An Alternative Approach to Asset-Liability Management

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Abstract

Traditionally, the potential financial impact of interest rate movements has been evaluated through duration measures or through computer simulations. The duration measure approach is analytical, and focuses on changes in market value; the computer simulation approach is computational and focuses on income statement and balance sheet impacts. In this paper a complementary approach for projecting the earnings impact of asset-liability mismatch is derived. This approach, referred to as the mismatch-earnings formula (MEF) is analytical but focuses on income statement impact of interest rate movements. The MEF approach can be used to gain insight into ALM mismatch exposure, to estimate the expected earnings and volatility of earnings arising from a mismatch position, and to optimize portfolios.

Keywords: ALM; Earnings volatility; Portfolio optimization.

1. Introduction

1.1 Background: Current Techniques for Evaluating the Financial Impact of Interest Rate Movements.

There are two main techniques for evaluating the financial impact of interest rate movements on insurance companies: duration measures and computer simulations. In many ways these approaches provide complementary viewpoints.

The duration approach to evaluating interest rate risk is based on a relatively simple theoretical model (Panjer (1998), chapter 3.4). The basic goal of the duration approach is to evaluate the impact of interest rate movements on the market value of assets and liabilities. Although the model has evolved since it was described by Redington[1] (Panjer (1998), chapters 3.5-3.9) its fundamental properties have remained unchanged: this model is essentially based on market value measures. On the other hand, earnings (GAAP, Statutory and Tax) are based on book value measures such as net investment income and change in reserves. The duration approach, even at its most complicated, can therefore only approximate the impact of interest rate movements on earnings.

In contrast to the duration approach, computer simulation based asset-liability management (ALM) models (both homegrown and commercially available) typically take a large amount of asset and liability data and apply borrowing and reinvestment strategies to project financial experience. Although these simulation models will often produce market value outputs, their focus tends to be on a balance sheet and income statement presentation. In other words, as opposed to the duration model computer simulations can be said to have a book value focus.

In comparing approaches there is a temptation to simplify and say that the market value focus captures the underlying "true" economics, while the book value focus is "only" based on accounting. In reality, of course, the situation is more complicated:

- The market value analysis is based on the present value of asset and liability cash flows. Typically these cash flows are assumed to be independent of book value accounting rules; however, this is not always a good assumption (e.g. tax reserves are required for tax cash flows, and distributable earnings are based on income statement calculations).
- Companies are managed to book value specifications; for example, there may be a desire to hold the book value surplus at some multiple of risk based capital (RBC)[2].
- On a practical level the book value focus is the preferred focus of the IRS, the regulatory agencies, the equity analysts and therefore senior management.

Based on the observations above, it is clear that both the market value and the book value viewpoints are necessary, both are useful in different ways, and both represent approximations of the true economics.

1.2 A Rationale Supporting the Development of an Additional Approach

Given the complementary nature of the existing models, what benefit is there in a new model for understanding the financial impact of interest rate movements? The answer to this question lies in the nature of theoretical models as compared to computer simulations.

Computer simulations tend to take as much information as is available as inputs, can use arbitrarily complex assumptions, and produce reams of data as output. In other words, these models attempt to approximate reality by using as much information as is available. On the other hand, these models tend to be extremely time consuming to maintain, and produce no simple explanation of their results (if correct) or the source of errors in their results (if incorrect). Invariably we turn to theory in order to convince ourselves of the sensibleness of simulations' results.

In contrast, theoretical models attempt to approximate reality by using a few well chosen but dramatically simplifying assumptions. These models are therefore able to organize the mass of inputs into just a few aggregate items. This creates a simpler understanding of the nature of the model results, and therefore (hopefully) of the real world. On the other hand, theoretical model will have few outputs. For example, for all its strengths the duration approach is limited to its market value focus.

This paper presents an attempt to build a book value focused, theoretical model to complement the two approaches discussed above. This new approach will be referred to as the Mismatch-Earnings Formula (MEF).

1.3 A Basic Description of the Mismatch-Earnings Formula (MEF)

The basic characteristics of the MEF can be most easily illustrated by describing its inputs and outputs. Table 1 provides a comparison of the two existing approaches and the MEF.

	Duration	**Computer Simulation**	**MEF (some of the terms are defined below)**
Model Input (all inputs are at time t)	Market value, duration and convexity of assets and liabilities	Exact description of all assets and liabilities	Asset-liability mismatch function (representing asset-liability mismatch along the curve)
Interest Rate Assumptions	Change in interest rates at time t	Interest rates at all times after t.	Interest rates at all times after t.
Additional Assumptions	None	Explicit strategy for investing new money (or borrowing to make up shortfalls)	Future changes in the asset-liability mismatch function.
Model Output	Market value of assets and liability at time t+1	Income statements and balance sheets for all times after t	Mismatch-earnings for all times after t
Description of Calculations	Simple algebraic formula	Multiple iterative calculations as part of a simulation	Integral formula (equation 2.7)

Table 1: A comparison of three approaches to evaluating the financial impact of interest rate movements.

Two of the terms mentioned in the MEF column require some explanation:

- The input to the MEF is described as the "asset-liability mismatch function". This function represents a new metric for describing asset-liability mismatch along the curve. The function is based on projections of asset book value rollover minus liability book value rollover. Philosophically, this function is similar to certain extensions of the duration measure (e.g., key rate duration – see Ho (1992)); however, this measure is book value focussed. The asset-liability mismatch function is defined explicitly and examined in detail in section 2.
- The output to the MEF is described as "mismatch-earnings". This is a new metric designed to quantify that part of earnings that varies with interest rate movements. The underlying rationale for this measure is based on the observation that if the asset-liability mismatch function were identically zero then earnings might be positive or negative, and would likely vary over time, but would not vary due to interest rate movements. This base earnings level would depend on the details of the assets and liabilities in the portfolio. On the other hand, if the asset-liability mismatch function is not zero at all points, earnings will vary due to interest rate movements. The "mismatch-earnings" represents the potential earnings variability and is largely independent of the exact details of the assets and liabilities.

A motivation for the two new metrics described above (the asset-liability mismatch function and the mismatch-earnings) is provided in section 2.1, and a detailed mathematical discussion is provided in section 2.2.

Beyond the basic version of the MEF described above, this paper goes on to develop a second approach to measuring the financial impact of interest rate movements (MEF-2). This second approach applies the basic insights of the MEF, makes additional assumptions and simplifications, and uses stochastic interest rate scenarios to change the form of the output. A comparison of the MEF and MEF-2 is provided in table 2.

	MEF (same description as above)	**MEF-2**
Model Input (all inputs are at time t)	Asset-liability mismatch function (representing asset-liability mismatch along the curve)	Simplified asset-liability mismatch function. (See section 3.1)
Interest Rate Assumptions	Interest rates at all times after t.	Multiple stochastic interest rate scenarios for all times after t.
Additional Assumptions	Future changes in the asset-liability mismatch functions	The asset-liability mismatch function does not change after time t.[3]
Model Output	Mismatch-earnings for all times after t	Expected mismatch-earnings and standard deviation of mismatch-earnings over specified time periods after t.
Description of Calculations	Integral formula (equation 2.7)	1) A matrix is generated by solving the MEF under multiple stochastic scenarios. This matrix does not depend on the asset-liability mismatch function. 2) Given any simplified mismatch function the MEF-2 outputs are then generated using simple matrix multiplication.

Table 2: A comparison of the two models developed in this paper. The mismatch earnings formula (MEF) is discussed in section 2, and is simplified and extended (MEF-2) in section 3.

To elaborate on table 2: As discussed above, the MEF provides the projected mismatch-earnings under a single interest rate scenario. MEF-2, on the other hand, provides the expected value and standard deviation of the mismatch-earnings under multiple stochastic interest rate scenarios.

Among the advantages of the MEF-2 approach is its ability to search all asset-liability mismatch functions to find an optimal solution[4]. The MEF-2 approach is described in section 3, and the portfolio optimization procedure is described in section 4.

1.4 Primary Results of this Paper

The primary results of this paper can be summarized as follows:

A. In section 2, the MEF, a theoretical model with a book-value focus will be derived (equation 2.7). This model is complementary to both the duration approach and to the computer simulation approach to examining the financial impact of yield curve movements.

B. Through this theoretical analysis a new asset-liability mismatch measure will be identified (equation 2.2).

C. Elaboration of the initial mathematical analysis yields a series of powerful separations:
   - In section 2, (equation 2.6) the impact on future earnings of future asset-liability mismatch decisions is separated from the impact on future earnings of past asset-liability mismatch decisions. Clearly only future asset-liability mismatch decisions can be altered, and therefore the impact of these decisions are properly the focus of asset-liability management.
   - In section 3, (the MEF-2 approach) the problem of projecting the financial implications of asset-liability mismatches is separated into the problem of evaluating the financial implication of a series of generic mismatches, and the problem of measuring the current mismatch position. The calculations required for these two items are shown to be independent of one another.
   - In section 4, the problem of portfolio optimization is separated into the problem of determining the optimal location for mismatching along the curve and the problem of determining the optimal level of mismatch.

D. Section 5 contains a brief discussion of some easily implemented practical applications of the ideas in this paper.

2. The Mismatch Earnings Formula

2.1 Initial Discussion and Motivation for a New Metric to Measure the Asset-Liability Mismatch

As discussed in the introduction, the goal of this paper is to build a theoretical model with a book value earnings focus. A simple example will help to introduce the formal mathematical development to follow. Consider a $100 3-year coupon-paying GIC crediting 5% backed by a $100 3-year coupon-paying bond earning 6%. It is clear that (ignoring bond defaults and taxes) this portfolio will throw off earnings of 100 basis points annually for 3 years. Furthermore, assume that these distributable earnings will in fact be distributed and will therefore not be reinvested. In this case the stream of earnings will be insensitive to yield curve movements. This insensitivity depends only on the fact that the book value rollover of the liability takes place at the same time as the book value rollover of the asset: if the asset were to earn 6.5% the projected earnings would be higher, but would still be insensitive to yield curve movements.

Now consider the same $100 3-year coupon paying GIC crediting 5% now backed by a $100 2-year coupon paying bond earning 6%. At the end of year 2 the bond will be reinvested into a new bond of some term (either a 1 year term to match the liability, or some other term which would not match the liability). Regardless of the reinvestment decision, the level of earnings would be locked in for 2 years; however, the earnings in year 3 will vary depending on the reinvestment strategy and yield curve at the end of year 2. Following the logic of the previous paragraph, the mismatch can be described by noting that the $100 asset rolls over at the end of year 2 while the $100 liability does not roll over until the end of year 3. Based on this discussion a "mismatch function" can be plotted (figure 1a). Note that the mismatch function shown in figure 1a only applies to one point in time. Assuming that no asset actions are undertaken then one year from now the mismatch function will have drifted left as the liability now has a 2-year term and the asset now has a 1-year term. The mismatch function at t=1 is plotted in figure 1b. Also note that the mismatch function will change dramatically at the end of the second year since the assets must now be reinvested. Assuming reinvestment in a 3-year bond, the mismatch function will appear as is plotted in figure 1c; however, since any reinvestment strategy is possible, the mismatch function can take on almost any shape at year 2. This situation, of course, is not unique to year 2 – at the end of year 1, for example, it is possible to dramatically change the mismatch function by shorting a 1 year bond and investing in a 5 year bond.

In discussing the simple examples above the concept of a "mismatch function" has been implicitly defined: The mismatch function is the projected asset rollovers minus the projected liability rollovers. The argument was made that if this mismatch function was zero at all points then earnings would not vary with yield curve movements. Further, it was stated that if the mismatch function was not zero then earnings would vary with yield curve movements (in our example they vary in year 3). Finally it was asserted that the mismatch function at any time represented a snapshot and would change as time progressed depending on asset actions such as borrowing and reinvesting.

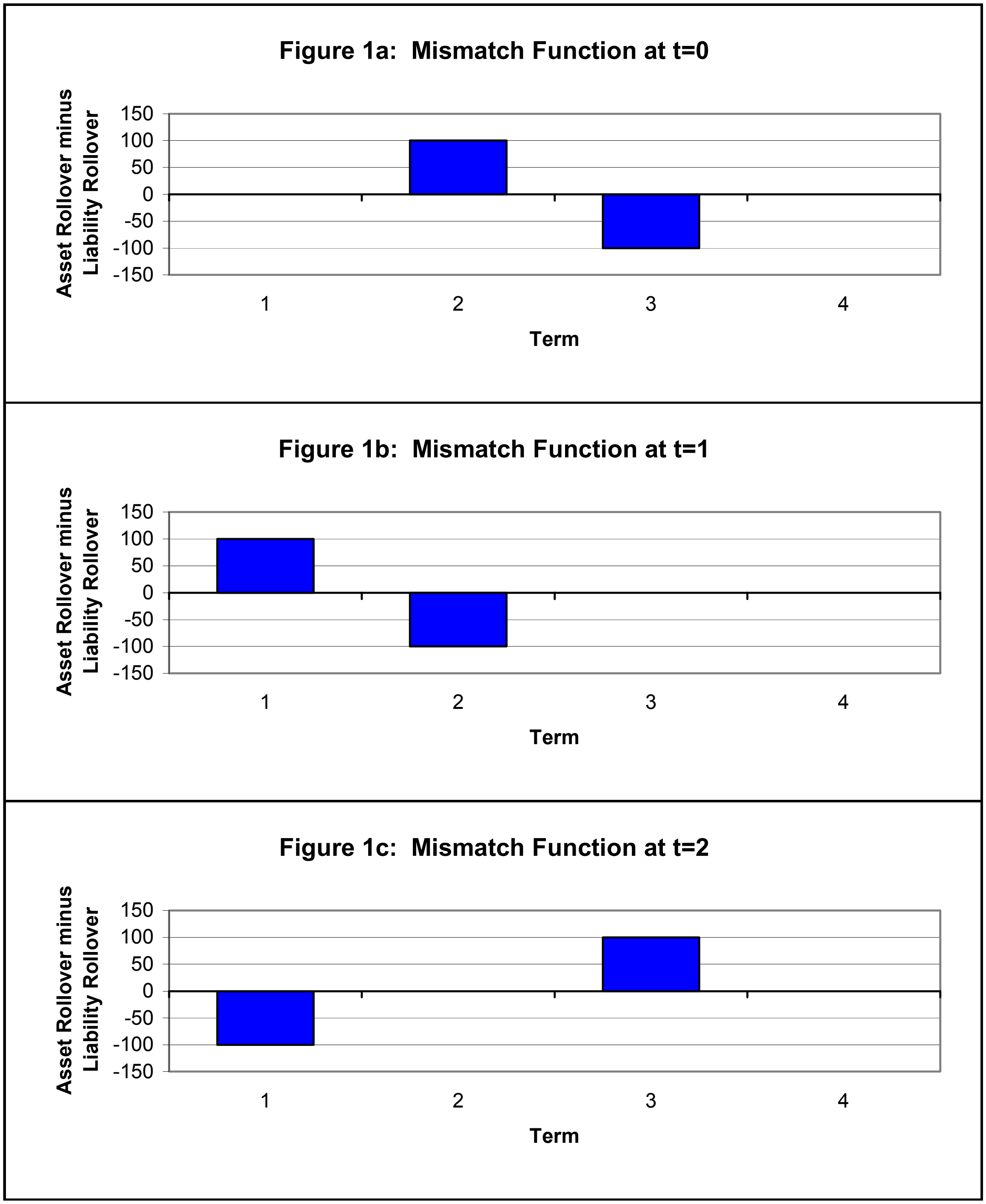


Figure 1: The mismatch function for a $100 3-year coupon paying GIC, backed (initially) by a 2-year bond. Figure 1a shows the initial mismatch function; 1 year later the mismatch function will naturally migrate left, as shown in figure 1b. Figure 1c shows the mismatch function at end of year 2 assuming reinvestment in a 3-year bond.

The mismatch function which has been described in this section is a key element in the discussion to follow. A typical computer simulation ALM model relies on setting a strategy for borrowing and for new investment. In section 2.2 it is proven that defining the mismatch function at all future times is equivalent to describing the strategy for new cash investments. In other words, in a traditional simulation model one starts with existing assets and liabilities, then, given an interest rate scenario, one projects cash flows. These cash flows are invested (or borrowed) using some investment strategy defined as one of the model's assumptions. The equations below, however, contain no reference to the investment assumption – instead they make use of the asset-liability mismatch function. In other words, in this model instead of explicitly specifying what is done with available cash, the model assumes that available cash is used to reach some specified level of asset-liability mismatch. This allows for unusually complex reinvestment strategies.

In section 2.2 it is also shown that given the mismatch function and the future yield curve changes one can produce a formula for future earnings that are due to mismatch. One dramatic element of this formula is that it clearly distinguishes between earnings that are due to asset-liability mismatching and hence are sensitive to yield curve movements and earnings that are insensitive to yield curve movements. By separating out the component of earnings which derive from the asset-liability mismatch the mismatch-earnings formula (MEF) can be derived and analyzed.

2.2 A Derivation of the Mismatch-Earnings Formula

This section consists of a derivation of a relatively simple mathematical formula for future earnings arising from asset-liability mismatch. In deriving the equations below, some simplifying assumptions have been made:

1. None of the liabilities has interest-sensitive cash flows. In other words, the portfolio (or company) in question is made up only of traditional insurance products such as fixed payout annuities, group long term disability, or whole life (with no loan provision).[5]
2. Similarly none of the assets in the portfolio has interest-sensitive cash flows. In other words, the assets in questions are non-callable coupon bearing bonds (extending to zeros is also straightforward).
3. No new business is included in the analysis.
4. The asset portfolio is managed using a buy and hold strategy. Cash deficiencies are therefore met using a borrowing strategy. On the other hand, the nature of the borrowing and new investment strategy is left free and can change over time.
5. Taxes are set equal to zero.
6. For calculational simplicity all financial transactions are assumed to take place in continuous time. Thus assets pay coupons continuously, and liabilities credit interest continuously.

Admittedly some of the assumptions above are unrealistic (perhaps even draconian). However, it is worth recalling that the purpose of this theoretical analysis is not to produce absolutely accurate predictions but to develop additional insight into the problem of asset-liability management. Similarly, although Redington's definition of duration is a full statement of the problem only if all non-parallel shifts of the yield curve are excluded from consideration, nonetheless, the concept of duration mismatch is a useful tool for understanding a company's level of risk. In addition, note that not all of these assumptions are vital to the mathematical development. A revised list detailing which assumptions can be relaxed is given at the end of this section. Finally, since the intent of this model is to quantify book value income volatility the emphasis of the development is on items that flow through the income statement (i.e., unrealized capital gains and losses are ignored.)

The derivation below will begin with a mathematical approximation of a typical ALM computer simulation model. In particular, we begin with the following inputs:

- a runout of liabilities
- a runout of existing assets
- an interest rate scenario.
- a strategy for investing available cash in future time periods

For every future time period, we then can calculate new cash available for investment (and invest it in the predefined strategy). This will allow us to project future earnings. Having defined a traditional ALM model, we then mathematically define the mismatch function discussed in section 2.1, and incorporate it into the equations.

The derivation make use of the following definitions:

Current time t = 0.

Projection of Liability:

- $R(t)$ = Reserve at time t. Since there is an assumption that no new liabilities are sold $R(t)$ can be fully known at t = 0 (up to variations due to mortality and morbidity experience). (In units of dollars.)
- $IC(t)$ = Interest credited at time t. As with $R(t)$ this function can be projected at time t = 0. (In units of dollars.)
- $CF_L(t)$ = Liability cash flow at time t. (In units of dollars.)
- It is possible to write a formula that relates $CF_L(t)$, $IC(t)$ and $R(t)$, so only two of the three need to be considered inputs.

Projection of Existing Assets:

- BV_0 = Book value of assets at time 0. (In units of dollars.) Typically we assume that $BV_0 = R(0)$.
- $B_0(t)$ = Book value of assets remaining at time t from the assets in the portfolio at time 0. (In units of dollars.)
- $f_0(t)$ gives the distribution of rollover times of assets in the portfolio at time 0 (the initial assets). (In units of percent.) Mathematically,
$$f_0(t) = -\frac{1}{BV_0}\frac{\partial B_0(t)}{\partial t}.$$
- $C_{EA}(t)$ = Coupon yield rates for the existing assets projected to roll over at time t.
- To clarify: At time t, BV_0 x $f_0(t)$ assets are projected to roll over. Between time 0 and time t these assets will throw off net investment income of BV_0 x $f_0(t)$ x $C_{EA}(t)$.
- $CF_{EA}(t)$ = Cash flows from existing assets at time t. (In units of dollars.)

Interest Rate Scenario: Future Yield Curves

- $C_\tau(T)$ = Coupon yield rate for a bond of term T purchased at time τ. (τ>=0).
- Note that $C_0(T)$ is not the same as $C_{EA}(T)$.

Future Investment Strategy:

- $n(\tau)$ = New cash (positive or negative) available for borrowing or new investments at time τ. (In units of dollars.) This needs to be calculated based on the runout of liabilities, existing assets and all previously purchased assets.
- $f_\tau(T)$ = Purchase strategy at time τ. (In units of percent.) For example, specify that at time 1 excess cash will be invested 50% in 5-year bonds and 50% in 10-year bonds. Or, in other words, purchase $n(\tau)f_\tau(T)$ bonds with term T at time τ (i.e., these bonds will mature at time T+τ).
- Note that $f_\tau(T)$ is defined consistently with $f_0(T)$ (the distribution of rollover times of assets in the portfolio at time 0, i.e., the initial assets.)

- $CFA_\tau(T)$ = Cash flows from new asset purchased at time τ with term T. In other words, these cash flows occur at time τ+T. (In units of dollars.) Given the assumption that all new assets are coupon bonds purchased at par, we can express $CFA_\tau(T)$ in terms of $n(\tau)$, $f_\tau(T)$, and $C_\tau(T)$.

Earnings

e(t) = Earnings at time t. (In units of dollars.)

Certain formulas follow from the above definitions. For example, since earnings are equal to coupon income on existing assets and new assets minus interest credited on liabilities:

$$e(t) = BV_0 \int_t^\infty dt' \, f_0(t') C_{EA}(t') + \int_0^t d\tau \, n(\tau) \int_{t-\tau}^\infty dt' \, f_\tau(t') C_\tau(t') - IC(t) \qquad (2.1)$$

As noted above, earnings do not include unrealized capital gains and losses. (Nor does e(t) include realized capital gains and losses because of the assumption that all bonds are held to maturity.)

It is also possible to write a formula for $n(\tau)$:

$$n(\tau) = CF_{EA}(\tau) - CF_L(\tau) + \int_0^\tau dt' \, CFA_{t'}(\tau - t') - e(\tau)$$

The first three terms simply sum up the asset and liability cash flows. The last term assumes that distributable earnings are, in fact, distributed and therefore represent an additional cash flow. One impact of this cash flow is that it keeps the book value of assets equal to the book value of liabilities.

There is not a lot of insight implicit in the above formulas – they simply repeat what would naturally emerge from a computer simulation. We now diverge from this path and mathematically define the mismatch function.

Asset-Liability Mismatch Function

As discussed in section 2.1, the mismatch function at time 0 is defined as the projected asset rollovers minus the liability rollovers. We will use the notation $\delta_0(T)$.

Mathematically, $\delta_0(T) = \frac{\partial}{\partial T}\left[R(T) - B_0(T)\right]$

Again, as discussed previously, this function can also be defined for some future time τ (and notated $\delta_\tau(T)$). That is, at time τ, $\delta_\tau(T)$ is the projected difference in asset and liability rollovers at time τ+T.

Now define,

- $B_\tau(T)$ = Book value of assets remaining at time T+τ from the assets in the portfolio at time τ. (In units of dollars.)

- $R_\tau(T)$ = Book value of liabilities remaining at time $T+\tau$ from the liabilities in the portfolio at time τ. (In units of dollars.) Since there are no new liabilities therefore $R_\tau(T) = R(\tau+T)$.

$$\delta_\tau(T) = \frac{\partial}{\partial T}\left[R_\tau(T) - B_\tau(T)\right] = \frac{\partial}{\partial T}\left[R(\tau + T) - B_\tau(T)\right] \tag{2.2}$$

Based on the definitions above, a formula for $B_\tau(T)$ can be written:

$$B_\tau(T) = BV_0 \int_{T+\tau}^{\infty} f_0(t')dt' + \int_0^{\tau} n(t')dt' \int_{T+\tau-t'}^{\infty} f_{t'}(s)ds \tag{2.3}$$

In other words, the book value of assets remaining at time $T+\tau$ from the assets in the portfolio at time τ is equal to the remaining book value of initial assets plus the book value projected to remain from assets purchased between times 0 and τ.

Using equations 2.2 and 2.3, and recalling the general theorem that:

$$\frac{\partial}{\partial x}\int_0^x g(y,x)dy = g(x,x) + \int_0^x \frac{\partial}{\partial x} g(y,x)dy$$

a relationship between the mismatch function and the new investment functions can be derived:

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial T}\right)\delta_\tau(T) = n(\tau) f_\tau(T) \tag{2.4}$$

Equation 2.4 provides a relationship between the new investment strategy and changes in the asset-liability mismatch function. In words it states (after some minor rearrangement) that the mismatch function at time τ changes in time due to the fact that the mismatch function rolls leftward along the curve (see figures 1a and 1b) and due to the fact that new cash flows can be invested anywhere along the curve (see figure 1c).

In equation 2.1 we were able to express future earnings using the general methodology of a computer simulation ALM model. Using equation 2.4, earnings can be restated in terms of the mismatch function.

Recall that equation 2.1 stated:

$$e(t) = BV_0 \int_t^{\infty} dt' f_0(t') C_{EA}(t') + \int_0^t d\tau\, n(\tau) \int_{t-\tau}^{\infty} dt' f_\tau(t') C_\tau(t') - IC(t)$$

Into equation 2.1 substitute:

- $C_{EA}(t) = C_{EA}(t) - C_0(t) + C_0(t)$
- $n(\tau) f_\tau(t') = \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t'}\right)\delta_\tau(t')$ (equation 2.4); and
- $\delta_0(t') = \frac{\partial R(t')}{\partial t'} + BV_0 f_0(t')$

where the final item is the $\tau = 0$ version of equations 2.2 and 2.3.

This substitution results in:

$$\begin{aligned}
e(t) &= \int_t^\infty dt'\left[-\frac{\partial R(t')}{\partial t'}\right]C_{EA}(t') - IC(t) \\
&+ \int_t^\infty dt'\delta_0(t')\left[C_{EA}(t') - C_0(t')\right] \\
&+ \int_t^\infty dt'\delta_0(t')C_0(t') \\
&+ \int_0^t d\tau \int_{t-\tau}^\infty dt'\left[\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t'}\right)\delta_\tau(t')\right]C_\tau(t')
\end{aligned} \tag{2.5}$$

Equation 2.5 can be made somewhat simpler. Integrating the last term by parts, and noting that:

$$\int_0^t d\tau \int_{t-\tau}^\infty dt' = \int_0^t dt' \int_{t-t'}^t d\tau + \int_t^\infty dt' \int_0^t d\tau$$

the final two terms of equation 2.5 can be rewritten, yielding the equation 2.6:

$$\begin{aligned}
e(t) &= \int_t^\infty dt'\left[-\frac{\partial R(t')}{\partial t'}\right]C_{EA}(t') - IC(t) \\
&+ \int_t^\infty dt'\delta_0(t')\left[C_{EA}(t') - C_0(t')\right] \\
&+ \int_0^\infty dt'\delta_t(t')C_t(t') \\
&+ \int_0^t d\tau \int_{t-\tau}^\infty dt'\delta_\tau(t')\left(\frac{\partial}{\partial t'} - \frac{\partial}{\partial \tau}\right)C_\tau(t')
\end{aligned} \tag{2.6}$$

At first glance equation 2.6 seems quite complicated, but on closer inspection the meaning of the various terms begin to emerge. The first two rows represent future earnings deriving from decisions made in the past, such as the initial profit margin at which the liabilities were sold, and previous asset-liability mismatch decisions. In other words, if action were to be taken at all times t>0 to reduce the future asset-liability mismatch to zero, then the first two rows would represent all future earnings, and those earnings would be insensitive to yield curve movements. The last comment is significant because it implies that from an asset-liability management perspective, the first two rows simply provide a baseline for projected earnings, and since they are based on decisions made in the past they should be ignored. As a result, from an asset-liability management perspective it makes sense to separate out the final two rows, and refer to them as $e_{mismatch}(t)$ – the earnings due to current and future asset-liability matching decisions.

$$\boxed{e_{mismatch}(t) = \int_0^{\infty} dt'\delta_t(t')C_t(t') + \int_0^t d\tau \int_{t-\tau}^{\infty} dt'\delta_\tau(t')\left(\frac{\partial}{\partial t'} - \frac{\partial}{\partial \tau}\right)C_\tau(t')} \qquad (2.7)$$

Equation 2.7 is the mismatch-earnings formula. The remainder of this paper will be devoted to discussing and modifying it. The first term in $e_{mismatch}(t)$ represents the earnings at time t from the asset-liability mismatch chosen at time t. The second term represents the earnings at time t from the asset-liability mismatch chosen from times 0 to t. In particular, the $\frac{\partial}{\partial t'}$ term represents the earnings arising from the shape of the yield curve from time 0 to t, while the $\frac{\partial}{\partial \tau}$ term represents earnings arising from yield curve shifts from time 0 to t.

What has been gained in the development of the mismatch-earnings formula? Although the formula itself can be used to project earnings under various scenarios, solving the integral would necessitate a computer approximation. In other words, by itself equation 2.7 does not seem to provide any obvious insight into portfolio behavior. (A more intuitive relationship between earnings and mismatch is based on an extension of this formula described in section 3 (MEF-2).) What is significant, however, is the fact that the MEF provides a separation of projected earnings into two components

- earnings that are insensitive to interest movements (of great interest to financial projections but of less interest from the perspective of ALM); and
- mismatch-earnings which are sensitive to interest rate movements (of fundamental interest from the perspective of ALM).

Furthermore, the MEF provides a direct mathematical relationship between the projected mismatch-earnings and the asset-liability mismatch function thus proving the significance of this new ALM metric.

Some additional comments about the MEF (equation 2.7), and the asset-liability mismatch function:

- The mismatch function depends only on projected book value rollovers of assets and liabilities, but not on total cash flows. As a result it is possible to have a mismatch function which is identically zero yet not be duration matched (although the duration mismatch would likely be extremely small).
- Practically, duration calculations for fixed income assets are much better defined than for liabilities, since assets have clearly defined market values while the definition of liability market value is still a subject of ongoing discussion (Vanderhoof (2000), Girard (2000), Girard (2002), FASB (1999), AAA (2000)). The mismatch function described above removes the

uncertainty in liability definition since reserve rollover is quite well defined and is easily projected.

- Similarly, the question of how to treat equities is somewhat better defined in the mismatch function above than it had been in terms of duration (Leibowitz (1993)). Rather than trying to estimate the interest rate sensitivity of an equity market value, to incorporate equities into the mismatch function one need “only” estimate the length of time one intends to hold the equity (and its projected book value growth until that time). Thus the final answer to the question of how to treat equities is not simple; however, the question is more easily explained.
- Note that there is no provision in the equations to allow the mismatch function to vary as a function of the current interest rate environment.
- Finally, note that equation 2.7 is linear in the mismatch function. This linearity will be made use of in section 3.1.

This section concludes with a discussion of which of the assumptions above may be relaxed, and which are vital to the mathematical derivation.

1. It is fairly straightforward to update equation 2.7 to include liabilities in which the interest credited is interest rate dependent (for example, floating rate funding agreements). Incorporating interest rate sensitivity into rollover rates is much more difficult.
2. Similarly, it is relatively straightforward to include floating rate coupon bonds; however, it is not yet possible to apply this method to mortgage backed securities, etc.
3. It is easy to incorporate new business into the analysis of mismatch earnings since $e_{mismatch}(t)$ depends only on the mismatch function. New business will, therefore, only change the base (non-interest sensitive) earnings.
4. Little progress has been made to incorporating asset sales into these equations.
5. Although no work has been done on this issue to date, it seems likely that taxes could be easily incorporated into these equations.

3. MEF-2: Extending the Mismatch Earnings Formula

3.1 Deriving MEF-2: Simplifications to the MEF and Stochastic Interest Rates

In this section, an extension to the MEF is derived. First, certain simplifying assumptions are applied to the MEF, then the single scenario basis of equation 2.7 is expanded to include a series of stochastic interest rate scenarios. In addition, as part of the process, the formulas are switched from continuous to discrete.

There are two primary simplifications that must be made to the equation 2.7:

1. It is assumed that the portfolio is managed such that the book value of assets is equal to the book value of liabilities at all times. This assumption is clearly realistic within those companies that subdivide their general account into portfolios and sweep earnings out of every portfolio into a surplus portfolio. It is also a realistic assumption for the company as a whole since (as was mentioned in the introduction) insurance companies are sometimes managed in order to maintain a specified RBC ratio. (If this method is applied to the company as a whole then the liability reserves must be replaced by reserves plus required surplus.) Mathematically this assumption is equivalent to assuming that for all times τ:

$$\int_0^\infty \delta_\tau(T)dT = 0 \tag{3.1}$$

2. More radically, it is assumed that the future portfolio management strategy is based on maintaining the current mismatch function. In other words, it is assumed that the current position of the assets vis-à-vis the liabilities was entered into intentionally, and that this position will be maintained in the future. Implicitly this also assumes that the company is willing to take strategic mismatch positions (e.g. always hold assets that are long to the liabilities) but not tactical mismatch positions (e.g. shortening because of the expectation of an interest rate increase). Mathematically, this is equivalent to assuming that the mismatch function is constant in time; i.e., $\delta_\tau(T)$ has no τ dependence and can be written simply as $\delta(T)$. Since the mismatch function is based on total dollars, this is equivalent to assuming that new business is replacing existing business as it rolls off the books (or that the mismatch is increasing as a percentage of portfolio size). This assumption may not be entirely realistic, but in the absence of other information is possibly a best estimate.

Next, in order to make the mismatch function more easily usable, the continuous function must be simplified. To this end, define a series of N subfunctions $g_i(T)$ that will allow us to approximate the mismatch function:

$$\delta_\tau(T) \approx \sum_{i=1}^{N} a_i g_i(T) \tag{3.2}$$

where the a_i's are constants that do not depend on τ (as per the second assumption above.)

The $g_i(T)$ subfunctions can be chosen arbitrarily; however, it is useful to scale them such that:

$$\int_0^\infty g_i(T)dT = 1 \tag{3.3}$$

Given this condition, then the first assumption above (i.e., equation 3.1) requires that:

$$\sum_{i=1}^{N} a_i = 0 \tag{3.4}$$

A simple example of a set of N=10 subfunctions is:

$$g_i(T) = \begin{cases} 1 & for\ i-1 < T \le i \\ 0 & otherwise \end{cases} \tag{3.5}$$

Since i is chosen to run from 1 to 10 this allows mismatches of terms up to 10 years. Once the $g_i(T)$ subfunctions are chosen specifying the a_i's is equivalent to specifying the mismatch function.

Finally, the output variable must be chosen. For example, to study the expected value and volatility of earnings two years from today one might choose to evaluate:

$$r[\delta, C] = \frac{\int_1^2 e_{mismatch}(t)dt}{\int_1^2 R(t)dt}$$

where r can be described as the year 2 earnings spread due to the asset-liability mismatch. (Recall that $e_{mismatch}(t)$ was defined in equation 2.7.) The explicitly notated dependence of r on δ and C serves as a reminder that r will vary based on the yield curve scenario and the nature of mismatch chosen.

A more general output variable which will be analyzed in section 3.2 is:

$$r[\delta, C, T_1, T_2] = \frac{\int_{T_1}^{T_2} e_{mismatch}(t)e^{-kt}dt}{\int_{T_1}^{T_2} R(t)e^{-kt}dt} \tag{3.6}$$

where k represents some hurdle rate. Typically, one would be interested in evaluating a number of output variables, as will be done in section 3.2. Note that this output variable can evaluate earnings over a long period as well as over a short period. A long period measure of expected earnings and earnings volatility can be considered a measure of the true economic risk and return.

Now given that the MEF (equation 2.7) is linear in $\delta_\tau(T)$, the subfunction breakdown can be used to develop a simple framework for evaluating the response of the output variable r to different mismatch functions. First define N new output variables (r_i) based on the original output variable *r*:

$$r_i[C] = r[\delta(T) = g_i(T), C] \tag{3.7}$$

Then, using a stochastic interest rate scenario generator, calculate the following standard statistical functions:

$$\bar{r}_i = E[r_i[C]] \tag{3.8}$$

$$\sigma_{ij} = E[(r_i[C] - \bar{r}_i)(r_j[C] - \bar{r}_j)] \tag{3.9}$$

where the expectations are taken over the distribution of C. Some additional comments on calculating $\bar{r}_i$ and σ_{ij} are given in the Appendix.

With these definitions and with the approximation to the mismatch function given in equation 3.2 it is simple to calculate the mean and variance of the output variable.

$$E[r[\delta, C]] = \sum_{i=1}^{N} a_i \bar{r}_i \tag{3.10}$$

$$Var(r[\delta, C]) = \sum_{i,j=1}^{N} a_i \sigma_{ij} a_j \tag{3.11}$$

Equations 3.10 and 3.11 result from the fact that *r* is a linear function of the mismatch function. As a result, when the mismatch function was broken into a linear combinations of subfunctions (in equation 3.2) then it became possible to describe E[*r*] as a linear combination of the average value of the subfunctions (equation 3.10), and Var[*r*] as a combination of the variances and covariances of the subfunctions (equation 3.11).

From the discussion above equations 3.8 through 3.11 seem standard, but note that something interesting has happened. The significant calculational complexity required to evaluate equations 3.8 and 3.9 requires no knowledge of the actual asset-liability mismatch which is being modeled (i.e. they do not depend on the a_i's). In other words, the variables $\bar{r}_i$ and σ_{ij} have no dependence on the portfolio's current ALM position, and need only be recalculated when the stochastic interest rate generator is updated. On the other hand, given $\bar{r}_i$ and σ_{ij} solving equations 3.10 and 3.11 is extremely simple, and quick calculations can be done on various portfolio asset-liability mismatch positions.

In summary, the procedure described above separates out the two complicated aspects of projecting the financial implications of asset-liability mismatches:

- The problem of managing a complex model under a large variety of interest rate scenarios is reduced to evaluating N(N+1) items (per output variable) every time the stochastic interest rate generator is updated.

- The problem of measuring the current mismatch is reduced to describing the current mismatch in terms of the N a_i's. Once these a_i's are measured (as often as one chooses – the timing need not be determined by the first step) it is extremely simple to produce the mean and standard deviation of the financial output variables.

3.2 Sample Results from MEF-2: Metrics for Expected Earnings Volatility Due to Asset-Liability Mismatch

In this section, an example of the procedure outlined in section 3.1 will be given. This example will provide an opportunity to discuss the nature of a typical calculation. For simplicity the analysis will be run assuming a short portfolio (for example a portfolio of term life liabilities), and therefore mismatches will be limited to terms of 5 years and earnings volatility will be projected for only 5 years.

The first step in measuring earnings volatility for a portfolio is defining the mismatch approximation subfunctions as described in section 3.1. Following equation 3.5, for this example the following subfunctions are defined:

$$g_i(T) = \begin{cases} 1 & \text{for } i-1 < T \le i \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

where i runs from 1 to N=5.

Given the subfunction definition, the next step is to choose an interest rate scenario generator. There are many interest rate generators discussed in the literature (for a summary see Christiansen (1992)) and this paper takes no position on which one to use. The scenario generator chosen for this example is the one provided with the TAS[6] software package. The following items describe the interest rate scenario generator model and inputs:

- Yield curves are generated stochastically using a lognormal probability density function with mean reversion at two points on the curve – the 90 day rate and the 10 year rate. All other points are interpolated.
- The volatility parameter for the 90-day rate is 16% and for the 10 year rate is 8%. The correlation between the two is 70%. The mean reversion factor is 5% for the 90-day rate and 2% for the 10-year rate.
- The initial yield curve is based on the January 2, 2003 90-day LIBOR, and 10-year LIBOR swap rate. (90 day rate = 1.32%, 10 year rate = 4.48%). The curve is also assumed to revert to these same rates.
- Projections are quarterly. Between quarters linear interpolation is assumed.
- 1,000 scenarios are generated.

A summary of the 1,000 scenarios is given by the following three figures. Figure 2a shows the expected value of the future yield curve (i.e. the average over the 1,000 scenarios). Since the mean reversion was set to current rates, the expected yield curve should not change. Figure 2b shows the mean and standard deviation (as error bars) of the yield curve 5 years from today. Figure 2c shows the standard deviation along the yield curve as it varies over the next 5 years.

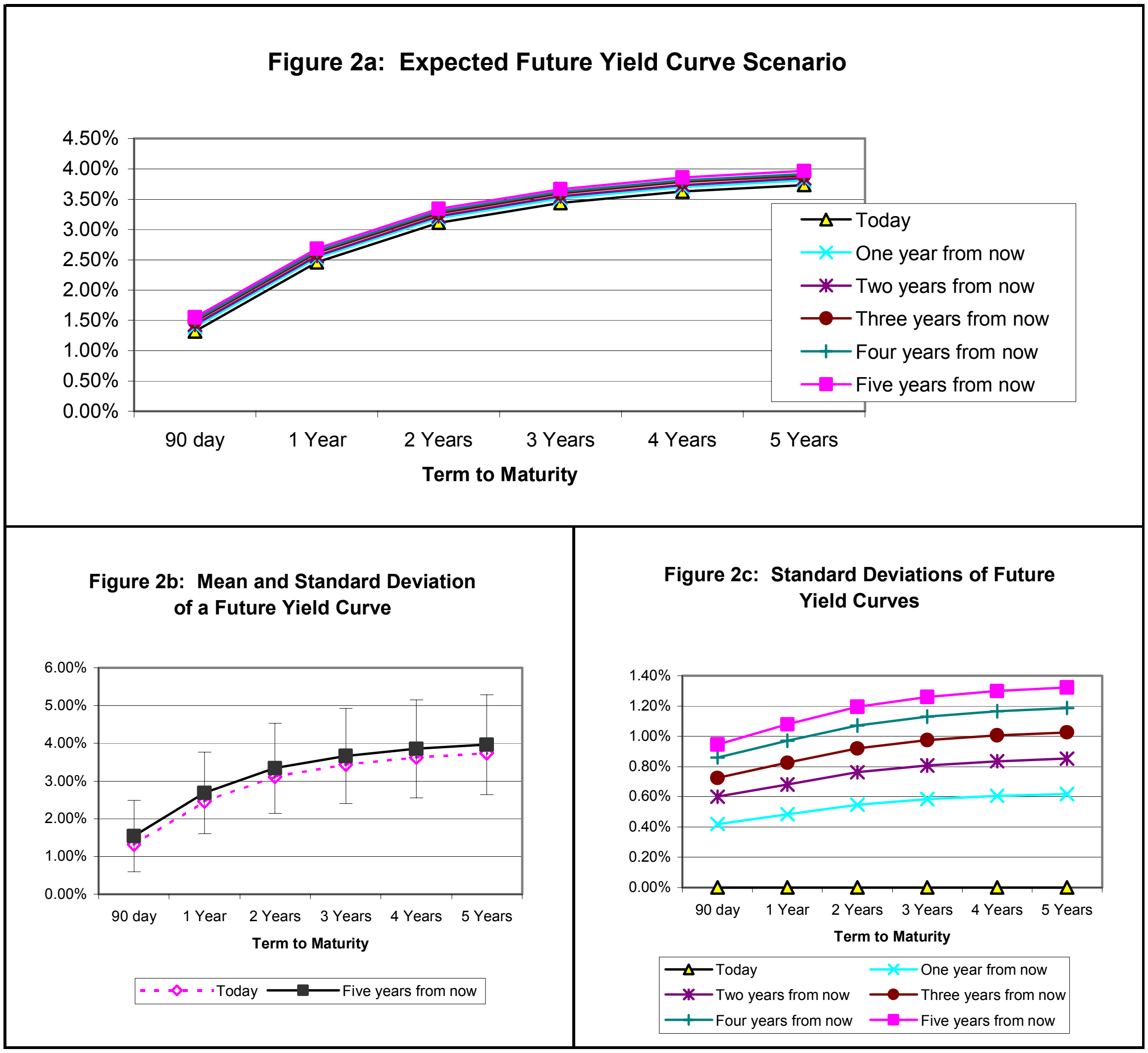


Figure 2: Summary of the 1,000 randomly generated interest rate scenarios. Figure 2a shows that the mean of the future yield curves is constant over time. Figures 2b and 2c illustrate the range of variation of future yield curves.

Having selected the 1,000 scenarios, the next step is to choose an appropriate output variable. For this example two separate output variables of the generic type described in equation 3.6 will be evaluated. Recall that equation 3.6 defines the following output variable:

$$r[\delta, C, T_1, T_2] = \frac{\int_{T_1}^{T_2} e_{mismatch}(t) e^{-kt} dt}{\int_{T_1}^{T_2} R(t) e^{-kt} dt}$$

where k is the hurdle rate (here set equal to 15%). Initially the output variable which will be analyzed is $(T_1, T_2) = (0,5)$. Subsequently, for comparison, $(T_1, T_2) = (1,2)$ will be examined (and eventually results for all yearly intervals will be shown).

For each output variable analyzed one must evaluate $\bar{r}_i$ and σ_{ij} as described in equations 3.8 and 3.9. Note that assumption 2 in section 3.1 required that the mismatch function be constant in time. This assumption does not necessarily require that R(t) be constant in time – i.e. that new business replace old business that rolls off the books – however, it is the simplest explanation. In keeping with this explanation assume that R(t) is constant; furthermore, assume (with no loss of generality) that R(t) = 1. A result of setting R(t)=1 is that the mismatch function will be expressed as a percentage of the portfolio rather than as a dollar amount.

Figure 3a shows the values of $\bar{r}_i$ and σ_{ij} for $(T_1, T_2) = (0,5)$. Additionally, the correlation coefficient (ρ_{ij}) corresponding to the σ_{ij} is given. Some points worth noting:

- The $\bar{r}_i$'s are monotonically increasing as one would expect given the interest rate curves in figure 2; however $\bar{r}_3$ through $\bar{r}_5$ are quite close together. This is due to the fact that the overall rate level is not the only important element in calculating this factor – the slope of the yield curve at the relevant time period is also quite important. In this case the yield curve is upward sloping but with a decreasing steepness which lessens the advantage of rolling down the curve.
- The ρ_{ij} have approximately the distribution that one would expect; however, note that the lowest correlation ($\rho_{1,5}$) is still a fairly significant 0.61.

For contrast, figure 3b shows the values of $\bar{r}_i$, σ_{ij} and ρ_{ij} for $(T_1, T_2) = (1,2)$.

- As in the case of the first output variable the $\bar{r}_i$'s are monotonically increasing with a decreasing difference in rates at the higher terms**.**
- Observe that the ρ_{ij} for i and j both greater than 2 are all 1.00. It is simple to prove that if the yield curve movements were constrained to only include parallel shifts these correlations should be exactly 1.00 and the standard deviations would be 0; the fact that the yield curve movements are controlled by two points on the curve leads to the non-zero standard deviations but maintains the correlations at 1.00.

Figure 3a: $\bar{r}_i$ and σ_{ij} and ρ_{ij} for output variable defined by $(T_1,T_2) = (0,5)$

$$\bar{r}_i = \begin{pmatrix} 2.45\% \\ 3.63\% \\ 3.87\% \\ 3.93\% \\ 3.95\% \end{pmatrix} \qquad \sigma_{ij} = \begin{pmatrix} 1.94 \times 10^{-5} & 1.62 \times 10^{-5} & 9.81 \times 10^{-6} & 4.45 \times 10^{-6} & 1.39 \times 10^{-6} \\ 1.62 \times 10^{-5} & 1.55 \times 10^{-5} & 9.98 \times 10^{-6} & 4.81 \times 10^{-6} & 1.66 \times 10^{-6} \\ 9.81 \times 10^{-6} & 9.98 \times 10^{-6} & 6.73 \times 10^{-6} & 3.38 \times 10^{-6} & 1.18 \times 10^{-6} \\ 4.45 \times 10^{-6} & 4.81 \times 10^{-6} & 3.38 \times 10^{-6} & 1.80 \times 10^{-6} & 6.47 \times 10^{-7} \\ 1.39 \times 10^{-6} & 1.66 \times 10^{-6} & 1.18 \times 10^{-6} & 6.47 \times 10^{-7} & 2.54 \times 10^{-7} \end{pmatrix}$$

$$\rho_{ij} = \begin{pmatrix} 1.00 & 0.93 & 0.86 & 0.75 & 0.62 \\ 0.93 & 1.00 & 0.98 & 0.91 & 0.84 \\ 0.86 & 0.98 & 1.00 & 0.97 & 0.90 \\ 0.75 & 0.91 & 0.97 & 1.00 & 0.96 \\ 0.62 & 0.84 & 0.90 & 0.96 & 1.00 \end{pmatrix}$$

Figure 3b: $\bar{r}_i$ and σ_{ij} and ρ_{ij} for output variable defined by $(T_1,T_2) = (1,2)$

$$\bar{r}_i = \begin{pmatrix} 2.49\% \\ 3.67\% \\ 3.78\% \\ 3.82\% \\ 3.86\% \end{pmatrix} \qquad \sigma_{ij} = \begin{pmatrix} 1.61 \times 10^{-5} & 6.01 \times 10^{-6} & 1.19 \times 10^{-6} & 6.61 \times 10^{-7} & 4.20 \times 10^{-7} \\ 6.01 \times 10^{-6} & 3.57 \times 10^{-6} & 9.18 \times 10^{-7} & 5.11 \times 10^{-7} & 3.25 \times 10^{-7} \\ 1.19 \times 10^{-6} & 9.18 \times 10^{-7} & 3.17 \times 10^{-7} & 1.78 \times 10^{-7} & 1.13 \times 10^{-7} \\ 6.61 \times 10^{-7} & 5.11 \times 10^{-7} & 1.78 \times 10^{-7} & 9.97 \times 10^{-8} & 6.33 \times 10^{-8} \\ 4.20 \times 10^{-7} & 3.25 \times 10^{-7} & 1.13 \times 10^{-7} & 6.33 \times 10^{-8} & 4.02 \times 10^{-8} \end{pmatrix}$$

$$\rho_{ij} = \begin{pmatrix} 1.00 & 0.79 & 0.53 & 0.52 & 0.52 \\ 0.79 & 1.00 & 0.86 & 0.86 & 0.86 \\ 0.53 & 0.86 & 1.00 & 1.00 & 1.00 \\ 0.52 & 0.86 & 1.00 & 1.00 & 1.00 \\ 0.52 & 0.86 & 1.00 & 1.00 & 1.00 \end{pmatrix}$$

Figure 3: Sample evaluation of statistical variables defined by equations 3.8 and 3.9 using the yield curves illustrated in figure 2 and the output (earnings) variable defined by equation 3.6. In the MEF-2 framework these statistical variables allow simple calculations to determine the expected earnings and standard deviation of earnings for arbitrary mismatch functions.

Having calculated the $\bar{r}_i$ and σ_{ij} for the general set of subfunctions described in equation 3.12, we can now evaluate the expected earnings and standard deviation of earnings for a portfolio. As shown in equations 3.10 and 3.11 this is a simple calculation once we have specified a_i's that approximate the actual asset-liability mismatch. In general, specifying the a_i's for the asset-liability mismatch requires an approximation to the mismatch function using the set of chosen subfunctions. Given the choice of subfunction in equation 3.12 the approximation is simple:

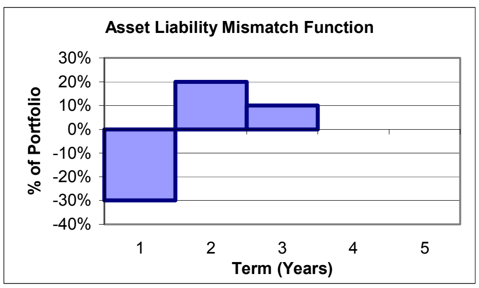
$$a_i = \left[BV(t = i) - BV(t = i\text{-}1)\right] - \left[R(t = i)\text{-}R(t = i\text{-}1)\right]$$

Where BV(t) represents the runout of the book value of the existing assets, and R(t) represents (as usual) the runout of the book value of the current liability. Recall also that BV(0) = R(0) (by assumption) and that in order to simplify the calculations the runouts are scaled down so that R(0) = 1.

It is simply a matter of matrix multiplication to estimate the expected earnings and earnings volatility over the two time periods discussed in figure 3 (using the matrices shown in the figure). This procedure can, of course, be extended to estimate expected earnings and volatility of earnings at every year from 1 to 5. The results are illustrated for two sample mismatch positions in figure 4.

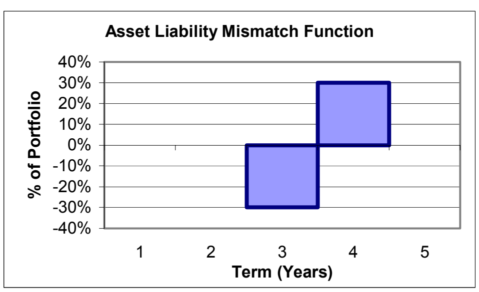
Extended Mismatch Earnings Formula (MEF-2) Output

Mismatch Example #1



	Expected Earnings From Mismatch	Std Dev of Earnings From Mismatch
Year 1	0.31%	0.03%
Year 2	0.37%	0.09%
Year 3	0.41%	0.11%
Year 4	0.42%	0.13%
Year 5	0.42%	0.15%
5 Year Average	*0.38%*	*0.06%*

Mismatch Example #2



	Expected Earnings From Mismatch	Std Dev of Earnings From Mismatch
Year 1	0.06%	0.00%
Year 2	0.01%	0.01%
Year 3	-0.02%	0.05%
Year 4	0.00%	0.12%
Year 5	0.02%	0.13%
5 Year Average	*0.02%*	*0.04%*

Figure 4: An illustration of the expected earnings and standard deviation of earnings arising from two potential asset-liability mismatch positions. Recall that the MEF-2 formulas assume that the mismatch position will be maintained over the entire 5 year simulation period. (This assumption can be relaxed; however, this would require a full evaluation of the MEF (equation 2.7) as opposed to the simplifications of section 3.)

Beyond measuring the impact of existing portfolio mismatches, one can also easily test alternative asset-liability mismatch strategies. This observation leads directly to the next section.

4. Optimizing Portfolios Using MEF-2

One of the most interesting implications of the theoretical analysis above is its application to optimizing portfolios. This aspect of the theory is also one of the most different from the traditional computer simulation models where a systematic search over a complete class of investment strategies is virtually impossible.

An optimal mismatch position is taken to be a position that has the maximum expected return for a given level of risk. This definition is similar to that used in an efficient frontier calculation. In this case, as is typical in efficient frontiers, risk is defined to be the standard deviation of return.

There are two aspects of the analyses in sections 2 and 3 that allow the possibility of an optimization formula.

1) The only restriction on the mismatch function is equation 3.1 which requires:

   $$\int_0^\infty \delta_\tau(T)dT = 0$$

   As a result, any given mismatch function can be multiplied by a constant and will still be a valid mismatch function.

2) In the mismatch-earnings formula (equation 2.7) $e_{mismatch}$ is linear in the mismatch function. Assume that the output variable is chosen to be linear in $e_{mismatch}$ (as is true of the class of output variables described by equation 3.6). As a result, if the mismatch function is multiplied by some constant, then both the expected value and standard deviation of the output variable will be multiplied by this constant as well.

Based on these two observations, it is possible to split the optimization problem into two segments: First choose the optimal mismatch position along the curve by optimizing the ratio of expected return to standard deviation of return (similar to the Sharpe ratio). As described above this ratio would be insensitive to multiplying the mismatch function by a constant; hence, the optimal mismatch location is insensitive to the mismatch level. Having determined the optimal mismatch position, some other measure (for example, utility theory) can be employed to determine the acceptable level of risk and therefore the desired level of mismatch (i.e., the desired multiple to apply to the mismatch function).

As an example of optimizing the mismatch position along the curve:

- Optimize the return to risk ratio using the output variable from section 3 based on $(T_1, T_2) = (0,5)$. The statistics ($\bar{r}_i$ and σ_{ij}) connected with this output variable were given in figure 3a.

- The variables over which the optimization takes place are the a_i's, and the relevant return and risk formulas are given by equations 3.10 and 3.11.
- The constraints are:

  $$\sum_{i=1}^{N} a_i = 0$$

  and

  $$\sum_{i=1}^{N} a_i^{\ 2} = 1$$

  where the first constraint is simply equation 3.4, and the second constraint sets an arbitrary level for the mismatch.
- Using the Excel solver on the equations described above, the optimal mismatch position is:

  (-42%, 73%, -46%, -11%, 26%).

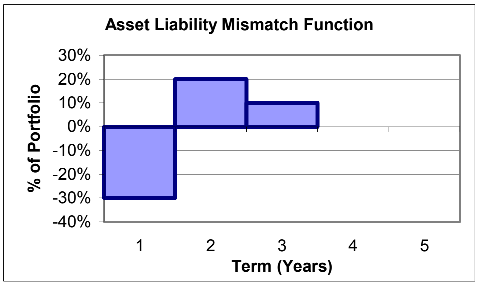
  This mismatch gives an expected return of 0.42%, and a standard deviation of 0.05% for a Sharpe ratio of 8.0. The final mismatch that would be applied to a portfolio would then be a constant multiple of the mismatch position given above. In figure 5 the financial impact of this optimal position is compared to the positions selected as examples in figure 4 above.

One must be extremely cautious in applying this type of optimization for two reasons. Most importantly, any optimization technique is likely to find mistakes in the assumptions rather than true optimal solutions. In the example above, the double peak found in the "optimal" solution is possibly an indicator that there is some inadequacy with the process. For example, the interest rate generator selected had only two degrees of freedom, but the process of searching for the optimal portfolio allowed three degrees of freedom (five variables and two constraints).

As a second reason for caution it is worth recalling that the optimal solution is non-interest rate sensitive. This lack of interest rate sensitivity is, of course, one of the constraints of the model; however, there may be more effective solutions that vary with the interest rate environment.

MEF-2 Output for Sample Portfolios and "Optimal" Portfolio

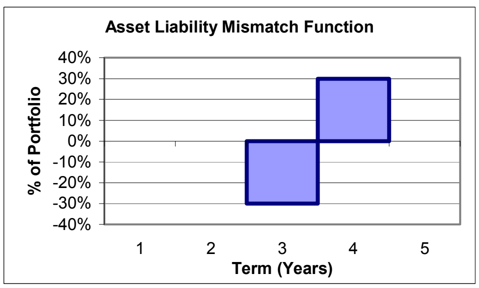
Mismatch Example #1



	Expected Earnings From Mismatch	Std Dev of Earnings From Mismatch
Year 1	0.31%	0.03%
Year 2	0.37%	0.09%
Year 3	0.41%	0.11%
Year 4	0.42%	0.13%
Year 5	0.42%	0.15%
5 Year Average	*0.38%*	*0.06%*

Sharpe Ratio: 6.9

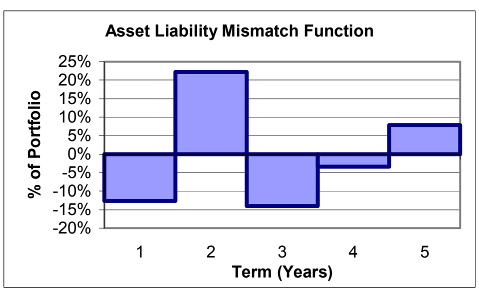
Mismatch Example #2



	Expected Earnings From Mismatch	Std Dev of Earnings From Mismatch
Year 1	0.06%	0.00%
Year 2	0.01%	0.01%
Year 3	-0.02%	0.05%
Year 4	0.00%	0.12%
Year 5	0.02%	0.13%
5 Year Average	*0.02%*	*0.04%*

Sharpe Ratio: 0.4

Mismatch Example #3 -- "Optimal" Asset Liability Mismatch*



	Expected Earnings From Mismatch	Std Dev of Earnings From Mismatch
Year 1	0.10%	0.01%
Year 2	0.14%	0.03%
Year 3	0.14%	0.05%
Year 4	0.13%	0.05%
Year 5	0.14%	0.06%
5 Year Average	*0.13%*	*0.02%*

Sharpe Ratio: 8.0

* The measure under which this mismatch is optimal is the ratio of expected earnings to standard deviation of earnings (5 year average). For purposes of comparison the mismatch expressed in the text is arbitrarily scaled down so that the total dollars mismatched is the same as in the first two examples above.

Figure 5: An illustration of the "optimal" portfolio mismatch arising from the methods in section 4. The bottom box shows the optimal mismatch position with a Sharpe ratio of 8.0, compared to the top two boxes which represent the mismatch examples from figure 4, and have Sharpe ratios of 6.9 and 0.4.

5. Practical Applications

The techniques and formulas described in sections 2 to 4 offer a range of potential applications.

A. Prior to any analysis, the mismatch-function conveys a great deal of insight into the level of potential earnings volatility due to interest rate movements at different time periods (much as duration mismatch gives insight into the level of market value volatility). It is also relatively easy to calculate the mismatch-function, so this function could be monitored over time as part of a risk management report.

B. Given the mismatch-function it is possible to provide the risk-return tradeoff for earnings over various time scales using the techniques in section 3. The shorter time scales will be useful from the CFO point of view, while longer time scales will provide some insight into economic risk. These risk-return measures can be updated as frequently as the mismatch-function is updated and would be a valuable addition to a risk management report.

C. If the current level of risk and return is unsatisfactory then it is relatively straightforward to test the impact of alternative strategies. As described in section 4, it is possible to optimize the risk return ratio, although this procedure may produce questionable results. However, the theoretical optimal mismatch also provides some clue as to how much improvement could be expected. In figure 5, for example, note that the Sharpe ratio in mismatch example #1 is close to the value of the optimal level (as opposed to mismatch example #2).

As has been mentioned frequently in this paper, despite the complicated nature of some of the equations, the actual application of the ideas described is relatively straightforward.

Appendix: Some additional comments on calculating $\bar{r}_i$ and σ_{ij}

Equations 3.8 and 3.9 describe the calculation of the variables $\bar{r}_i$ and σ_{ij}. In order to perform this calculation we must determine the value of $r_i[C]$ (defined in equations 3.6 and 3.7) under a large number of interest rate scenarios. Essentially, this requires the evaluation of $e_{mismatch}(t)$ under a variety of interest rate scenarios.

Instead of starting with equation 2.7 (the MEF) it is easier to evaluate $e_{mismatch}(t)$ based on a prior step in the derivation: equation 2.5. In other words, evaluate:

$$e_{mismatch}(t) = \int_t^\infty dt'\,\delta_0(t')C_0(t') + \int_0^t d\tau \int_{t-\tau}^\infty dt' \left[\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial t'}\right)\delta_\tau(t')\right]C_\tau(t') \qquad \text{(A.1)}$$

In addition, recall that for the purpose of evaluating $\bar{r}_i$ and σ_{ij}, $\delta_\tau(t')$ is replaced by $g_i(t')$ (which has no τ dependence). Therefore evaluate:

$$e_{mismatch}(t) = \int_t^\infty dt'\,\delta_0(t')C_0(t') + \int_0^t d\tau \int_{t-\tau}^\infty dt' \left[\left(-\frac{\partial}{\partial t'}\right)g_i(t')\right]C_\tau(t') \qquad \text{(A.2)}$$

There are three reasons that equation A.2 is simpler to evaluate than equation 2.7:

1. The first term is independent of the interest rate scenario and therefore has no impact on σ_{ij}.
2. One of the derivatives internal to the second term has been eliminated.
3. There is no need to worry about simulating a yield curve that has sensible derivatives; instead the derivative of $g_i(t')$ is all that is of concern.

One can further note that the $g_i(t)$ functions chosen in equation 3.5 have as their derivative a pair of Dirac delta function which reduces the double integral in the second term of equation A.2 to two single integrals.

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[1] For example, as described in Panjer (1998), Redington's initial assumption (in 1952) of flat yield curves and parallel yield curve shifts were relaxed in papers by Fisher and Weil (in 1971) and Shiu (in 1987). Ho (1992) and Reitano (1993) redefined the duration concept for practical immunization against changes in the shape of the yield curve. Stochastic interest rate modeling was also used in Reitano (1993).

[2] In a paper discussing the fair value of liabilities, Girard (2000) explicitly assumes that the statutory value of liabilities is equal to the statutory value of assets at all future times.

[3] As will be described in section 3, the assumption that the mismatch is constant over time allows for a significant simplification in the model calculations. This also seems to be a reasonable "best-guess" assumption for future ALM strategy. It is possible to run specialized versions of the MEF-2 model in which the mismatch changes in a specified way over time; however, this is more calculationally intensive.

[4] Optimal, in this case, means that one is able to identify the mismatch position that maximizes expected return given a level of standard deviation of return.

[5] This assumption excludes liabilities in which the company can reset the crediting rate, such as UL, and liabilities that depend on the level of interest rates, such as GMIB and GMDB.

[6] Tillinghast Actuarial Software.