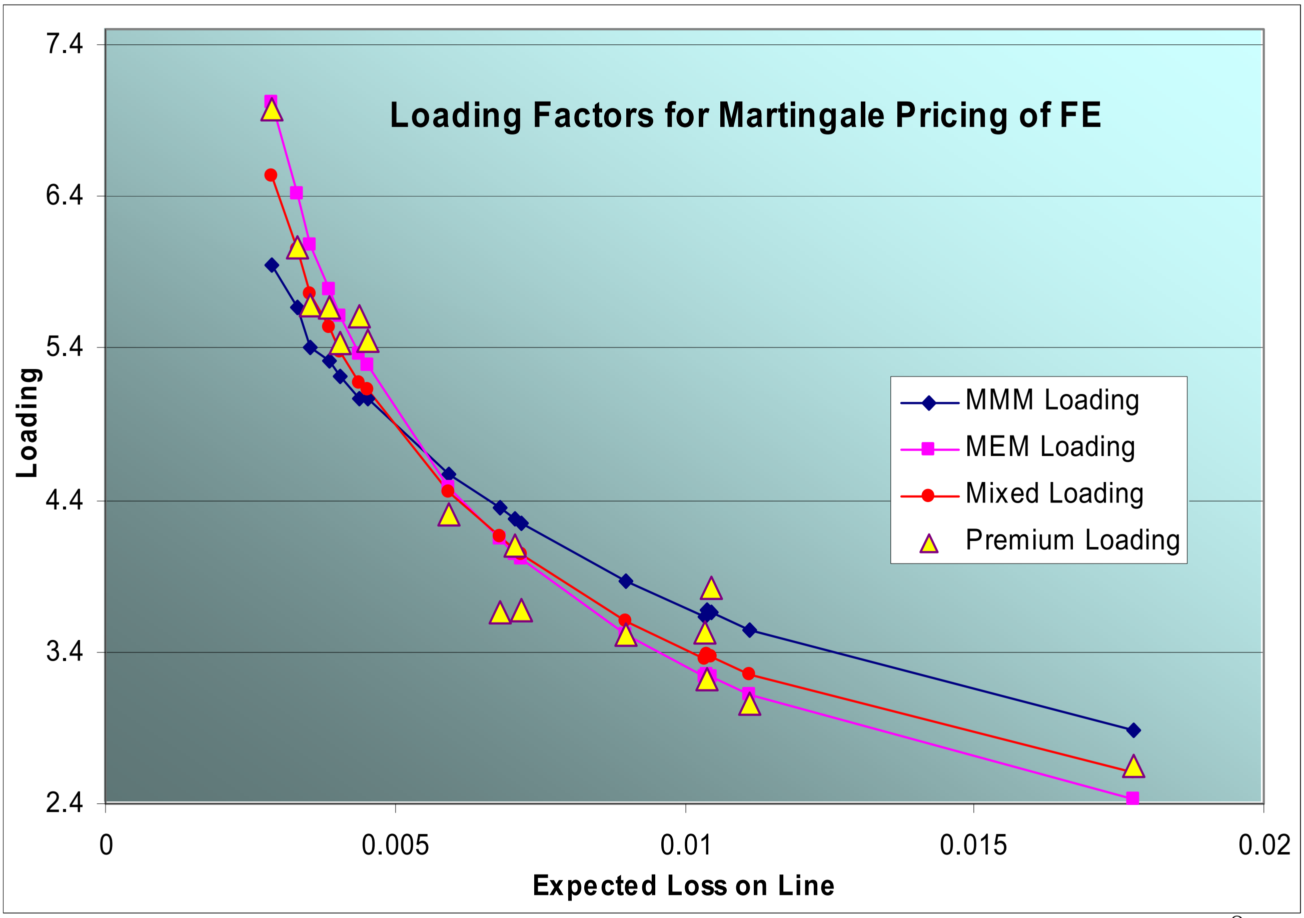
The error measures are not comparable across perils because there are different numbers of treaties for each peril. Within each peril the MEM fits are noticeably better than those from the MMM. The mixture has three parameters (k, s, c) so it is not clear that it fits enough better.

A graphical representation may be the easiest way to see how well the fits vary across the layers. The actual vs. fitted loading factors ROL/LOL are shown for each peril.



MMM misses the high loadings for the low LOL layers for all perils. This is not atypical for quadratic risk measures in general. MEM does a reasonable job in this area and overall. Thus it appears to be a good candidate for a model of market pricing of reinsurance risk. Its use of an exponential moment suggests that the market has a high degree of aversion to the largest losses.


Loading Factors for Martingale Pricing of Earthquake
Loading
Expected Loss on Line
5
4.25
3.5
2.75
2
1.25
0
0.01
0.02
0.03
0.04
0.05
MMM Loading
MEM Loading
Mixed Loading
Premium Loading


Loading Factors for Martingale Pricing of FE
Loading
Expected Loss on Line
7.4
6.4
5.4
4.4
3.4
2.4
0
0.005
0.01
0.015
0.02
MMM Loading
MEM Loading
Mixed Loading
Premium Loading