Capital Allocation in Insurance: Economic Capital and the Allocation of the Default Option Value*

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Abstract

The determination and allocation of economic capital is important for pricing, risk management and related insurer financial decision making. This paper considers the allocation of economic capital to lines of business in insurance. We show how to derive closed form results for the complete markets, arbitrage-free allocation of the insurer default option value, also referred to as the insolvency exchange option, to lines of business. We assume that individual lines of business and the surplus ratio are joint log-normal although the method we adopt allows other assumptions. The allocation of the default option value is required for fair pricing in the multi-line insurer. We illustrate some other methods of capital allocation and give numerical examples for the capital allocation of the default option value based on explicit payoffs by line.

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Introduction

Capital allocation in insurance is a topic of great interest to both practitioners and academics. Butsic (1999) [2], Meyers (2003) [7], Myers and Read (2001) [10], Merton and Perold (1993) [8], Panjer (2001) [11], Phillips, Cummins and Allen (1998) [12], Valdez and Chernih (2003), Venter (2004) [16], Zanjani (2002) [17] and Zeppetella (2002) [18] are amongst an increasing number of authors addressing the issue of capital allocation in insurance.

Venter (2004) [16] gives a comprehensive review and discussion of approaches for allocating capital to lines of business including a discussion of the Myers and Read (2001) [10] approach based on Butsic (1999) [2]. Cummins (2000) [3] provides an overview of capital allocation in insurance including the methods of allocating capital proposed by Merton and Perold (1993) [8] and Myers and Read (2001) [10]. An often stated benefit of the Myers and Read (2001) [10] approach is that all capital is allocated whereas the Merton and Perold (1993) [8] approach leaves some capital unallocated. Myers and Read (2001) [10] identify the importance of the insurer default option in determining capital allocation. They show that a marginal default option value can be used to determine the capital required to maintain a specified insurer default option value to liability ratio for small changes in lines of business.

Cummins (2000) [3] states (page 24)

> Because the amounts of capital allocated to each line of business differ *substantially* between the M-R and M-P methods, the two methods will not yield the same pricing and project decisions when used with a method that employs by-line capital allocations. Consequently, it is important to determine which method is correct.

Cummins (2000) [3] also states in the final sentence of his conclusion

> Finally, the winning firms in the twenty-first century will be the ones that successfully implement capital allocation and other financial decision-making techniques. Such firms will make better pricing, underwriting and entry/exit decisions and create value for shareholders.

Mildenhall (2002) [9] shows that the Myers and Read (2001) [10] method will only allocate the default option value in practice so that the values "add-up" for insurance loss distributions by line of business that are homogeneous. This is important since capital allocation is often used in practice for determining the amount of capital required as lines of business are grown or exited.

Phillips, Cummins and Allen (1998) [12] argue that it is not appropriate to allocate capital to lines of business. Gründl and Schmeiser (2003) [4] make the case that there is no need to allocate capital for the purposes of pricing insurance contracts and determining surplus requirements.

Sherris (2004) [14] discusses capital allocation in a complete markets and frictionless model. He shows how economic capital of the total insurer balance

sheet can be allocated by line of business. He gives results for the allocation of the default option value allowing explicitly for the share of the asset shortfall for each line of business in the event of insolvency. His approach fully allocates the total insurer default option value to lines of business regardless of the loss distribution provided arbitrage-free values can be determined.

The allocation of the insurer level default option value has economic significance since this determines the fair price of the insurance by line. Prices of lines of business reflect the risk of the liability as well as the allocation of the default option value to the line of business. Thus the allocation of the default option value by line of business is critical to fair pricing in the multi-line insurer. We discuss the implications of the default option value and its allocation to lines of business for pricing and capital management. For pricing it is necessary to allocate the default option value to lines of business in order for insurance prices to reflect the impact of insolvency on claims. For capital management it is necessary to consider the capital required to maintain the default option value in a dynamic model in order to properly account for the impact of changes in lines of business.

Allocation of total insurer capital requires an allocation of assets as well as the default option value. Sherris (2004) [14] shows there is no unique or optimal way to allocate the assets to lines of business without additional criteria.

In this paper we derive closed form expressions for the default option value by line of business under the assumption that lines of business and the ratio of assets to liabilities are joint log-normal. The approach used can be generalised to other distributions for lines of business.

We provide a review of the Myers and Read (2001) [10] results and other approaches to capital allocation in insurance. Our approach allows for the use of more realistic assumptions than those in Myers and Read (2001) [10]. A major difference between our approach and other approaches is that we explicitly determine the value of the default option by line of business based on payoffs in insolvency. We also recognise that there is no unique allocation of assets to line of business without additional criteria.

We conclude with some numerical examples.

1 Allocation of Economic Capital and the Insurer Balance Sheet

This section is based on Sherris (2004) [14]. More details including numerical examples are provided in his paper. The model is a single period model and we will denote the terminal date by T. The full economic balance sheet of the insurer consists of the assets, $V(t)$, the liabilities, consisting of $L_i(t)$ for line of business i and the default option value, $D(t)$. Surplus is defined as

$$S(t) = V(t) - L(t)$$

At time zero we also have

$$s = \frac{S(0)}{L(0)}$$

where s is the initial known solvency ratio for the insurer.

The insurer economic balance sheet is as follows:

Balance Sheet	**Initial Value**	**End of Period Payoff**
Assets	V	$V(T)$
Liabilities	$L - D$	$L(T) - (L(T) - V(T))^{+}$
Equity	$S + D$	$\boxed{V(T) - L(T) + (L(T) - V(T))^{+}}$

Table 1: *Total Insurer Economic Balance Sheet*

where $V = V(0)$, $L = L(0)$, $D = D(0)$ and $S = V - L$. The total economic capital of the insurer is $S(t) + D(t)$ and the market, or economic, value of the liabilities allowing for the insolvency of the insurer is $L(t) - D(t)$.

In the next section we consider how to allocate the default option value component of the economic capital to lines of business for the initial balance sheet of the insurer. We derive expressions for $D_i = D_i(0)$, the default option value allocated to line of business i, based on payoffs allowing for insolvency and equal priority shown by Sherris (2004) [14] to be given by $L_i(T)\left(1 - \frac{V(T)}{L(T)}\right)^{+}$. The allocated values add up to the total insurer default option value. The other component of the insurer economic capital is the surplus $S(0)$. Since this equals the initial value of the assets minus the initial value of the liabilities, we need to allocate assets to lines of business in order to allocate initial surplus. There is no unique way to allocate the assets to line of business without additional criteria such as requiring constant expected return to line of business based on allocated capital.

We can determine the solvency ratio for each line of business based on an arbitrary allocation of assets to line of business. This will be given by

$$s_i = \frac{S_i}{L_i} = \frac{V_i - L_i}{L_i}$$

where V_i is the value of the assets allocated to line of business i at time 0. The values of the payoff to each line of business in the event of insolvency determines the internal balance sheet for line of business i:

Balance Sheet	**Initial Value**	**End of Period Payoff**
Assets	V_i	$V_i(T)$
Liabilities	$L_i - D_i$	$L_i(T) - L_i(T)\left(1 - \frac{V(T)}{L(T)}\right)^{+}$
Equity	$S_i + D_i$	$\boxed{V_i(T) - L_i(T) + L_i(T)\left(1 - \frac{V(T)}{L(T)}\right)^{+}}$

Table 2: *Internal Balance Sheet for Line of Business i*

The by line allocation of the default option value is not arbitrary and should equal the value of the loss in the payoff to the line of business in the event of insolvency. The total surplus is allocated to lines of business so that $S = \sum_{i=1}^{M} S_i$ with

$$S = \sum_{i=1}^{M} s_i L_i$$

so that if $x_i \triangleq \frac{L_i}{L}$ then

$$s = \sum_{i=1}^{M} x_i s_i$$

where $\sum_{i=1}^{M} x_i = 1$. The allocation of S_i is arbitrary unless other criteria are specified. For instance, if allocated capital is to be used for pricing purposes, then it should be allocated so that each line of business has an equal expected return to capital.

We define

$$d_i = \frac{D_i}{L_i}$$

so that

$$D = \sum_{i=1}^{M} d_i L_i$$

We have allocated the default option value and the surplus so that the line of business allocations add to the total economic capital with

$$S + D = \sum_{i=1}^{M} (s_i + d_i) L_i$$

The allocation of capital to lines of business is an internal allocation that has no direct economic implications for the solvency of the insurer. It is the total insurer level of capital that determines the future likelihood of insolvency of the insurer - how it is allocated to lines of business has no bearing on the solvency of the insurer. The economic impact of insolvency is determined by the insurer default option value. The allocation of the insurer default option value to line of business is important since it is included in the fair price of insurance by line of business. This by line price is an entity specific price since it adjusts the market price of an insurance risk, assuming no default by the insurer, for the default cost of the insurance entity.

In Phillips, Cummins and Allen (1998) [12] and in Sherris (2003) [13], the assumption is made that the insurer default option value can be allocated to lines of business in proportion to the liability values for pricing purposes. Thus it is assumed that

$$D_i = \frac{L_i}{L} D \quad \text{for all } i$$

which implies that

$$d_i = d \quad \text{for all } i$$

This is also an assumption made in Myers and Read (2001) [10]. However, as shown in Sherris (2003) [13], this allocation does not give the fair allocation of the insurer default option value by line of business.

2 The Insurer Default Option Value and Allocation to Line of Business

For a given insurer balance sheet, Sherris (2004) [14] shows how the insurer default option value can be allocated to lines of business and that this can be done uniquely by line of business. The liability payoffs by line of business in the event of insolvency can be explicitly determined allowing for the ranking of different lines of business, which for an insurer is normally an equal priority of policyholders with claims outstanding at the date of insolvency. He also shows that default option values by line of business do not depend on the surplus allocation to line of business as implied by the results of Myers and Read (2001) [10]. He shows that, in a complete and frictionless market model, there is no unique allocation of surplus without imposing additional criteria such as requiring each line of business to have an equal expected return on capital or an equal solvency ratio.

We derive closed form expressions for the default option value and the allocation by line of business. Our assumptions are different to those of Myers and Read (2001) [10] and the approach provides a more flexible and general method of determining the insurer default option value as well as its allocation to lines of business. We will formally determine the default option value and the allocation of the default option and surplus to line of business. We will then discuss the assumptions and results of Myers and Read (2001) [10].

Denote the value of the liabilities of line $i = 1, \ldots, M$ at time t by $L_i(t)$ for $0 \leq t \leq T$ where T is the end of period. Assume that the risk-neutral dynamics of $L_i(t)$ are

$$dL_i(t) = \mu_i L_i(t)\, dt + \sigma_i L_i(t)\, dB^i(t) \quad \text{for } i = 1, \ldots, M \tag{1}$$

and that the amount $L_i(T)$ is the claim amount paid at time T, so that claims for each line have a log-normal distribution at time T. $B^i(t)$; $i = 1, \ldots, M$, are Brownian motions under the risk-neutral dynamics. We also assume $dB^i(t)\, dB^j(t) = \rho_{ij} dt$. If L_i can be replicated by traded assets and there are no claim payments made other than at the end of the period then $\mu_i = r$, the risk free rate. The total liabilities are given by

$$L(t) \triangleq \sum_{i=1}^{M} L_i(t)$$

These values for the liabilities assume claims are paid in full and ignore the effect of insolvency of the insurer on claim payments. The insurance policies are contingent claims on the value of the liabilities with payoff that depends on the insurer solvency.

The value of the assets of the insurer at time t are denoted by $V(t)$ for $0 \leq t \leq T$. We assume that the ratio of assets to liabilities

$$\Lambda(t) \triangleq \frac{V(t)}{L(t)} \tag{2}$$

follows geometric Brownian motion with risk neutral dynamics given by

$$d\Lambda(t) = \mu_\Lambda \Lambda(t)\, dt + \sigma_\Lambda \Lambda(t)\, dB^\Lambda(t) \tag{3}$$

with

$$\Lambda(0) = \frac{V(0)}{L(0)} = (1+s)$$

where s is the solvency ratio at time 0. $B^\Lambda(t)$ is a Brownian motion under the risk-neutral dynamics. Note that the initial balance sheet values of the total assets and liabilities are $V(0)$ and $L(0)$ respectively. We assume that we know the parameters μ_Λ and σ_Λ for the dynamics of the ratio of assets to liabilities and that these are not approximated from the dynamics of the individual assets and liabilities. Later in this paper, when we consider the Myers and Read (2001) [10] results, we will need to derive approximate expressions for these parameters in terms of the dynamics of the individual assets and liabilities.

Myers and Read (2001) [10] assume that both L and V are log-normal in order to apply the Margrabe [6] exchange option formula. In our assumptions, the ratio of assets to liabilities, $\Lambda(t)$ is assumed to be log-normal. We know that if L and V are portfolios of individual assets and lines of business respectively, each with log-normal dynamics, then the portfolio can not be log-normal unless the portfolio is dynamically rebalanced to ensure constant weights. In the Myers and Read (2001) [10] model the proportions of the lines of business are assumed fixed at the start of the period and not continuously rebalanced.

In our approach it is possible to consider a wider range of processes for the individual lines of business for which it will be reasonable to assume that the ratio of assets to liabilities, $\Lambda(t)$, is approximately log-normal. If assets are selected at the start of the period to closely match the liabilities then the ratio of assets to liabilities at the end of the period should be well approximated with a log-normal distribution. The distribution to be used for these assumptions in practice needs to be based on empirical data.

The payoff for the insurer default option at the end of the period is

$$[L(T) - V(T)]^+ \tag{4}$$

We assume that all lines of business rank equally in the event of default, so that policyholders who have claims due and payable in line of business i will be entitled to a share $\frac{L_i(T)}{L(T)}$ of the assets of the company, where the total outstanding claim amount is $L(T) = \sum_{i=1}^{M} L_i(T)$. Sherris (2004) [14] shows that the end-of-period payoff to line of business i is well defined based on this equal priority and given by

$$\begin{array}{ll} \frac{L_i(T)}{L(T)} V(T) & \text{if } L(T) > V(T) \ \ (\text{or } \frac{V(T)}{L(T)} \leq 1) \\ L_i(T) & \text{if } L(T) \leq V(T) \ \ (\text{or } \frac{V(T)}{L(T)} > 1) \end{array} \tag{5}$$

This is the normal situation for policyholders of insurers. They rank equally for outstanding claim payments in the event of default of the insurer.

The value of the exchange option allocated to line of business i is denoted by $D_i(t)$. This is given by the value of the pay-off to the line of business allowing for the payoffs in the event of insurer default. Assuming no-arbitrage, this is

$$\begin{aligned} D_i(t) &= E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\frac{V(T)}{L(T)}\right]^+|\mathcal{F}_t\right] \\ &= E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\Lambda(T)\right]^+|\mathcal{F}_t\right] \end{aligned} \tag{6}$$

where $(\mathcal{F}_t)$ is the filtration defined by the Brownian motions $B^\Lambda(t)$, $B^i(t)$; $i=1,\ldots,M$, and Q indicates that the expectation is under the risk neutral dynamics.

The total company level insolvency exchange option for the insurer is

$$\begin{aligned} D(t) &= E^Q\left[e^{-r(T-t)}\left[L(T)-V(T)\right]^+|\mathcal{F}_t\right] \\ &= \sum_{i=1}^{M} E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\frac{V(T)}{L(T)}\right]^+|\mathcal{F}_t\right] \\ &= \sum_{i=1}^{M} E^Q\left[e^{-r(T-t)}L_i(T)\left[1-\Lambda(T)\right]^+|\mathcal{F}_t\right] \end{aligned}$$

The value of the insolvency exchange option allocated to each line of business "adds up" to the total insurer value since the total insurer insolvency payoff is the sum of the amounts allocated to line of business using the equal priority for outstanding claims given in (5).

Since we assume individual lines of business are log-normal and the ratio of assets to liabilities is log-normal we can derive the value of the default option value for each line of business in closed form. The value of the insurer total default option is then given by the sum of the default option values for each line of business.

The insurer default option value for line of business i can be derived using a change of numeraire from the risk free bank account to $L_i(t)$ and corresponding change of measure. First note that under the risk neutral measure Q

$$L_i(T) = L_i(0)\,e^{\mu_i T}Z_i(T)$$

where

$$Z_i(T) = \exp\left[\sigma_i B^i(T) - \frac{1}{2}\sigma_i^2 T\right]$$

Define Q^i on $\mathcal{F}_T$ by

$$\frac{dQ^i}{dQ}|\mathcal{F}_T = Z_i(T) \tag{7}$$

and note that

$$E\left[Z_i(T)|\mathcal{F}_t\right] = Z_i(t) \quad t \le T$$

Now

$$\begin{aligned} & E^{Q_i}\left[e^{-r(T-t)}\left(1-\Lambda\left(T\right)\right)^{+}|\mathcal{F}_t\right] \\ = & \frac{E^{Q}\left[e^{-r(T-t)}Z_i\left(T\right)\left(1-\Lambda\left(T\right)\right)^{+}|\mathcal{F}_t\right]}{E^{Q}\left[Z_i\left(T\right)|\mathcal{F}_t\right]} \end{aligned}$$

so that

$$\begin{aligned} D_i\left(t\right) &= E^{Q}\left[e^{-r(T-t)}L_i\left(T\right)\left[1-\Lambda\left(T\right)\right]^{+}|\mathcal{F}_t\right] \\ &= E^{Q_i}\left[e^{-r(T-t)}\left(1-\Lambda\left(T\right)\right)^{+}|\mathcal{F}_t\right]Z_i\left(t\right)L_i\left(0\right)e^{\mu_i T} \\ &= L_i\left(t\right)e^{\mu_i(T-t)}E^{Q_i}\left[e^{-r(T-t)}\left(1-\Lambda\left(T\right)\right)^{+}|\mathcal{F}_t\right] \end{aligned} \tag{8}$$

We can make the change of numeraire involved in the above more explicit by noting that the default option value for line of business i can be written as

$$\begin{aligned} \frac{D_i\left(t\right)}{L_i\left(t\right)e^{(r-\mu_i)t}} &= \frac{e^{rt}}{L_i\left(t\right)e^{(r-\mu_i)t}}E^{Q}\left[e^{-rT}L_i\left(T\right)\left[1-\Lambda\left(T\right)\right]^{+}|\mathcal{F}_t\right] \\ &= \frac{e^{rt}}{L_i\left(t\right)e^{(r-\mu_i)t}}E^{Q}\left[\frac{L_i\left(T\right)e^{(r-\mu_i)T}}{e^{rT}}\frac{L_i\left(T\right)\left[1-\Lambda\left(T\right)\right]^{+}}{L_i\left(T\right)e^{(r-\mu_i)T}}|\mathcal{F}_t\right] \\ &= E^{Q}\left[\frac{Z_i\left(T\right)}{Z_i\left(t\right)}\frac{L_i\left(T\right)\left[1-\Lambda\left(T\right)\right]^{+}}{L_i\left(T\right)e^{(r-\mu_i)T}}|\mathcal{F}_t\right] \end{aligned}$$

where $Z_i\left(t\right)=\frac{1}{L_i(0)}\frac{L_i(t)e^{(r-\mu_i)t}}{e^{rt}}$. Using the Radon-Nikodym density process as before given by

$$\frac{dQ^i}{dQ}|\mathcal{F}_T = Z_i\left(T\right)$$

where $Z_i\left(t\right)$ is a non-negative Q-martingale with initial value $Z_i\left(0\right)=1$, under the changed probability measure we have

$$\frac{D_i\left(t\right)}{L_i\left(t\right)e^{(r-\mu_i)t}} = E^{Q_i}\left[\frac{L_i\left(T\right)\left[1-\Lambda\left(T\right)\right]^{+}}{L_i\left(T\right)e^{(r-\mu_i)T}}|\mathcal{F}_t\right]$$

or

$$D_i\left(t\right) = L_i\left(t\right)e^{\mu_i(T-t)}E^{Q_i}\left[e^{-r(T-t)}\left(1-\Lambda\left(T\right)\right)^{+}|\mathcal{F}_t\right]$$

as derived in Equation (8).

Note that $B^{\Lambda}\left(t\right), B^{j}\left(t\right); j=1,\ldots,M$, are Brownian motions under Q. We can show that $\widetilde{B}^{\Lambda}\left(t\right)=B^{\Lambda}\left(t\right)-\rho_{i\Lambda}\sigma_i t$ and $\widetilde{B}^{j}\left(t\right)=B^{j}\left(t\right)-\rho_{ij}\sigma_i t$ are Brownian motions under Q_i. Details are given in the Appendix.

In order to derive a closed form for the value of the insurer default option for line of business i consider

$$M\left(t\right) = E^{Q}\left[e^{-r(T-t)}\left[1-\Lambda\left(T\right)\right]^{+}|\mathcal{F}_t\right]$$

This is the value at time t of a European put option on an underlying asset paying a continuous dividend at rate $r-\mu_\Lambda$ with current price $\Lambda(t)$, exercise date T, and strike price 1. By assumption $\Lambda(T)$ has a log-normal distribution. Using the classical Black-Scholes result we have the closed form

$$M(t) = e^{-r(T-t)} N(-d_{2t}) - \Lambda(t) e^{-(r-\mu_\Lambda)(T-t)} N(-d_{1t}) \tag{9}$$

where

$$d_{1t} = \frac{\ln \Lambda(t) + \left(\mu_\Lambda + \frac{1}{2}\sigma_\Lambda^2\right)(T-t)}{\sigma_\Lambda \sqrt{(T-t)}} \tag{10}$$

and

$$d_{2t} = \frac{\ln \Lambda(t) + \left(\mu_\Lambda - \frac{1}{2}\sigma_\Lambda^2\right)(T-t)}{\sigma_\Lambda \sqrt{(T-t)}} = d_{1t} - \sigma_\Lambda \sqrt{(T-t)} \tag{11}$$

The default option value for line of business i is then given by

$$\begin{aligned} D_i(t) &= L_i(t) e^{\mu_i (T-t)} E^{Q_i}\left[e^{-r(T-t)} (1-\Lambda(T))^+ |\mathcal{F}_t\right] \\ &= L_i(t) e^{\mu_i (T-t)} M^i(t) \end{aligned} \tag{12}$$

where $M^i(t)$ is evaluated with the same formula as for $M(t)$ but with μ_Λ replaced by $\mu_\Lambda^i = \mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda$ where

$$dB^i(t)\, dB^\Lambda(t) = \rho_{i\Lambda} dt$$

This follows since

$$\begin{aligned} d\Lambda(t) &= \mu_\Lambda \Lambda(t)\, dt + \sigma_\Lambda \Lambda(t) \left(d\widetilde{B}^\Lambda(t) + \rho_{i\Lambda}\sigma_i dt\right) \\ &= (\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda)\Lambda(t)\, dt + \sigma_\Lambda \Lambda(t)\, d\widetilde{B}^\Lambda(t) \end{aligned}$$

The (instantaneous) correlation between each line of business and the asset to liability ratio is required to evaluate the default option value for line of business i. Later in this paper we will consider evaluating this default option value using our approach to compare with the results from Myers and Read (2001) [10].

The total insurer default option value is then given by

$$\begin{aligned} D(t) &= E^Q\left[e^{-r(T-t)} [L(T) - V(T)]^+ |\mathcal{F}_t\right] \\ &= \sum_{i=1}^M D_i(t) = \sum_{i=1}^M L_i(t) e^{\mu_i (T-t)} M^i(t) \end{aligned}$$

Note that if $L_i = L_i(0)$, $D_i = D_i(0)$ and $D = D(0)$ then

$$M^i(0) = e^{-rT} N(-d_{2i}) - \Lambda(0) e^{-(r-(\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda))T} N(-d_{1i}) \tag{13}$$

where

$$d_{1i} = \frac{\ln \Lambda(0) + \left(\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda + \frac{1}{2}\sigma_\Lambda^2\right) T}{\sigma_\Lambda \sqrt{T}} \tag{14}$$

and

$$d_{2i} = \frac{\ln \Lambda(0) + \left(\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda - \frac{1}{2}\sigma_\Lambda^2\right)T}{\sigma_\Lambda\sqrt{T}} = d_1 - \sigma_\Lambda\sqrt{T} \tag{15}$$

We then have

$$\begin{aligned}
\frac{\partial D}{\partial L_i} &= \frac{\partial\left(\sum_{i=1}^M D_i\right)}{\partial L_i} \\
&= \frac{\partial\left(\sum_{i=1}^M L_i e^{\mu_i T} M^i(0)\right)}{\partial L_i} \\
&= e^{\mu_i T} M^i(0) + L_i e^{\mu_i T}\frac{\partial M^i(0)}{\partial L_i} + \sum_{j=1, j\neq i}^M L_j e^{\mu_j T}\frac{\partial M^j(0)}{\partial L_i} \\
&= \frac{D_i}{L_i} + \sum_{j=1}^M L_j e^{\mu_j T}\frac{\partial M^j(0)}{\partial L_i} \\
&= d_i + \sum_{j=1}^M L_j e^{\mu_j T}\frac{\partial M^j(0)}{\partial L_i}
\end{aligned}$$

In Myers and Read (2001) [10] d_i is defined to be $\frac{\partial D}{\partial L_i}$. We have determined the d_i based on the explicit payoffs for each line of business. As we will show later in a numerical example these results differ.

For the total insurer balance sheet

$$\begin{aligned}
d &= \frac{D}{L} = \frac{1}{L}\sum_{i=1}^M L_i e^{\mu_i T} M^i(0) \\
&= \sum_{i=1}^M x_i d_i
\end{aligned}$$

Now consider the insurer surplus. By definition

$$S(t) = E^Q\left[e^{-r(T-t)}\left[V(T) - L(T)\right]|\mathcal{F}_t\right]$$

Note that even if $S(t) \leq 0$ for $t < T$, the insurer is not regarded as insolvent. It is only at the end of the period, $t = T$, that solvency is assessed in this model. If we assume that a fraction α_i of all of the assets is apportioned to line of business i with $\sum_{i=1}^M \alpha_i = 1$, we then have

$$\begin{aligned}
S(t) &= E^Q\left[e^{-r(T-t)}\sum_{i=1}^M\left[\alpha_i V(T) - L_i(T)\right]|\mathcal{F}_t\right] \\
&= E^Q\left[e^{-r(T-t)}\sum_{i=1}^M \alpha_i V(T)|\mathcal{F}_t\right] - E^Q\left[e^{-r(T-t)}\sum_{i=1}^M L_i(T)|\mathcal{F}_t\right] \\
&= \sum_{i=1}^M\left[\alpha_i e^{(\mu_A - r)(T-t)}V(t) - e^{(\mu_i - r)(T-t)}L_i(t)\right]
\end{aligned}$$

If the assets and liabilities pay no intermediate cash flows then this becomes

$$S = \sum_{i=1}^{M} (\alpha_i V - L_i) = \sum_{i=1}^{M} S_i$$

We also have

$$s_i = \alpha_i \frac{(1+s)\,L}{L_i} - 1$$

3 Myers and Read Revisited

Butsic (1999) [2] and Myers and Read (2001) [10] determine an allocation of the default option value by considering marginal changes to the total default insurer option value for changes in the initial value of each line of business. We will only consider the case of their log-normal assumptions although both normal and log-normal results are given in Myers and Read (2001) [10]. They give formulae for the case where the aggregate losses and the asset values are assumed joint log-normal. Similar results are given in Butsic (1999) [2].

They derive the default option value per unit of initial liability value, $\frac{D}{L}$, under the joint log-normal assumption as

$$d = f(s, \sigma) = N\{z\} - (1+s)\,N\{z - \sigma\}$$

where

$$z = \frac{-\ln(1+s)}{\sigma} + \frac{1}{2}\sigma$$

$$\sigma = \sqrt{\sigma_L^2 + \sigma_V^2 - 2\sigma_{LV}}$$

$$x_i = \frac{L_i}{L}$$

$$\sigma_L^2 = \sum_{i=1}^{M} x_i x_j \rho_{ij} \sigma_i \sigma_j$$

and

$$\sigma_{LV} = \sum_{i=1}^{M} x_i \rho_{iV} \sigma_i \sigma_V$$

Correlations between log losses for lines of business i and j are denoted by ρ_{ij} and correlations between log asset values and log losses in a single line are denoted by ρ_{iV}. For the Myers and Read (2001) [10] assumptions it is important to note that the value of the default option for the insurer depends only on the surplus ratio s and the volatility of the surplus ratio. The default value will not change if an insurer makes changes to its business mix, its assets or its capital structure as long as the surplus ratio and its volatility are maintained.

Under the joint log-normal assumptions, Myers and Read (2001) [10] define the marginal default value as

$$d_i = \frac{\partial D}{\partial L_i}$$

and derive the result that

$$d_i = d + \left(\frac{\partial d}{\partial s}\right)(s_i - s) + \left(\frac{\partial d}{\partial \sigma}\right)\left(\frac{1}{\sigma}\left[\left(\sigma_{iL} - \sigma_L^2\right)\right] - \left(\sigma_{iV} - \sigma_{LV}\right)\right) \tag{16}$$

They then consider what happens when the insurer expands or contracts business in a single line. They consider two assumptions. One is that the company maintains a constant surplus to liability ratio for every line so that

$$s_i = \frac{\partial S}{\partial L_i} = s \quad \text{for all } i$$

and obtain marginal default values of

$$d_i = d + \left(\frac{\partial d}{\partial \sigma}\right)\left(\frac{1}{\sigma}\left[\left(\sigma_{iL} - \sigma_L^2\right)\right] - \left(\sigma_{iV} - \sigma_{LV}\right)\right)$$

for this case. This implies a different allocation of default risk to each line of business and they state that this does not make sense since if the company defaults on one policy then it defaults on all policies.

They state that surplus should be allocated to lines of business to equalize marginal default values so that

$$d_i = \frac{\partial D}{\partial L_i} = d \quad \text{for all } i$$

and the surplus allocation is given by

$$s_i = s - \left(\frac{\partial d}{\partial s}\right)^{-1}\left(\frac{\partial d}{\partial \sigma}\right)\left(\frac{1}{\sigma}\left[\left(\sigma_{iL} - \sigma_L^2\right)\right] - \left(\sigma_{iV} - \sigma_{LV}\right)\right) \tag{17}$$

This gives the capital allocation proposed by Myers and Read (2001) [10] where total capital equals surplus plus the default option value. Note that if $d_i = d$ for all i then $D_i = \frac{L_i}{L} D$ and the allocation of the default option value given by Myers and Read (2001) [10] is in proportion to the value of the liabilities by line of business.

For their log-normal case, they use the assumption that the total assets and the total liabilities are log-normal to derive a closed form for the total insurer default option value. The capital allocation to line of business i will be

$$(s_i + d_i) L_i$$

with total capital of

$$\sum_{i=1}^{M} (s_i + d_i) L_i = (s + d) L$$

The results of Sherris (2004) [14] give the allocation of the insurer default option to line of business based on payoffs in the event of insolvency. We have derived a closed form for the insurer default option values based on the assumption that each line of business is log-normal and the ratio of the assets to liabilities is log-normal. In order to compare our results with the assumptions for the log-normal case in Myers and Read (2001) [10] we need to derive approximations to the parameters for $d\Lambda(t)$.

Consider the derivation of the parameters used in Myers and Read (2001) [10] for the total liabilities. In our case we assume individual lines of business are log-normal, but not the total of the liabilities. If we consider the total liabilities then we can write

$$\begin{aligned} dL(t) &= \sum_{i=1}^{M} dL_i(t) \\ &= \sum_{i=1}^{M} \mu_i L_i(t)\, dt + \sigma_i L_i(t)\, dB^i(t) \\ &= L(t)\left[\sum_{i=1}^{M} x_{it}\mu_i dt + x_{it}\sigma_i dB^i(t)\right] \end{aligned}$$

where

$$x_{it} = \frac{L_i(t)}{L(t)}$$

or

$$\frac{dL}{L} = \left(\sum_{i=1}^{M} x_{it}\mu_i\right) dt + \sum_{i=1}^{M} x_{it}\sigma_i dB^i(t)$$

We can immediately see that unless we rebalance the proportion of each line of business in the insurer liabilities the $x_{it} = \frac{L_i(t)}{L(t)}$ will not be constant. Thus $\left(\sum_{i=1}^{M} x_{it}\mu_i\right)$ and $\sum_{i=1}^{M} x_{it}\sigma_i$ will not be constant and will depend on $L_i(t)$ and $L(t)$. In order for the total liability to be log-normal these drifts and diffusions need to be deterministic.

In order to value the insurer default option, Myers and Read (2001) [10] implicitly assume that the aggregate losses have risk neutral dynamics (log-normal)

$$\frac{dL}{L} = rdt + \sigma_L dB^L(t)$$

and the assets are also log-normal with risk neutral dynamics

$$\frac{dV}{V} = rdt + \sigma_V dB^V(t)$$

with instantaneous correlation given by $dB^L(t)\, dB^V(t) = \rho_{LV} dt$. Myers and Read (2001) [10] equate the moments of the aggregate losses to the moments of

the sum of the individual losses. If we make the assumption that

$$x_{it} = x_{i0} \triangleq \frac{L_i(0)}{L(0)}$$

for all t then

$$\frac{dL}{L} = \left[rdt + \sum_{i=1}^{M} x_i \sigma_i dB^i(t) \right]$$

This gives expressions for the variance of the aggregate losses in terms of the individual losses as follows

$$\sigma_L^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} x_i x_j \rho_{ij} \sigma_i \sigma_j \tag{18}$$

For Myers and Read (2001) [10]

$$\begin{aligned} d\Lambda(t) &= d\left(\frac{V(t)}{L(t)}\right) \\ &= \frac{dV(t)}{L(t)} - \frac{V(t)}{L(t)^2} dL(t) - \frac{1}{L(t)^2} dV(t)\, dL(t) + \frac{V(t)}{L(t)^3} dL(t)^2 \\ &= \left[\Lambda(t) \left[\begin{array}{c} \left(\sigma_L^2 - \sigma_L \sigma_V \rho_{LV}\right) dt \\ +\sigma_V dB^V(t) - \sigma_L dB^L(t) \end{array} \right] \right] \end{aligned} \tag{19}$$

so that

$$\mu_\Lambda = \sigma_L^2 - \sigma_L \sigma_V \rho_{LV} \tag{20}$$

and

$$\sigma_\Lambda^2 = \sigma_V^2 + \sigma_L^2 - 2\sigma_L \sigma_V \rho_{LV} \tag{21}$$

We can evaluate ρ_{LV} using the Myers and Read (2001) [10] assumptions by noting that

$$\sigma_L dB^L(t) = \sum_{i=1}^{M} x_i \sigma_i dB^i(t)$$

so that

$$\begin{aligned} \sigma_V \sigma_L dB^L(t)\, dB^V(t) &= \sum_{i=1}^{M} x_i \sigma_i dB^i(t)\, dB^V(t) \\ &= \sum_{i=1}^{M} x_i \sigma_i \sigma_V \rho_{iV} dt \end{aligned}$$

hence

$$\sigma_L \sigma_V \rho_{LV} = \sum_{i=1}^{M} x_i \sigma_i \sigma_V \rho_{iV} \tag{22}$$

We derived a closed form for the default option value for line of business i in equation (8). If we assume that $t = 0$, $T = 1$ and $\mu_i = r$, as in Myers and Read (2001) [10] then

$$D_i = L_i e^r M^i(0)$$

where $M^i(0)$ is evaluated with the same formula as for $M(t)$ in equation (9) with μ_Λ replaced by $\mu_\Lambda^i = \mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda$ and $T = 1$ with

$$dB^i(t)\, dB^\Lambda(t) = \rho_{i\Lambda} dt$$

Hence

$$M^i(0) = e^{-r} N(-d_{2i}) - \Lambda(0)\, e^{-(r-(\mu_\Lambda+\rho_{i\Lambda}\sigma_i\sigma_\Lambda))} N(-d_{1i}) \tag{23}$$

where

$$d_{1i} = \frac{\ln\Lambda(0) + \left(\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda + \frac{1}{2}\sigma_\Lambda^2\right)}{\sigma_\Lambda} \tag{24}$$

and

$$d_{2i} = \frac{\ln\Lambda(0) + \left(\mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda - \frac{1}{2}\sigma_\Lambda^2\right)}{\sigma_\Lambda} = d_1 - \sigma_\Lambda \tag{25}$$

This then gives

$$D_i = L_i N(-d_{2i}) - L_i\Lambda(0)\, e^{(\mu_\Lambda+\rho_{i\Lambda}\sigma_i\sigma_\Lambda)} N(-d_{1i})$$

and

$$d_i = \frac{D_i}{L_i} = N(-d_{2i}) - \Lambda(0)\, e^{(\mu_\Lambda+\rho_{i\Lambda}\sigma_i\sigma_\Lambda)} N(-d_{1i})$$

We can derive an expression for μ_Λ^i as follows.

$$\begin{aligned}
\sigma_i\sigma_\Lambda dB^i(t)\, dB^\Lambda(t) &= \sigma_i dB^i(t)\left[\sigma_V dB^V(t) - \sigma_L dB^L(t)\right] \\
&= \left[\sigma_i\sigma_V\rho_{iV} - \sigma_i\sigma_L\rho_{iL}\right] dt
\end{aligned}$$

and hence

$$\begin{aligned}
\mu_\Lambda^i &= \mu_\Lambda + \rho_{i\Lambda}\sigma_i\sigma_\Lambda \\
&= \sigma_L^2 - \sigma_L\sigma_V\rho_{LV} + \sigma_i\sigma_V\rho_{iV} - \sigma_i\sigma_L\rho_{iL}
\end{aligned} \tag{26}$$

The default option value as a function of $L_1, L_2, \ldots, L_M, V$ is homogeneous of degree 1 so that

$$D = \sum_{i=1}^{M}\left(\frac{\partial D}{\partial L_i}\right) L_i + \left(\frac{\partial D}{\partial V}\right) V$$

and, since $L = \sum_{i=1}^{M} L_i$ we also have that

$$D = \left(\frac{\partial D}{\partial L}\right) L + \left(\frac{\partial D}{\partial V}\right) V$$

This implies that

$$\left(\frac{\partial D}{\partial L}\right) L = \sum_{i=1}^{M} \left(\frac{\partial D}{\partial L_i}\right) L_i$$

Note that, as demonstrated in the numerical examples below, $d_i = \frac{\partial D}{\partial L_i} \neq \frac{D_i}{L_i}$ where D_i is the by line allocation of the default option value based on equal priority of line of business in the event of insolvency in this paper.

The assumptions that we have made result in a total value of the default option under the Myers and Read (2001) [10] log-normal assumption and our total value that are similar. However, as already noted, the value allocated to line of business differs. To illustrate this we compare values for line-by-line allocations based on the data in Table 2 of Myers and Read (2001) [10]. The assumptions for line of business are given in Table 3. Derived values are given in Table 4.

		Ratio to Liabilities	Standard Deviation	Correlations		
				Line 1	Line 2	Line 3
Line 1	\$100	33%	10%	1.00	0.50	0.50
Line 2	\$100	33%	15%	0.50	1.00	0.50
Line 3	\$100	33%	20%	0.50	0.50	1.00
Liabilities	\$300	100%	12.36%	0.74	0.81	0.88
Assets	\$450	150%	15%	-0.20	-0.20	-0.20
Surplus	\$150	50%				

Table 3: *Data from Table 2 of Myers and Read*

		Covariance with Liabilities	Covariance with Assets	μ_Λ^i
Line 1	\$100	0.0092	-0.0030	0.0076
Line 2	\$100	0.0150	-0.0045	0.0003
Line 3	\$100	0.0217	-0.0060	-0.0079
Liabilities	\$300	0.0153	-0.0045	0.0000
Assets	\$450		0.0225	
Surplus	\$150		σ_Λ	0.2163

Table 4: *Parameters for Table 2 Data of Myers and Read*

The line-by-line allocations from Myers and Read (2001) [10] assuming uniform surplus are given in Table 5. The line-by-line allocations from assuming uniform default value are given in Table 6. The corresponding values using the formulae in this paper are given in Table 7.

From these Tables, note that the total default value under both formulae are similar but that the line by line allocations are significantly different. In this

	Myers and Read Lognormal Uniform Surplus		
	d_i %	D_i	S_i
Line 1	0.0163	0.0163	50
Line 2	0.3005	0.3005	50
Line 3	0.6169	0.6169	50
Total	0.3112	0.9336	150

Table 5: *Line by Line Allocations Table 2 Data of Myers and Read - Uniform Surplus*

	Myers and Read Lognormal Uniform Default Value		
	s_i %	D_i	S_i
Line 1	0.3775	0.3112	37.55
Line 2	0.4955	0.3112	49.55
Line 3	0.6290	0.3112	62.90
Total	0.50	0.9336	150

Table 6: *Line by Line Allocations Table 2 Data of Myers and Read - Uniform Default Value*

	Sherris allocations Uniform Surplus		
	d_i %	D_i	S_i
Line 1	0.2852	0.2852	50
Line 2	0.3102	0.3102	50
Line 3	0.3404	0.3404	50
Total	0.3119	0.9358	150

Table 7: *Sherris Line by Line Allocations Table 2 Data of Myers and Read*

example the Myers and Read (2001) [10] allocations may add up to the total surplus and default option value but the allocation will not give fair (arbitrage-free) prices for individual lines of business that reflect the allocation of the default option value based on equal priority.

4 Capital allocation - Numerical examples

Many different methods for allocating capital by line of business have been proposed. Venter (2004) [16], provides a review of these methods including that of Myers and Read (2001) [10]. Panjer (2001) [11] also summarises some of these and develops a covariance based method similar in concept to the CAPM beta. He includes an example with data based on 10 lines of business. The methods of allocation include allocating capital in proportion to variance, in proportion to VaR (Value at Risk), and in proportion to TailVaR by line of business. In these methods the assets and the default option value are not usually explicitly included. This is also the case in the method proposed by Panjer (2001) [11].

We use the data given in Panjer (2001) [11] for correlations by line of business. These correlations are given in Table 8. The value for the liabilities and variance of liabilities that we use are those given for the premiums in Panjer (2001) [11] and we assume that these represent the market value of the liabilities. For the surplus we use the total surplus derived in Panjer (2001) [11] for his example. The data used are given in Table 9.

Line 1	Line 2	Line 3	Line 4	Line 5	Line 6	Line 7	Line 8	Line 9	Line 10
1.00	0.00	0.12	-0.02	0.18	-0.26	-0.12	0.11	0.08	-0.03
0.00	1.00	0.05	0.27	0.02	0.08	0.16	-0.21	-0.17	-0.15
0.12	0.05	1.00	0.01	-0.11	0.10	0.03	-0.12	-0.09	-0.12
-0.02	0.27	0.01	1.00	0.22	0.05	0.09	-0.11	0.13	-0.23
0.18	0.02	-0.11	0.22	1.00	-0.11	0.01	-0.03	0.14	-0.01
-0.26	0.08	0.10	0.05	-0.11	1.00	0.07	-0.09	-0.46	-0.16
-0.12	0.16	0.03	0.09	0.01	0.07	1.00	-0.25	0.08	0.14
0.11	-0.21	-0.12	-0.11	-0.03	-0.09	-0.25	1.00	-0.16	-0.16
0.08	-0.17	-0.09	0.13	0.14	-0.46	0.08	-0.16	1.00	0.21
-0.03	-0.15	-0.12	-0.23	-0.01	-0.16	0.14	-0.16	0.21	1.00
0.25	0.69	0.09	0.35	0.16	0.40	0.39	-0.18	-0.08	0.18

Table 8: *Correlations by line of business for Panjer Data*

To begin with we show the allocations that would result from allocating surplus only in proportion to various risk measures by line of business assuming a multivariate normal distribution. The risk measures used are the standard deviation, the Value at Risk (or VaR), the TailVaR and the beta measure proposed by Panjer (2001) [11]. The results are given in Figure 1. This shows a very wide variation in the resulting surplus allocations to lines of business for the different risk measures. The allocations do not include the value of the insurer default option value.

Line	Amount	PerCent	Standard Deviation (Dollar)	Standard Deviation (Percent)
Line 1	36.00	9.6	2.69	7.47
Line 2	120.40	32.3	4.49	3.73
Line 3	1.30	0.3	0.21	16.12
Line 4	52.42	14.0	1.32	2.51
Line 5	0.70	0.2	0.57	82.14
Line 6	48.09	12.9	3.87	8.05
Line 7	47.40	12.7	1.59	3.36
Line 8	8.08	2.2	0.96	11.85
Line 9	8.64	2.3	1.06	12.29
Line 10	50.15	13.4	2.59	5.17
Liabilities	373.18	100	6.73	1.80
Assets	400.42	107	60.06	15.00
Surplus	27.24	7		

Table 9: *Liabilities and Standard Deviations*

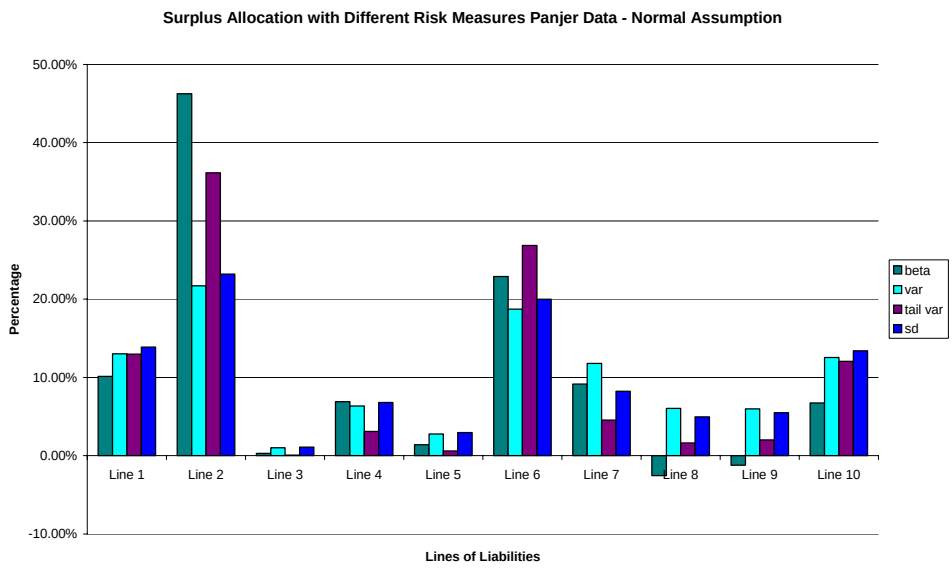

Figure 1: Surplus Allocations For Lines of Business

Next we illustrate the Myers and Read (2001) [10] allocations using the Panjer (2001) [11] data. To do this we explicitly include assets equal to the total of the liabilities plus the surplus. We assume that all lines of business have the same correlation with the assets for simplicity, although there is no difficulty in assuming different correlations with the assets for different lines of business. We first used the Myers and Read (2001) [10] normal distribution results with a uniform default value to determine by line surplus allocations. We then used the lognormal distribution results assuming the correlations given in the Panjer (2001) [11] data. This is effectively the same as using log-normal marginals with a normal copula with correlations given by the Panjer (2001) [11] data. There turned out to be very little difference in the capital allocations between these two distributional assumptions. As a result we show the log-normal distribution results in order to compare with the formula for allocating capital given in this paper.

The allocations for the different lines of business using Myers and Read (2001) [10] lognormal distribution results for an asset correlation for all lines of business ranging from -1 to +1 are given in Figure 2. We show the distribution across lines for each assumed asset correlation as well as the per dollar values and the actual dollar values. These figures include in the capital the surplus and the insurer default value. The default value varies with different correlations and the distribution by line of business varies significantly as well.

	Myers-Read distribution of (Si+Di) with di constant										
%	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Line 1	11.29%	11.07%	10.81%	10.50%	10.12%	9.66%	9.07%	8.28%	7.18%	5.50%	2.57%
Line 2	28.99%	29.49%	30.07%	30.76%	31.59%	32.62%	33.94%	35.69%	38.14%	41.89%	48.42%
Line 3	0.63%	0.59%	0.55%	0.49%	0.43%	0.35%	0.24%	0.11%	-0.08%	-0.38%	-0.89%
Line 4	11.13%	11.51%	11.95%	12.46%	13.09%	13.86%	14.85%	16.16%	18.00%	20.80%	25.69%
Line 5	1.28%	1.14%	0.97%	0.76%	0.52%	0.22%	-0.17%	-0.68%	-1.39%	-2.49%	-4.39%
Line 6	15.79%	15.43%	15.01%	14.50%	13.89%	13.14%	12.18%	10.91%	9.13%	6.41%	1.67%
Line 7	10.92%	11.15%	11.42%	11.74%	12.13%	12.61%	13.22%	14.04%	15.18%	16.92%	19.95%
Line 8	3.16%	3.01%	2.83%	2.62%	2.36%	2.04%	1.64%	1.10%	0.34%	-0.81%	-2.82%
Line 9	3.48%	3.31%	3.11%	2.87%	2.58%	2.22%	1.77%	1.16%	0.31%	-0.98%	-3.24%
Line 10	13.31%	13.31%	13.30%	13.29%	13.28%	13.27%	13.25%	13.23%	13.19%	13.14%	13.04%
Total	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%
	Myers-Read (si+di) with di constant										
$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Line 1	0.14	0.14	0.13	0.12	0.11	0.11	0.10	0.08	0.07	0.05	0.02
Line 2	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.12
Line 3	0.22	0.20	0.18	0.16	0.13	0.11	0.07	0.03	-0.02	-0.10	-0.21
Line 4	0.10	0.10	0.10	0.10	0.10	0.10	0.11	0.11	0.12	0.13	0.15
Line 5	0.84	0.72	0.60	0.46	0.30	0.12	-0.09	-0.35	-0.69	-1.17	-1.94
Line 6	0.15	0.14	0.14	0.13	0.12	0.11	0.10	0.08	0.07	0.04	0.01
Line 7	0.11	0.10	0.10	0.10	0.10	0.10	0.11	0.11	0.11	0.12	0.13
Line 8	0.18	0.17	0.15	0.14	0.12	0.10	0.08	0.05	0.01	-0.03	-0.11
Line 9	0.18	0.17	0.16	0.14	0.12	0.10	0.08	0.05	0.01	-0.04	-0.12
Line 10	0.12	0.12	0.11	0.11	0.11	0.10	0.10	0.10	0.09	0.09	0.08
Total	0.12	0.12	0.12	0.11	0.11	0.11	0.10	0.10	0.09	0.09	0.08
	Myers-Read (Si+Di) with di constant										
$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Line 1	$5.16	$4.93	$4.69	$4.42	$4.13	$3.81	$3.44	$3.01	$2.49	$1.81	$0.80
Line 2	$13.26	$13.14	$13.04	$12.95	$12.88	$12.85	$12.87	$12.98	$13.24	$13.80	$15.02
Line 3	$0.29	$0.26	$0.24	$0.21	$0.17	$0.14	$0.09	$0.04	-$0.03	-$0.12	-$0.27
Line 4	$5.09	$5.13	$5.18	$5.25	$5.34	$5.46	$5.63	$5.88	$6.25	$6.85	$7.97
Line 5	$0.59	$0.51	$0.42	$0.32	$0.21	$0.09	-$0.06	-$0.25	-$0.48	-$0.82	-$1.36
Line 6	$7.22	$6.88	$6.51	$6.10	$5.67	$5.18	$4.62	$3.97	$3.17	$2.11	$0.52
Line 7	$4.99	$4.97	$4.95	$4.94	$4.95	$4.97	$5.02	$5.11	$5.27	$5.57	$6.19
Line 8	$1.45	$1.34	$1.23	$1.10	$0.96	$0.81	$0.62	$0.40	$0.12	-$0.27	-$0.88
Line 9	$1.59	$1.48	$1.35	$1.21	$1.05	$0.88	$0.67	$0.42	$0.11	-$0.32	-$1.01
Line 10	$6.09	$5.93	$5.77	$5.60	$5.42	$5.23	$5.03	$4.81	$4.58	$4.33	$4.05
Total	$45.73	$44.56	$43.36	$42.10	$40.78	$39.39	$37.93	$36.38	$34.72	$32.94	$31.02

Figure 2: Myers and Read Lognormal Capital Allocations - Uniform Default Value

The Myers and Read (2001) [10] capital allocation is shown in Figure 3. the tremendous variation across the lines of business for the different assumed asset correlations is clearly evident.

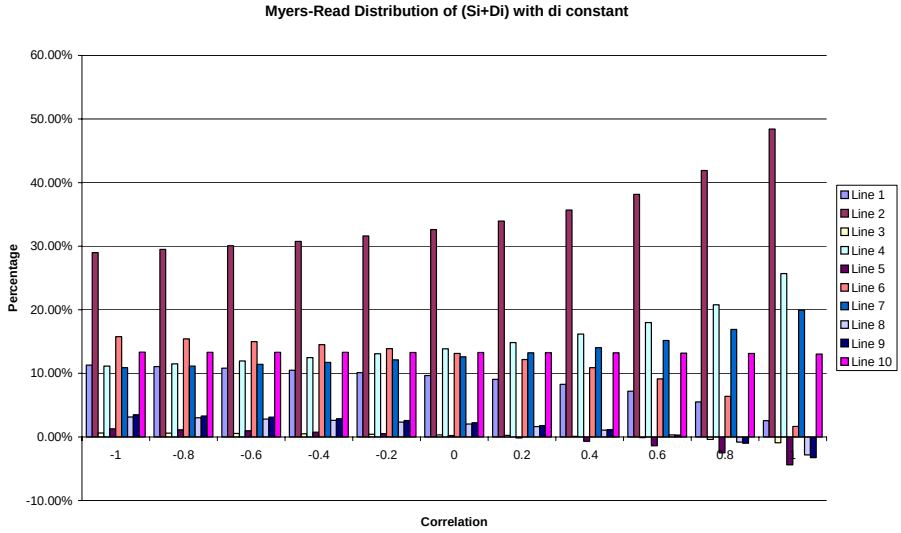

Figure 3: Myers and Read Capital Allocation - Lognormal and Uniform Default Value

Using the log-normal results in Myers and Read (2001) [10] and assuming that the correlations in Table 8 are for the lognormal case we derive allocations by line-of-business for the approach in this paper. We compare our results for the by line default option value, as a percentage of the liabilities and for the Myers and Read (2001) [10] case for the assumption that the surplus ratio is constant by line. The results are given in Figure 4.

The total insurer default values using the formulae in this paper and for the log-normal Myers and Read (2001) [10] case are almost identical. The Myers and Read (2001) [10] values for the default option value by line of business differ substantially in all but the zero asset correlation case compared to the default option value determined based on the by line insolvency payoffs given by the formula in this paper. The default option values by line of business based on Sherris (2004) [14] and the formula in this paper are shown in Figure 5.

Given the differences in allocations for these different methods we urge caution to those using by-line allocations for insurer financial decision making. From the results it is important to reflect the correlation of the liabilities with the assets in any capital allocation and it is also important to determine the fair allocation of the default option based on assumed payoffs in the event of insolvency by line of business if these are to be used for pricing in the multi-line

	Myers-Read di, Constant si										
$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Line 1	5.63%	5.21%	4.76%	4.29%	3.79%	3.26%	2.69%	2.08%	1.40%	0.66%	-0.16%
Line 2	4.55%	4.31%	4.07%	3.82%	3.56%	3.29%	3.01%	2.73%	2.43%	2.13%	1.81%
Line 3	8.17%	7.31%	6.40%	5.42%	4.38%	3.24%	2.00%	0.63%	-0.92%	-2.68%	-4.65%
Line 4	4.13%	3.95%	3.77%	3.59%	3.40%	3.22%	3.03%	2.85%	2.67%	2.50%	2.34%
Line 5	28.08%	23.92%	19.48%	14.69%	9.48%	3.77%	-2.57%	-9.70%	-17.84%	-27.26%	-38.00%
Line 6	5.85%	5.40%	4.92%	4.42%	3.89%	3.32%	2.71%	2.05%	1.32%	0.51%	-0.38%
Line 7	4.40%	4.18%	3.95%	3.72%	3.48%	3.23%	2.98%	2.73%	2.46%	2.20%	1.92%
Line 8	6.78%	6.13%	5.44%	4.71%	3.93%	3.08%	2.16%	1.15%	0.02%	-1.25%	-2.67%
Line 9	6.95%	6.28%	5.57%	4.82%	4.01%	3.14%	2.19%	1.14%	-0.03%	-1.35%	-2.83%
Line 10	4.92%	4.61%	4.28%	3.94%	3.59%	3.22%	2.82%	2.41%	1.96%	1.48%	0.96%
Total	4.95%	4.64%	4.32%	3.98%	3.63%	3.26%	2.87%	2.45%	2.01%	1.53%	1.01%
	Sherris di										
$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
Line 1	5.07%	4.74%	4.39%	4.03%	3.65%	3.26%	2.84%	2.41%	1.95%	1.46%	0.94%
Line 2	4.88%	4.59%	4.28%	3.96%	3.62%	3.26%	2.88%	2.48%	2.05%	1.57%	1.06%
Line 3	5.54%	5.10%	4.66%	4.20%	3.73%	3.25%	2.76%	2.26%	1.75%	1.23%	0.71%
Line 4	4.81%	4.53%	4.23%	3.92%	3.60%	3.25%	2.88%	2.49%	2.07%	1.60%	1.09%
Line 5	10.14%	8.64%	7.19%	5.80%	4.51%	3.32%	2.28%	1.40%	0.72%	0.27%	0.05%
Line 6	5.11%	4.77%	4.41%	4.05%	3.66%	3.26%	2.85%	2.41%	1.94%	1.45%	0.93%
Line 7	4.86%	4.56%	4.26%	3.94%	3.61%	3.25%	2.88%	2.48%	2.05%	1.58%	1.07%
Line 8	5.28%	4.90%	4.50%	4.09%	3.67%	3.23%	2.78%	2.32%	1.83%	1.33%	0.81%
Line 9	5.32%	4.92%	4.52%	4.11%	3.68%	3.24%	2.79%	2.31%	1.82%	1.32%	0.80%
Line 10	4.95%	4.64%	4.31%	3.97%	3.62%	3.25%	2.86%	2.45%	2.00%	1.52%	1.01%
Total	4.96%	4.64%	4.32%	3.98%	3.63%	3.26%	2.87%	2.45%	2.01%	1.53%	1.02%

Figure 4: Myers and Read and Sherris Default Option Allocations - Lognormal assumption

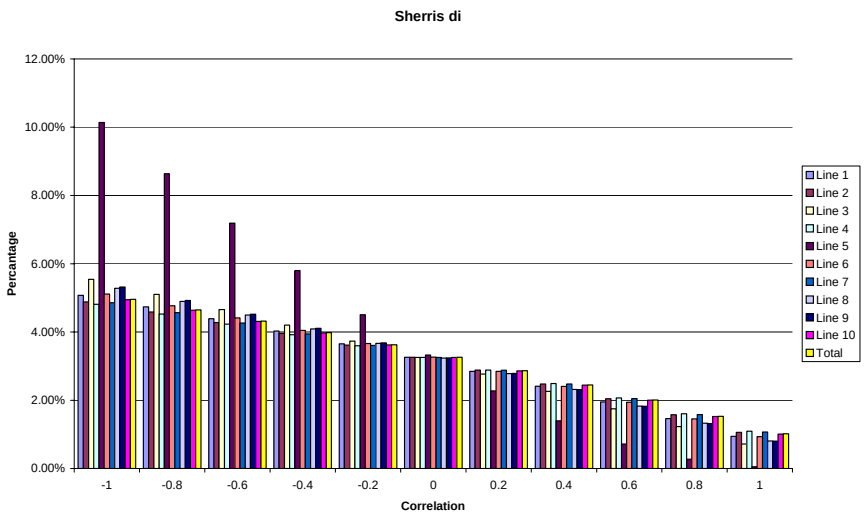

Figure 5: Sherris Default Option Values

insurer.

5 Conclusion

We have developed expressions for the insurer default option value that reflect the actual payoff by line in the event of insolvency. We have used the assumption of log-normal lines of business and a log-normal ratio of asset to liabilities to derive closed form results and illustrated these with examples from Myers and Read (2001) [10] and data from Panjer (2001) [11]. The approach we have used will give fair default option values for lines of business that can be used in pricing for the mulit-line insurer.

We have shown how the results of applying different methods of allocation can give significantly different results so that the approach used is important. Although our total value for the insurer default option value is similar to that of Myers and Read (2001) [10], the allocation by line of business differs significantly. We advise caution to those using the formula in Myers and Read (2001) [10] for capital allocation, particularly those using the results for pricing in a multi-line insurer.

6 Appendix - Change of Measure

Lemma 1 *If $B^j(t)$ $0 \leq t \leq T$ is a Brownian motion under Q then*

$$\widetilde{B}^j(t) = B^j(t) - \rho_{ij}\sigma_i t$$

is Brownian motion under Q_i.

Proof. Let $B^j(t)$ be Brownian motion under Q and $\widetilde{B}^j(t) \equiv B^j(t) - \int_0^t \varphi^{ij}(u)\,du$ be Brownian motion under Q_i. We need to derive $\varphi^{ij}(t)$ for $0 \leq t \leq T$. We have

$$E^{Q_i}\left[\widetilde{B}^j(t)\,|\mathcal{F}_s\right] = E^Q\left[Z_i(t)\,\widetilde{B}^j(t)\,|\mathcal{F}_s\right] \quad t > s$$

Now

$$L_i(t) = L_i(0)\exp\left[\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i B^i(t)\right]$$

so

$$\begin{aligned} Z_i(t) &= \frac{1}{L_i(0)}\frac{L_i(t)\,e^{(r-\mu_i)t}}{e^{rt}} \\ &= \exp\left[-\frac{1}{2}\sigma_i^2 t + \sigma_i B^i(t)\right] \end{aligned}$$

and

$$dZ_i(t) = Z_i(t)\,\sigma_i dB^i(t)$$

It follows that

$$\begin{aligned} & d\left(Z_i(t)\,\widetilde{B}^j(t)\right) \\ = \;& Z_i(t)\,d\widetilde{B}^j(t) + \widetilde{B}^j(t)\,dZ_i(t) + d\widetilde{B}^j(t)\,dZ_i(t) \\ = \;& Z_i(t)\,dB^j(t) - Z_i(t)\,\varphi^{ij}(t)\,dt + \widetilde{B}^j(t)\,dZ_i(t) + d\widetilde{B}^j(t)\,dZ_i(t) \\ = \;& Z_i(t)\,dB^j(t) + \left[Z_i(t)\,\rho_{ij}\sigma_i - Z_i(t)\,\varphi^{ij}(t)\right]dt + \widetilde{B}^j(t)\,dZ_i(t) \end{aligned}$$

hence

$$Z_i(t)\,\widetilde{B}^j(t) = \left[\begin{array}{c} Z_i(s)\,\widetilde{B}^j(s) + \int_s^t \left[\widetilde{B}^j(u)\,dZ_i(u) + Z_i(u)\,dB^j(u)\right] \\ + \int_s^t Z_i(u)\left[\rho_{ij}\sigma_i - \varphi^{ij}(u)\right]du \end{array}\right]$$

and

$$E^Q\left[Z_i(t)\,\widetilde{B}^j(t)\,|\mathcal{F}_s\right] = Z_i(s)\,\widetilde{B}^j(s) + E^Q\left[\int_s^t Z_i(u)\left[\rho_{ij}\sigma_i - \varphi^{ij}(u)\right]du|\mathcal{F}_s\right]$$

We also have

$$E^{Q_i}\left[\left(\widetilde{B}^j(t)^2 - t\right)|\mathcal{F}_s\right] = E^Q\left[Z_i(t)\left(\widetilde{B}^j(t)^2 - t\right)|\mathcal{F}_s\right]$$

and

$$
\begin{aligned}
& d\left(Z_i(t)\left(\widetilde{B}^j(t)^2-t\right)\right) \\
= & \left[\begin{array}{c} 2Z_i(t)\widetilde{B}^j(t)\left[d\widetilde{B}^j(t)-\varphi^{ij}(t)\,dt\right] \\ +\left(\widetilde{B}^j(t)^2-t\right)dZ_i(t)+2\widetilde{B}^j(t)\,d\widetilde{B}^j(t)\,Z_i(t)\,\sigma_i dB^i(t) \end{array}\right] \\
= & \left[\begin{array}{c} 2Z_i(t)\widetilde{B}^j(t)\,d\widetilde{B}^j(t)+\left(\widetilde{B}^j(t)^2-t\right)dZ_i(t) \\ +2Z_i(t)\widetilde{B}^j(t)\left[\sigma_i\rho_{ij}-\varphi^{ij}(t)\right]dt \end{array}\right]
\end{aligned}
$$

so

$$
\begin{aligned}
& E^Q\left[Z_i(t)\left(\widetilde{B}^j(t)^2-t\right)|\mathcal{F}_s\right] \\
= & Z_i(s)\left(\widetilde{B}^j(s)^2-s\right)+E^Q\left[\int_s^t 2Z_i(u)\widetilde{B}^j(u)\left[\sigma_i\rho_{ij}-\varphi^{ij}(u)\right]du|\mathcal{F}_s\right]
\end{aligned}
$$

Therefore we have that

$$
\varphi^{ij}(t)=\sigma_i\rho_{ij} \quad j=1,2,\ldots,M,\Lambda
$$

since in this case

$$
E^{Q_i}\left[\widetilde{B}^j(t)|\mathcal{F}_s\right]=\widetilde{B}^j(s)
$$

$$
E^{Q_i}\left[\left(\widetilde{B}^j(t)^2-t\right)|\mathcal{F}_s\right]=\left(\widetilde{B}^j(s)^2-s\right)
$$

and from Levy's Theorem (1948), subject to some technical conditions, $\widetilde{B}^j(t)$ is Brownian motion under Q_i (see for example Karatzas and Shreve (1988) [5] pages 156-157).

Under the change of measure Q_i

$$
\begin{aligned}
dL_j(t) &= \mu_j L_j(t)\,dt+\sigma_j L_j(t)\,dB^j(t) \\
&= \mu_j L_j(t)\,dt+\sigma_j L_j(t)\left[d\widetilde{B}^j(t)+\sigma_i\rho_{ij}dt\right] \\
&= \left(\mu_j+\sigma_i\sigma_j\rho_{ij}\right)L_j(t)\,dt+\sigma_j L_j(t)\,d\widetilde{B}^j(t) \quad j=1,\ldots,M \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
d\Lambda(t) &= \mu_\Lambda\Lambda(t)\,dt+\sigma_\Lambda\Lambda(t)\,dB^\Lambda(t) \\
&= \mu_\Lambda\Lambda(t)\,dt+\sigma_\Lambda\Lambda(t)\left[d\widetilde{B}^\Lambda(t)+\sigma_i\rho_{i\Lambda}dt\right] \\
&= \left(\mu_\Lambda+\sigma_i\sigma_\Lambda\rho_{i\Lambda}\right)\Lambda(t)\,dt+\sigma_\Lambda\Lambda(t)\,d\widetilde{B}^\Lambda(t) \qquad (28)
\end{aligned}
$$

■

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