Market Value of Life Insurance Contracts under Stochastic Interest Rates and Default Risk

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Abstract: The purpose of this article is to value some life insurance contracts in a stochastic interest rate environment taking into account the default risk of the underlying insurance company. The participating life insurance contracts considered here can be expressed as portfolios of barrier options as shown by Grosen and Jørgensen [1997]. In order to price these options, the Longstaff and Schwartz [1995] methodology is used with the Collin-Dufresne and Goldstein [2001] correction.

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Introduction

Life insurance companies offer complex contracts written with the following many covenants: interest rate guarantees, bonus and surrender options, equity-linked policies, choice of a reference portfolio, participating policies. Every clause have a value and is a part of the company liabilities. These embedded options should not be ignored and must be priced. Many companies have neglected these values for a long time, and, for this reason, have increased the difficulties they faced in the nineties.

Most of the recent studies rely on the Briys and de Varenne [1997a, 1997b] model. These authors aim at providing a fair valuation of liabilities. By this, it is meant that market value is the reference. More precisely, the computed prices must be arbitrage free. The life insurance contracts are thus considered as purely financial assets traded on a liquid market among perfectly informed investors. This fact is taken as a fundamental assumption in these studies, and it is the basis hypothesis we make in this article. Note that this principle is in line with the Financial Accounting Standards Board (FASB) and International Accounting Standard Board (IASB) directives.

Although Briys and de Varenne [1994, 1997a, 1997b] work in continuous time, their model is essentially a single-period one, and furthermore does not take into account the mortality risk. They value the assets and liabilities of an insurance company which sells only one type of contract. The default can occur only at maturity. Their framework is of the Merton type, and they can therefore obtain closed-form formulae which permit to adjust the different parameters involved in a fair contract. Nevertheless, this model can be considered as a prototype in the valuation of life insurance contract.

Miltersen and Persson [2003] propose a multi-period extension and also provide closed form formulae. Bacinello [2001] analyzes the most sold life insurance contract in Italy. She takes into account mortality and suggests a contract which offers the choice among different triplets of technical rate, participation level and volatility. Paying each year a premium, the insured customer gets the guarantee to recover his initial investment accrued at a fixed rate and can possibly benefit from a bonus indexed on a reference portfolio. The pricing is achieved under the standard Black and Scholes model and assuming independence between mortality risk and financial risk.

Taaksanen and Lukkarinen [2003] consider general participating life insurance contracts. Their contract values depend on the evolution of a reference portfolio at different dates. These authors incorporate the following features: minimum interest rate guaranteed each year, right to change each year the reference portfolio, as well as possibility to surrender each year the contract - giving it a Bermudian aspect. They work with constant interest rates and a constant volatility.

Because there are various kinds of contracts and modeling frameworks, the pricing methodologies are diverse. In fact, mortality, a stochastic interest rate environment and stochastic volatilities, for instance, can be taken into account.
as well as the right to sell back the contract. Participating policies are also multiple. It must be noted that closed form solutions are obtained in the simple Black and Scholes setting. Tanskanen and Lukkarinen [2003] use a numerical procedure to solve their partial differential equation in order to compute the surrender option.

Jørgensen [2001] and Grosen and Jørgensen [2002] show that a life insurance contract with a minimum interest rate guarantee can be expressed in four terms, the final guarantee (equivalent to a zero-coupon bond), the European bonus option associated with a percentage of the positive performance of the company’s asset portfolio, if any, a put option linked to the default risk, and finally a fourth term which is a rebate given to the policyholders in case of default prior to the maturity date.

In Grosen and Jørgensen [1997], the possibility of an early payment is envisaged. To treat this American-style contract they use a binomial lattice whereas Jensen, Jørgensen and Grosen [2001] use a finite difference approach. Grosen et Jørgensen [2002] take into account a default barrier of an exponential type. They obtain closed form formulae in the case of a constant interest rate. Jørgensen [2001] extends this study to the more difficult case of stochastic interest rates, using a Monte Carlo approach.

This study is devoted to the valuation of life insurance contracts in the presence of a stochastic term structure of interest rates, it also takes into account the company’s default risk. We provide an alternative method to trees, numerical solutions of PDE and Monte-Carlo simulations, schemes usually used to price such contracts. The term structure of interest rates considered here stems from the classical Heath Jarrow Morton [1992] framework. Amongst the two standard choices of zero-coupon volatilities making the instantaneous risk-free rate Markovian - linear volatility as in the Ho and Lee model or exponential volatility as in the Hull and White model - we take the second one. Our model is therefore a Vasicek one. Note that we could have considered in our paper a full Hull and White or generalized Vasicek framework by relying on a purely exogenously specified (by a set of zero-coupons) initial term structure of interest rates. The extension of our computations to a Ho and Lee choice of zero-coupon volatility is also straightforward. Our valuation method relies on Collin-Dufresne and Goldstein’s [2001] article which is an outgrowth of Fortet’s [1943] algorithm used by Longstaff and Schwartz [1995] to approximate the first passage time density to a given level by a log-normal process.

Firstly, we give the general setting of our model. Then we explicit the adopted methodology, and finally we present some numerical applications giving the market price of our life insurance contract and we explain how to choose the parameters leading to a fair value contract.

1 Framework

We wish to study how to price a participating life insurance contract with a minimum guaranteed rate in presence of default risk of the issuing company.
We begin by the definition of the contract and the default process, before moving on to modeling the interest rate.

1.1 Contract and Default Model

We consider an insurance company with two types of agents: policyholders and shareholders. The policyholders possess the same unique contract which will be defined precisely in the following. The considered life-insurance company has no debt and its planning horizon is finite with $T$ as maturity, being also the expiry date of the contract. Let $A_0$ be the assets initial value, $L_0 = \alpha A_0$ the initial investment by policyholders, and $E_0 = (1 - \alpha)A_0$ is the initial equity.

The policyholder is guaranteed a fixed interest rate $r_g$. So, the guaranteed amount at $T$ is a priori $L^g_T = L_0 e^{r_g T}$. However, when the firm defaults, this amount will be lowered, on the contrary it will be raised if exceptional results of the company occur. The next step is to express these payments according to the firm’s assets dynamics. We refer to a continuous time economy with a perfect financial market into which our life insurance company is included.

Payment at maturity

Let us look at what happens at $T$: if $A_T \geq L^g_T$, the company is able to fulfill its commitments, otherwise $A_T < L^g_T$ and it is insolvent. In this case, policyholders receive $A_T$ and equityholders nothing. Because we assume a participating policy, when the assets generate value such that $A_T > L^g_T/\alpha$ with $\alpha < 1$, the policyholder is given a bonus, say $\delta$, a contractual part of the surplus, this proportion is known as the participation coefficient. To sum up, policyholders receive at $T$, assuming no prior bankruptcy:

$$\Theta_L(T) = \begin{cases} 
A_T & \text{if } A_T < L^g_T \\
L^g_T & \text{if } L^g_T \leq A_T \leq \frac{L^g_T}{\alpha} \\
L^g_T + \delta(\alpha A_T - L^g_T) & \text{if } A_T > \frac{L^g_T}{\alpha}
\end{cases}$$

In this paragraph we have mimicked the Merton [1974] default approach. We can rewrite the payoff in a more concise form:

$$\Theta_L(T) = L^g_T + \delta(\alpha A_T - L^g_T)^+ - (L^g_T - A_T)^+ \quad (1)$$

The first term is the promised amount, the second term - called "Bonus Option" - is linked to the participating clause, the third one is a put option associated with the default risk.

These last payoffs share the same features as usual European options. According to our fundamental hypothesis and assuming that the assets dynamics follows a Brownian geometric motion it is easy to price them. For more details and closed form solutions, we refer to Briys and de Varenne [1994].
Company Early Default

Now we consider that default can occur prior to the maturity $T$. The default mechanism we choose is of a structural type, so we introduce an activating barrier on the firm’s assets. From now on, bankruptcy can occur at any time $t$ before $T$. The contract value depends on the assets price until contract expiry and not only on their price at $T$. The barrier is chosen exponential and is denoted by $B_t$.

The firm pursues its activities until $T$ if:

$$\forall t \in [0, T], \quad A_t > \lambda L_0 e^{rs t} \triangleq B_t$$

(2)

If it is not so, it is declared bankrupt. Let $\tau$ be the default time; it is the first time when $A_t$ hits the barrier $B_t$, otherwise stated:

$$\tau = \inf \{ t \in [0, T] \mid A_t < B_t \}$$

(3)

With $\lambda$ greater than 1, the firm is able, even when going bankrupt, to pay back policyholders their investments accrued at the guaranteed rate $r_g$. The residual capital (equal to $(\lambda - 1)L_0 e^{rs \tau}$) can be used to pay bankruptcy costs or can be distributed to shareholders. The situation $\lambda \geq 1$ is therefore very favorable to policyholders and regulators. Theoretically it is a risk free position. On the contrary, in the case when $\lambda < 1$, the firm is totally insolvent in the case of bankruptcy and unable to meet its commitments.

So, policyholder will receive in case of early default:

$$\Theta_L(\tau) = \begin{cases} 
L_0 e^{rs \tau} & \text{si } \lambda \geq 1 \\
\lambda L_0 e^{rs \tau} & \text{si } \lambda < 1
\end{cases}
= \min(\lambda, 1)L_0 e^{rs \tau} = \min(\lambda, 1)L_0^\tau$$

(4)

Contract value

Using the standard machinery of arbitrage theory in continuous time and denoting by $Q$ the risk-neutral probability measure, the arbitrage free price of our life insurance contract (hereafter Lic), writes at $t$:

$$V_L(t) = \mathbb{E}_Q^t \left[ e^{-\int_t^T r_s ds} \left[ L_T^g + \delta (\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \right] \mathbb{1}_{\tau \geq T} \\
+ e^{-\int_t^\tau r_s ds} \min(\lambda, 1)L_0^\tau \mathbb{1}_{\tau < T} \right]$$

(5)

This contract can be split up into four simpler subcontracts:

$$V_L(t) = GF_t + B\hat{O}_t - P\hat{O}_t + LR_t$$

(6)

where $GF$ corresponds to the final guarantee, $B\hat{O}$ stands for the "bonus option", $P\hat{O}$ for the default put on which policyholders are short, and, at last, $LR$ is the rebate paid to policyholders in case of early default. Individually these four subcontracts can be written as:
\[
\begin{align*}
\hat{G}t &= \mathbb{E}_Q^t[e^{-\int_t^T r_s ds} 1_{\tau \geq T} L_T^\theta ] \\
\hat{B}O_t &= \mathbb{E}_Q^t[e^{-\int_t^T r_s ds} 1_{\tau \geq T} \delta (\alpha A_T - L_T^\theta )^+] \\
\hat{P}O_t &= \mathbb{E}_Q^t[e^{-\int_t^T r_s ds} 1_{\tau \geq T} (L_T^\theta - A_T)^+] \\
\hat{L}R_t &= \mathbb{E}_Q^t[e^{-\int_t^T r_s ds} 1_{\tau < T} \min(\lambda, 1) L_T^\theta ]
\end{align*}
\]

Note that closed form formulae are available with constant interest rates (see Grosen and Jørgensen [2002]). Our aim in this article is to value our Lic in a reasonably sound stochastic interest rate environment. Of course this problem is rather complex and will lead us to semi-closed formulae. Let us now turn back to the term structure of interest rate.

1.2 Assets Dynamics and Interest Rate Modeling

The most efficient way to price options in a stochastic interest rates environment is to use the change of numéraire technique and to choose an ad hoc zero-coupon bond as new numéraire. So, forward-neutral probability measures technically play a key role. We need to know the T-forward-neutral assets dynamics as well as the dynamics of a default free zero-coupon bond with expiry date T. We denote by \( P(t, T) \) its price at current time \( t \). We assume that the assets price follows a geometric Brownian motion in the risk-neutral universe and we use a one factor Heath, Jarrow and Morton [1992] interest rate model with a deterministic volatility for the T-Zero-coupon bond of an exponential type (this is the Hull and White choice). With \( \nu > 0 \) and \( \alpha > 0 \),the volatility structure then writes

\[
\sigma_P(t, T) = \frac{\nu}{\alpha} \left( 1 - e^{-\alpha(T-t)} \right)
\]

In this case, the dynamics of the instantaneous interest rate \( r \) under the forward-neutral probability \( Q_T \) can be written like:

\[
dr_t = \alpha(\theta_t - r_t)dt + \nu dZ^{Q_T}(t)
\]

where \( \theta_t = \theta - \frac{\nu^2}{2\alpha} \left( 1 - e^{-\alpha(T-t)} \right) \).

Under the risk-neutral probability measure \( Q \), the assets value, \( A_t \), and the zero-coupon bond price with expiry date \( T \), \( P(t, T) \), follow the stochastic diffusions

\[
\frac{dA_t}{A_t} = r_t dt + \sigma dZ^Q(t)
\]

and

\[
\frac{dP(t, T)}{P(t, T)} = r_t dt - \sigma_P(t, T) dZ^Q(t)
\]
where $Z^Q(t)$ and $Z^Q_1(t)$ are $Q$-standard Brownian motions. Let $\rho$ be the correlation coefficient between these two Brownian movements ($dZ^Q dZ^Q_1 = \rho dt$).

Let us now consider a Brownian motion $Z^Q_2$ independent from $Z^Q_1$ (such that $dZ^Q_1 dZ^Q_2 = 0$); the Brownian motion $Z^Q$ can be expressed as

$$dZ^Q(t) = \rho dZ^Q_1(t) + \sqrt{1 - \rho^2} dZ^Q_2(t)$$

In this way we decorrelate the interest rate risk from the firm assets risk.

The assets dynamics (10) then writes:

$$\frac{dA_t}{A_t} = r_t dt + \sigma \rho dZ^Q_1(t) + \sigma \sqrt{1 - \rho^2} dZ^Q_2(t)$$

Let us now denote by $Q_T$ the $T$-forward-neutral measure. It is defined through its Radon-Nikodym derivative

$$\frac{dQ_T}{dQ} = e^{-\int_0^T \sigma_P(s,T) dZ^Q_1(s) - \frac{1}{2} \int_0^T \sigma_P^2(s,T) ds}$$

From Girsanov theorem the process $Z^Q_1$ defined by $dZ^Q_1 = dZ^Q + \sigma_P(t,T) dt$ is a $Q_T$-Brownian motion. The process $Z^Q_2$ is then built such that $Z^Q_1$ and $Z^Q_2$ are $Q_T$-non correlated standard Brownian motions. Under $Q_T$ the prices $P(t,T)$ and $A_t$ follow the stochastic differential equations

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T)) dt - \sigma_P(t,T) dZ^Q_1$$

and

$$\frac{dA_t}{A_t} = (r_t - \sigma \rho \sigma_P(t,T)) dt + \sigma \left( \rho dZ^Q_1 + \sqrt{1 - \rho^2} dZ^Q_2 \right)$$

After integration, one obtains

$$A_t = \frac{A_0}{P(0,t)} \exp \left( \int_0^t (\sigma_P(u,t) + \rho \sigma) dZ^Q_1(u) + \int_0^t \sigma \sqrt{1 - \rho^2} dZ^Q_2(u) + \int_0^t \left( -\sigma_P(u,T) (\sigma_P(u,t) + \rho \sigma) + \frac{\sigma_P^2(u,t) - \sigma^2}{2} \right) du \right)$$

This formula will be useful to simulate the process $A_t$ as well as to study the moments of $\ln(A_T)$; we shall see next that it is a prerequisite to solve our problem.
1.3 The Valuation

We now present the valuation of our Lic under the setting defined above. For the sake of simplicity, we set the current time to zero \((t = 0)\). Using the fact that the relative prices are martingale under the T-forward-neutral equivalent martingale measure, we can rewrite formula (5) according as:

\[
V_L(0) = P(0, T) \mathbb{E}_{Q_T} \left[ \left( L_T^q + \delta(A_T - L_T^q) + (L_T^q - A_T)^+ - L_T^q \right) \mathbb{1}_{\tau \geq T} + e^{\int_0^\tau r_s ds} \min(\lambda, 1) L_T^q \mathbb{1}_{\tau < T} \right]
\]

Using the relation \(\mathbb{1}_{\tau \geq T} = 1 - \mathbb{1}_{\tau < T}\), the expression of the subcontracts in (7) lead in a very simple way in the T-forward-neutral-universe to:

\[
V_L(0) = P(0, T) \left( GF + BO - PO + LR \right)
\]

where

\[
\begin{align*}
GF & = L_T^q (1 - E_1) \\
BO & = \alpha \delta(E_7 - E_2) - \delta L_T^q (E_8 - E_3) \\
PO & = L_T^q (E_9 - E_4) - E_{10} + E_5 \\
LR & = \min(\lambda, 1) L_0 E_6
\end{align*}
\]

and where we introduce the following quantities

\[
\begin{align*}
E_1 &= Q_T [\tau < T] & E_6 &= E_{Q_T} \left[ e^{\int_0^\tau r_s ds} e^{r \tau} \mathbb{1}_{\tau < T} \right] \\
E_2 &= E_{Q_T} \left[ A_T \mathbb{1}_{\left\{ A_T > \frac{L_T^q}{\alpha} , \tau < T \right\}} \right] & E_7 &= E_{Q_T} \left[ A_T \mathbb{1}_{A_T > \frac{L_T^q}{\alpha}} \right] \\
E_3 &= Q_T \left[ A_T > \frac{L_T^q}{\alpha} , \tau < T \right] & E_8 &= Q_T \left[ A_T > \frac{L_T^q}{\alpha} \right] \\
E_4 &= Q_T [A_T < L_T^q , \tau < T] & E_9 &= Q_T[A_T < L_T^q] \\
E_5 &= E_{Q_T} \left[ A_T \mathbb{1}_{A_T < L_T^q} \mathbb{1}_{\tau < T} \right] & E_{10} &= E_{Q_T} \left[ A_T \mathbb{1}_{A_T < L_T^q} \right]
\end{align*}
\]

In the next section, we show how to compute these expressions. The ones in which the default time \(\tau\) does not intervene lead to closed form formulae. For the others, as far as we know, closed form formulae are not available, hence, in order to compute them, we use an approximation of the distribution of \(\tau\). This is the object of the following paragraph and constitutes the core of our pricing methodology.
2 Valuation Methodology

To price our Lic we need to compute each expectation $E_i$ in (16). We have to know the law of the default time $\tau$ - first passage time of the lognormal process of the assets $A_t$ at the exponential barrier, given in (2).

Longstaff and Schwartz [1995] use Fortet’s result [1943] to approximate the law of $\tau$ in a similar problem to ours: the pricing of defaultable bonds and defaultable floating rate notes. However, the Longstaff et Schwartz [1995] approximation is not satisfactory and mathematically not valid. Collin-Dufresne and Goldstein [2001] have given a correction to the previous approximation which validates the method for problems of the kind we encounter. We call this corrected method the extended Fortet’s method. It is the key solution to the pricing of our Lic in this article; let us now explain this method.

Firstly, adopt the following convention: $l_t = \ln(\chi_t) = \ln(A_t) - r_g t$. For this process, the default barrier then becomes $h = \ln(\lambda L_0)$; we assume it is below $l_0$, the initial value of the process under study. Besides, it can be shown that the process $l_t$ obeys under $Q_T$ the following stochastic differential equation (applying Itô’s lemma to equation (12)):

$$dl_t = \left( r_t - r_g - \frac{\sigma^2}{2} - \sigma \rho \sigma P(t, T) \right) dt + \sigma \rho dZ_1^{Q_T} + \sigma \sqrt{1 - \rho^2} dZ_2^{Q_T}$$

So, we have to study the first passage time of $l_t$ to the constant level $h$, put more explicitly

$$\tau = \inf\{t \in [0, T] \mid l_t \leq h\}.$$

In order to compute the expectations in formula (14), we choose to approximate the law of $\tau$.

Next, we need to compute the probability of the event $\tau \in [t_j, t_{j+1}]$ when the interest rate is between $r_i$ and $r_{i+1}$. We denote this density by

$$p( r_i, t_j ) \quad j = 0 \ldots n_T - 1 \quad i = 0 \ldots n_r.$$

We give a recursive approximation of the density of $\tau$ as a piecewise constant function on $[t_j, t_{j+1}]$ when the interest rate is between $r_i$ and $r_{i+1}$. We denote this density by

$$q(i, j) = \delta_t \delta_r p( r_i, t_j )$$
Let \( f(l_t, r_t, t \mid l_s = a, r_s = r, s) \) be the conditional law of \((l_t, r_t)\) given \(\{l_s = a, r_s = r\}\).

Define respectively \(\Phi\), \(\Psi\) and \(g\) by:

\[
\Phi(r_t, t) = \int_{-\infty}^{h} f(l_t, r_t, t \mid l_0, r_0, 0)dl_t
\]

\[
\Psi(r_t, t \mid r_s, s) = \int_{-\infty}^{h} f(l_t, r_t, t \mid l_s = h, r_s, s)dl_t
\]

\[
g(r_s, s) = p_r(l_s = h, r_s, s \mid l_0, r_0, 0)
\]

It can be shown (for further details, see Collin-Dufresne and Goldstein) that:

\[
q(i, 1) = \Phi(r_i, t_1) = \sum_{u=0}^{n_r} q(u, 1) \Psi(r_i, t_1 \mid r_u, t_1)
\]

The quantities \(q(i, 1)\) are computed for every \(i\); from them the quantities \(q(i, j)\) for \(j \geq 2\) are recursively obtained:

\[
q(i,j) = \Phi(r_i, t_j) - \sum_{v=1}^{j-1} \sum_{u=0}^{n_r} q(u, v) \Psi(r_i, t_j \mid r_u, t_v)
\]  \hspace{1cm} (17)

To calculate \(q(i,j)\), the expressions \(\Phi(r_t, t)\) and \(\Psi(r_t, t \mid r_s, s)\) are needed. Since the processes \(l_t\) and \(r_t\) are Gaussian, the conditional law of \(l_t\) given the \(\sigma\)-tribe generated by the information available at time \(s\) and given \(r_t\), is Gaussian, with mean \(\mu(r_t, l_s, r_s)\) and variance \(\Sigma^2(r_t, l_s, r_s)\). The computations and results are given in the appendix as well as the centered moments of order 1 and 2 of the processes \(l_t\) et \(r_t\).

Let us denote, as usual, by \(N\) the cumulative distribution function of the standard normal law. Using the previous Gaussian conditional law and the Bayes’ rule we obtain

\[
\Phi(r_t, t) = f_r(r_t, t \mid l_0, r_0, 0) N\left(\frac{h - \mu(r_t, l_0, r_0)}{\sqrt{\Sigma^2(r_t, l_0, r_0)}}\right)
\]

\[
\Psi(r_t, t \mid r_s, s) = f_r(r_t, t \mid l_s = h, r_s, s) N\left(\frac{h - \mu(r_t, l_s = h, r_s)}{\sqrt{\Sigma^2(r_t, l_s = h, r_s)}}\right)
\]

where we have an explicit formula for the transition density \(f_r\) of \(r\) (which is a Gaussian process):

\[
f_r(r_t, t \mid l_s = h, r_s, s) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(r_t - m)^2}{2v}}
\]
where \( m = \mathbb{E}[r_t | r_s] \) and \( v = \text{Var}[r_t | r_s] \) respectively stand for the conditional moments of \( r_t \) given \( r_s \). They are also provided in the appendix.

To sum up, we have now, with formula (17) the possibility to compute the \( q(i, j) \) terms, which give us the density of \( \tau \) we were looking for. Now we are equipped to obtain our expectations.

\[
Q_T [\tau < T] = \int_{0}^{+\infty} \int_{-\infty}^{0} p_r(l_s = h, r_s, s | l_0, r_0, 0)dr_sds
\]

Figure 1: Empirical and extended Fortet’s approximate density

Graph 1 illustrates the fact that this corrected method gives an approximated density \( p_j = \sum_i q(i, j) \) which adjusts satisfactorily the empirical density of \( \tau \) (obtained here by Monte Carlo simulation). The extended Fortet’s method is, of course, longer to use than the ordinary Fortet’s method, because of the double discretization; however it is far less time consuming than Monte Carlo simulations.

2.1 The Quasi-closed Form Formula for the Lic

At present, we have to apply our method to compute the expectations in (16) in order to get \( V_L(0) \). Each term involving \( \tau \) is computed using the extended Fortet’s method.

For this goal, we need to know precisely the moments of \( l_t \) (the formulae in (16), which are expressed as functions of \( A \), can indeed be rewritten as functions of \( l \)) and the moments of \( r_t \) and also the conditional moments of \( l_t \) given \( r_t \). These calculations are provided in the appendix.

Let us begin with the computation of \( E_l \). From its definition, it writes in the following integral form:

\[
Q_T [\tau < T] = \int_{0}^{+\infty} \int_{-\infty}^{0} p_r(l_s = h, r_s, s | l_0, r_0, 0)dr_sds
\]
We then discretize according to time and rate and replace the exact density \( p_\tau \) of \( \tau \) by its approximation \( q(i,j) \):

\[
E_1 = \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} q(i,j).
\]

We also detail the computation of \( E_2 \), the other approximated \( E_i \) will be obtained in a similar manner.

\[
E_2 = \mathbb{E}_{Q_T} \left[ A_T e^{-r_{rT}} e^{r_{sT}} \mathbb{1}_{\{A_T e^{-r_{sT}} > \frac{L_0}{\alpha}, \tau < T\}} \right]
\]

\[
= \mathbb{E}_{Q_T} \left[ \chi_T e^{r_{sT}} \mathbb{1}_{\tau > \frac{L_0}{\alpha}} \mathbb{1}_{\tau < T} \right]
\]

\[
= e^{r_{sT}} \mathbb{E}_{Q_T} \left[ e^{r_{T}} \mathbb{1}_{l_T > \ln(\frac{L_0}{\alpha})} \mathbb{1}_{\tau < T} \right].
\]

Conditioning is the key tool:

\[
E_2 = e^{r_{sT}} \int_0^T ds \int_{-\infty}^{+\infty} dr_s \ g(r_s, s) \mathbb{E}_{Q_T} \left[ e^{r_{T}} \mathbb{1}_{l_T > \ln(\frac{L_0}{\alpha})} | r_s \right]
\]

In this last formula, the expectation only concerns \( l_T \). But we do not know the density of \( l_T \), we only know the conditional law of \( l_T \) given \( r_T \), and the transition density of an Ornstein-Uhlenbeck process, denoted by \( f_r \). Therefore:

\[
E_2 = e^{r_{sT}} \int_0^T ds \int_{-\infty}^{+\infty} dr_s \ g(r_s, s) \int_{-\infty}^{+\infty} dr_T \ f_r(r_T | r_s, s, l_s) \mathbb{E}_{Q_T} \left[ e^{r_{T}} \mathbb{1}_{l_T > \ln(\frac{L_0}{\alpha})} | r_T, F_s \right]
\]

The law of \( l_T \) conditional on \( F_s \) and given \( r_T \) is Gaussian ; its first two centered moments are \( \hat{\mu}_{s,T} = \mu(r_T, l_s, r_s) \) and \( \hat{\Sigma}_{s,T}^2 = \Sigma^2(r_T, l_s, r_s) \).

Let \( X \) be the Gaussian random variable \( \mathcal{N}(m, \sigma^2) \), we define:

\[
\Phi_1(m; \sigma; a) = \mathbb{E}[e^{X} \mathbb{1}_{e^X > a}] = \exp \left( m + \frac{\sigma^2}{2} \right) \mathcal{N} \left( \frac{m + \sigma^2 - \ln(a)}{\sigma} \right)
\]

The expectation \( E_2 \) can be rewritten as:

\[
E_2 = e^{r_{sT}} \int_0^T ds \int_{-\infty}^{+\infty} dr_s \ g(r_s, s) \int_{-\infty}^{+\infty} dr_T \ f_r(r_T | r_s, s, l_s) \Phi_1 \left( \hat{\mu}_{s,T}; \hat{\Sigma}_{s,T}; \frac{L_0}{\alpha} \right)
\]

Then, the extended Fortet’s approximation for \( E_2 \) is:

\[
E_2 = e^{r_{sT}} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k | r_{i}, t_{j}, l_{t_j}) \Phi_1 \left( \hat{\mu}_{t_j,T}; \hat{\Sigma}_{t_j,T}; \frac{L_0}{\alpha} \right) q(i,j)
\]

With the same scheme, we give the formulae for the others \( E_i \) given in (16).
It can be shown that

\[ E_3 = e^{r_s T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k \mid r_i, t_j, l_t) \mathcal{N} \left( \frac{\hat{\mu}_{ij,T} - \ln \left( \frac{L_0}{\alpha} \right)}{\sqrt{\hat{\Sigma}_{ij,T}^2}} \right) q(i, j) \]

and

\[ E_4 = e^{r_s T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k \mid r_i, t_j, l_t) \mathcal{N} \left( \frac{\ln (L_0) - \hat{\mu}_{ij,T}}{\sqrt{\hat{\Sigma}_{ij,T}^2}} \right) q(i, j) \]

For the computation of \( E_5 \), we define

\[ \Phi_2(m; \sigma; a) = \mathbb{E} \left[ e^{X \mathbf{1}_{X < a}} \right] = \exp \left( m + \frac{\sigma^2}{2} \right) \mathcal{N} \left( \frac{\ln(a) - m - \sigma^2}{\sigma} \right) \]

where \( X \) is the Gaussian \( \mathcal{N}(m, \sigma^2) \).

We then obtain

\[ E_5 = e^{r_s T} \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} \sum_{k=0}^{n_r} \delta_r f_r(r_k \mid r_i, t_j, l_t) \Phi_2 \left( \hat{\mu}_{ij,T}; \hat{\Sigma}_{ij,T}; L_0 \right) q(i, j) \]

and

\[ E_6 = \sum_{j=1}^{n_T} \sum_{i=0}^{n_r} e^{r_s t_j} \mathbb{E}_{Q_x} \left[ e^{f_{ij, r, \alpha, u} \mid r_{t_j} = r_i, t_j, l_t} \right] q(i, j) \]

In expectations \( E_7, E_8, E_9, \) and \( E_{10} \), the random time \( \tau \) does not intervene. Furthermore the random variable \( \chi_T \) is lognormal with moments \( M_T \) and \( V_T \) (computed in the appendix). Hence, explicit formulae for the last four expectations can be obtained.

Indeed, applying the properties associated with the functions \( \Phi_1 \) et \( \Phi_2 \), we obtain

\[ E_7 = e^{r_s T} \Phi_1 (M_T; \sqrt{V_T}; \frac{L_0}{\alpha}) \quad E_8 = \mathcal{N} \left( \frac{M_T - \ln \left( \frac{L_0}{\alpha} \right)}{\sqrt{V_T}} \right) \]

\[ E_9 = \mathcal{N} \left( \frac{\ln (L_0) - M_T}{\sqrt{V_T}} \right) \quad E_{10} = e^{r_s T} \Phi_2 (M_T; \sqrt{V_T}; L_0) \]

(18)

To sum up, in order to compute the different \( E_i \), we need to know \( \Phi_1, \Phi_2, f_r, \) and the different moments given above which are explicited in the appendix, as well as the probabilities \( q(i, j) \). To accurately obtained the desired results, it is sufficient to use a grid with a thin mesh, which can be done with \( n_T \) and \( n_r \) large enough.
3 Numerical Analysis

In this section we make a numerical analysis on our Lic. Two parameters will happen to play a key role: the guaranteed rate \( r_g \) and the participating coefficient \( \delta \).

The parameters \( r_g \) and \( \delta \) cannot be fixed arbitrarily. The guaranteed rate must be neither too high (bankruptcy risk would be too much important in case of falling interest rates), nor too low (unfavorable contract to policyholders). Besides, it cannot go beyond a legal threshold limit. In France, this threshold is typically around 2.75\%. As far as the participating level is concerned, it is calculated such that Lics be fair to both sides. The participating level necessarily varies contrary to the guaranteed rate: the higher the former the lower the latter and reciprocally.

There are infinitely many couples \((\delta, r_g)\) leading to a fair contract. These parameters depend, of course, on the company’s investment policy. That is to say, in our model, they depend on the assets volatility \( \sigma \) and on the default barrier level \( \lambda \). However, all these contracts are not acceptable, \( \delta \) must be between 0 and 1. Besides, the participating coefficient must obey legal constraints; for example, \( \delta \) must be greater than 85\% in France (cf. Briys and de Varenne [1997b]).

As a first step, we recapitulate the values we choose for the parameters involved in our study. We then turn to the numerical valuation of our Lics and make a comparison of the extended Fortet’s method and Monte Carlo simulations. We also show how to calculate the participating level. Finally, we conclude this section by a sensitivity analysis of the contact price to the assets volatility.

3.1 Data

We give in table 1 the chosen parameters values. Some will be changed after, in particular volatility \( \sigma \) and barrier level \( \lambda \).

<table>
<thead>
<tr>
<th>( A_0 )</th>
<th>( a )</th>
<th>( \nu )</th>
<th>( \theta )</th>
<th>( r_0 )</th>
<th>( \rho )</th>
<th>( \sigma )</th>
<th>( T )</th>
<th>( \lambda )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.4</td>
<td>0.008</td>
<td>0.06</td>
<td>0.03</td>
<td>-0.02</td>
<td>0.1</td>
<td>10</td>
<td>0.8</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 1: Data

Recall that \( A_0 \) stands for the initial assets value of our company, \( a, \nu, \theta \) and \( r_0 \) determine the instantaneous interest rate process, and \( \rho \) is the correlation coefficient between the assets process and the instantaneous interest rate process. The small value for \( \sigma \), which is set to 10\%, corresponds to a standard investment approximately half in stocks and half in bonds by the life insurance company. At last, the maturity contract \( T \) is set at 10 years to begin, and \( \alpha \) is the initial proportion of investment by the insured on the total liabilities of the firm.
3.2 Numerical Results

We now examine in the following the numerical results we could obtain for the contract value and the fair participating level.

Contract Valuation

Tables 2 and 3 display the Lic contract and subcontracts numerical estimations, done with the extended Fortet and Monte-Carlo methods respectively, using the parameters defined in the previous subsection and taking $r_g = 2.6\%$ and $\delta = 89.8\%$. Five million sample paths have been used in Monte-Carlo simulations for each valuation.

The first remark we must emphasize on is that the extended Fortet method is by far faster than the Monte-Carlo method. Ten minutes of computation time is not instantaneous (as is the case with a closed form formula) but is extremely efficient in the numerical valuation of a complex contract submitted to both interest rate risk and default risk.

<table>
<thead>
<tr>
<th>Extended Fortet</th>
<th>GF</th>
<th>BO</th>
<th>PO</th>
<th>LR</th>
<th>Contract</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_T = 200$, $n_F = 50$</td>
<td>28.11</td>
<td>89.05</td>
<td>0.09</td>
<td>1.27</td>
<td>60.9967</td>
<td>2 min</td>
</tr>
<tr>
<td>$n_T = 500$, $n_F = 50$</td>
<td>28.11</td>
<td>89.03</td>
<td>0.09</td>
<td>1.29</td>
<td>69.9996</td>
<td>10 min</td>
</tr>
</tbody>
</table>

Table 2: Contract and subcontracts values

Furthermore, we observe rather rapidly a convergence for the contract and subcontracts prices when using the extended Fortet’s method, while Monte-Carlo converges poorly for some subcontracts such that the default put $PO$. Hence to obtain a sufficient precision with Monte-Carlo, it would be necessary to launch simulations lasting many days, which is unacceptable for practical use.

<table>
<thead>
<tr>
<th>Monte-Carlo</th>
<th>GF</th>
<th>BO</th>
<th>PO</th>
<th>LR</th>
<th>Contract</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$step = 1/12$</td>
<td>28.10</td>
<td>89.28</td>
<td>0.14</td>
<td>1.30</td>
<td>70.1108</td>
<td>15 min</td>
</tr>
<tr>
<td>$step = 1/52$</td>
<td>28.11</td>
<td>89.14</td>
<td>0.13</td>
<td>1.31</td>
<td>70.0451</td>
<td>1 h 20 min</td>
</tr>
<tr>
<td>$step = 1/365$</td>
<td>28.14</td>
<td>89.07</td>
<td>0.13</td>
<td>1.30</td>
<td>70.0201</td>
<td>1 day</td>
</tr>
</tbody>
</table>

Table 3: Contract and subcontracts values

Our numerical experiments show, as confirmed by Monte-Carlo simulations, that the prices obtained with the extended Fortet’s method are reliable, and in a quite short computation time. The contract fair value is 70 and the extended Fortet’s method provides an accuracy of 3 digits in ten minutes. On the contrary, the Monte-Carlo method is very slow in converging: indeed our path-dependent problem requires a very fine discretization (many time steps) for each - amongst many - sample path. The implementation of both methods has been done with an extensive use of Matlab vectorization tools, on a 3 GHz...
Computation of the Participating Level

We are looking for participating levels fair to both policyholders and the company. This will be done under the following equilibrium condition: a contract is said to be fair if the policyholders’ initial investment $L_0 = \alpha A_0$ is equal to the total value of subscribed contracts.

We present in figures 2 and 3 the contract value as a function of $\delta$ and the guaranteed rate $r_g$. Note that the level 70 corresponds to the $L_0$ value. Figure 2 is obtained with a guaranteed rate set at 2.6%. The higher the participating level, the higher the contract value. Let us note that only one value of $\delta$ corresponds to a fair value contract, which is the initial investment $L_0$. Figure 3 is obtained with a participating level set to $\delta = 0.898$, and represents the contract value as a function of the guaranteed rate. Here again, only one value of $r_g$
leads to a fair contract.

Now, let us explain the employed procedure. If one wants to determine the participating coefficient with a given guaranteed rate, one has to compute:

$$\delta = \frac{L_0}{P(0,T)} - GF + PO - LR \alpha(E_7 - E_2) - L_T^2(E_8 - E_3)$$

The calculation of the guaranteed rate given the participating coefficient is more difficult. One has to use a root searching algorithm with the constraint that the contract initial value is equal to $L_0$.

### 3.3 Sensitivity to Volatility

We examine now the sensitivity of the participating level $\delta$ - and guaranteed rate $r_g$ - to the assets volatility. These sensitivities are displayed in figure 4 and 5.

**Figure 4:** $r_g$ as a function of $\sigma$

**Figure 5:** $\delta$ as a function of $\sigma$

Figure 4 shows that the weaker the participating level is, the more it is necessary to compensate with a big guaranteed interest rate. On the graph the curves are presented in descending order with respect to $\delta$.

On the opposite, we remark in figure 5 that (fair participating curves are presented in descending order with respect to $r_g$) a low guaranteed rate must be compensated by a high level of the participating coefficient.

Let us examine now the impact of volatility. It is clear from figure 5 that the guaranteed rate begins to fall before moving up as volatility increases. When the volatility $\sigma$ is low, the default risk is negligible, a rising volatility corresponds to a rising return. Given a fixed participating coefficient $\delta$, the guaranteed rate must necessarily decrease to preserve a fair contract. Should $r_g$ remain constant, the contract would be more and more advantageous when $\sigma$ increases. On the contrary, when $\sigma$ is above 10%, the default risk becomes important and
the probability for policyholders to get back their guaranteed investment diminishes; it is then necessary to compensate with a higher guaranteed rate.

At last, let us analyze figure 5. Here again we have a similar behavior. With a fixed guaranteed rate, the participating level begins to decrease before rising, as long as volatility increases. When the volatility is low, in other words, when we can consider that no default risk exists, a volatility rise implies a better return; in order to limit the policyholders advantage, the participating level must decrease. On the contrary, when volatility is high, default risk is important, and necessarily the participating level has to be raised up, given the guaranteed rate, to preserve fairness (policyholders bearing the risk not to recover their initial investment).

Conclusion

In this article we have proposed a new method to value typical participating life insurance contracts, with minimum guaranteed rate, in the presence of default risk, and in a stochastic interest rate environment. We have determined the fair participating level, which is a delicate and important point for a life insurance company. We have also analyzed the sensitivity of the main parameters to volatility.

The suggested method relies on Fortet’s equation [1943] giving the first passage time of the assets process to the default barrier, and consequently paving the way for computing diverse exotic options embedded in the contract involving this random time. This method has been used in Finance for the first time by Longstaff and Schwartz [1995] then by Collin-Dufresne and Goldstein [2001]. These last authors have amended the Longstaff and Schwartz approach extending it in a rigorous way to two dimensional continuous Markov processes. It is this method we used under the name of extended Fortet’s method.

Confronting with Monte-Carlo method, we have proved that the extended Fortet’s method performs very well to value typical life insurance contracts in a rather general context. More than that, the extended Fortet’s method permits to value these contracts in a very fast computing-time, which constitutes certainly a convincing argument for practioners.

Because the fair participating coefficient asks for a root searching algorithm, it is important to have a rapid and efficient method to value Lics. Once again one can perceive the advantage of the proposed method with respect to Monte Carlo simulations routinely used.
References


Moments and Conditional Moments of $\chi_T$

Recall that the process $\chi$ is defined by $\ln(\chi_t) = \ln(A_t) - r_g t$. For a fixed $t$, $\chi_t$ is a log-normal random variable described by its two first centered moments $M_t = \mathbb{E}[\ln(\chi_t)]$ and $V_t = \text{Var}[\ln(\chi_t)]$ that can easily be computed:

$$M_t = \ln \left( \frac{A_0}{P(0,t)} \right) + \int_0^t \left( -\sigma_P(u,T)(\sigma_P(u,t) + \rho \sigma) + \frac{\sigma_P^2(u,t) - \sigma^2}{2} - r_g \right) du.$$

and

$$V_t = \int_0^t (\sigma^2 + \sigma_P^2(u,t) + 2 \rho \sigma \sigma_P(u,t)) du$$

Let us give the moments of $\ln(\chi_t)$ for an exponential volatility structure

$$M_t = \ln \left( \frac{A_0}{P(0,t)} \right) + \nu^2 \frac{\sigma^2}{2a^2} - \frac{\sigma^2}{2} + \frac{\sigma}{a} e^{-at}$$

$$V_t = 2\nu + \frac{\rho \sigma \nu}{a^3} e^{-at} - \frac{\nu^2}{2a^3} e^{-2at} - \frac{3 \nu^2}{2a^3} - \frac{2 \nu^2}{a^3} + \left( \frac{\sigma^2 + 2 \rho \sigma \nu}{a} + \frac{\nu^2}{a^3} \right) t$$

We need to compute the covariance between $\ln(\chi_t)$ and $\ln(\chi_s)$:

$$C(s,t) = \int_0^t (\sigma^2 + \rho \sigma (\sigma_P(u,t) + \sigma_P(u,s)) + \sigma_P(u,s) \sigma_P(u,t)) du$$

In the case of the Hull and White volatility, we obtain (with $s < t$):

$$C(s,t) = -\left( \frac{\rho \sigma \nu}{a^2} + \frac{\nu^2}{a^3} \right) + \left( \frac{\sigma^2 + 2 \rho \sigma \nu}{a^2} + \frac{\nu^2}{a^3} \right) s + \left( \frac{\rho \sigma \nu}{a^2} + \frac{\nu^2}{a^3} \right) e^{-as}$$

$$+ \left( \frac{\rho \sigma \nu}{a^2} + \frac{\nu^2}{a^3} \right) e^{-at} - \left( \frac{\rho \sigma \nu}{a^2} + \frac{\nu^2}{2a^3} \right) e^{-a(t-s)} - \frac{\nu^2}{2a^3} e^{-a(t+s)}$$

Besides, the conditional law of $\ln(\chi_t)$ given $\ln(\chi_s)$ is Gaussian with mean $\hat{M}(s,t)$ and variance $\hat{V}(s,t)$. The conditional moments of $\ln(\chi_t)$ are

$$\hat{M}(s,t) = M_t + \frac{C(s,t)}{V_s} (\ln(\chi_s) - M_s)$$

$$\hat{V}(s,t) = V_t - \frac{C(s,t)^2}{V_s}$$

(19)
Moments of the Processes \( r_t \) and \( l_t \)

We work under the forward-neutral measure. The instantaneous interest rate \( r \) is an Ornstein-Uhlenbeck process. We compute its moments and those of \( l \) associated with the assets process. Define \( B_a \) by:

\[
B_a(u) = \frac{1}{a} \left( 1 - e^{-au} \right)
\]

\( r \) is a Gaussian process, therefore it is possible (after integrating (9)) to compute its two centered conditional moments with respect to the tribe \( \mathcal{F} \) generated by \( r \):

\[
E[ r_t | \mathcal{F}_u ] = e^{-a(t-u)} r_u + \left( \theta a - \frac{\nu^2}{a} \right) B_a(t-u) + \frac{\nu^2}{a} e^{-a(T-t)} B_{2a}(t-u)
\]

and

\[
\text{Var}[ r_t | \mathcal{F}_u ] = \nu^2 B_{2a}(t-u)
\]

and for \( s < t \)

\[
\text{Cov}(r_s, r_t | \mathcal{F}_u ) = \frac{\nu^2}{2a} e^{-a(s+t)} (e^{2as} - e^{2au}) = \nu^2 e^{-a(s-t)} B_{2a}(s-u)
\]

Let us now examine the moments of the process \( l_t = \ln(\chi_t) = \ln(A_t) - r_g t \) obeying the SDE

\[
dl_t = \left( r_t - r_g - \frac{\sigma^2}{2} - \sigma \rho \nu B_a(T-t) \right) dt + \sigma \rho \sqrt{1 - \rho^2} dZ_{1Q_T} + \sigma \sqrt{1 - \rho^2} dZ_{2Q_T} \quad (20)
\]

where \( Z_{1Q_T} \) and \( Z_{2Q_T} \) are two independent Brownian under the \( T \)-forward neutral measure.

We integrate \( l_t \); it can be expressed in terms of \( r_t, Z_{1Q_T}^T \) and \( Z_{2Q_T}^T \). \( l \) is a Gaussian process. after some computations we obtain:

\[
E[ l_t | \mathcal{F}_u ] = l_u - \left( r_g + \frac{\sigma^2}{2} + \frac{\sigma \rho \nu}{a} - \theta + \frac{\nu^2}{a^2} \right) (t-u) - \frac{\nu^2}{a^2} e^{-a(T-t)} B_{2a}(t-u) + \left( r_u - \theta + \frac{\nu^2}{a^2} + \frac{\nu^2}{a^2} e^{-a(T-t)} + \frac{\sigma \rho \nu}{a} e^{-a(T-t)} \right) B_a(t-u)
\]

\[
\text{Var}[ l_t | \mathcal{F}_u ] = \left( \sigma^2 + \frac{\nu^2}{a^2} + 2 \frac{\sigma \rho \nu}{a} \right) (t-u) - 2 \left( \frac{\nu^2}{a^2} + \frac{\sigma \rho \nu}{a} \right) B_a(t-u) + \frac{\nu^2}{a^2} B_{2a}(t-u).
\]

If \( s < t \):

\[
\text{Cov}(l_s, l_t | \mathcal{F}_u ) = \frac{\nu^2}{a^2} e^{-a(t-s)} B_{2a}(s-u) + \left( \sigma^2 + \frac{2 \sigma \rho \nu}{a} + \frac{\nu^2}{a^2} \right) (s-u) - \left( \frac{\nu^2}{a^2} + \frac{\sigma \rho \nu}{a} \right) (e^{-a(t-s)} + 1) B_a(s-u).
\]
Covariances between $l_t$ and $r_t$:

The processes $l_t$ and $r_t$ are correlated through $Z_t^{Q_T}$ and have the conditional covariance

$$\text{Cov}(l_t, r_t | \mathcal{F}_u) = -\frac{\nu^2}{a} B_{2a}(t - u) + \left(\frac{\nu^2}{a} + \rho \sigma \nu\right) B_{a}(t - u).$$

Besides, we need

$$\mu(r_t, l_s, r_s) = \mathbb{E}[l_t | \mathcal{F}_s] + \frac{\text{Cov}(l_t, r_t | \mathcal{F}_s)}{\text{Var}[r_t | \mathcal{F}_s]} (r_t - \mathbb{E}[r_t | \mathcal{F}_s])$$

$$\Sigma^2(r_t, l_s, r_s) = \text{Var}[l_t | \mathcal{F}_s] - \frac{\text{Cov}(l_t, r_t | \mathcal{F}_s)^2}{\text{Var}[r_t | \mathcal{F}_s]}$$

Finally, we also need the following expectation to compute $E_6$:

$$\mathbb{E}\left[\int_u^t r_s ds | \mathcal{F}_u\right] = (r_u - \theta) B_{a}(t - u) + \frac{\nu^2}{a} B_{a}(t - u)^2$$

$$+ \frac{\nu^2}{a^2} e^{(t-u-T)} B_{2a}(t - u) + \left(\theta - \frac{\nu^2}{a^2}\right) (t - u)$$