The Actuarial Profession
making financial sense of the future

Statistical Estimation
Connecting Regulation to Theory

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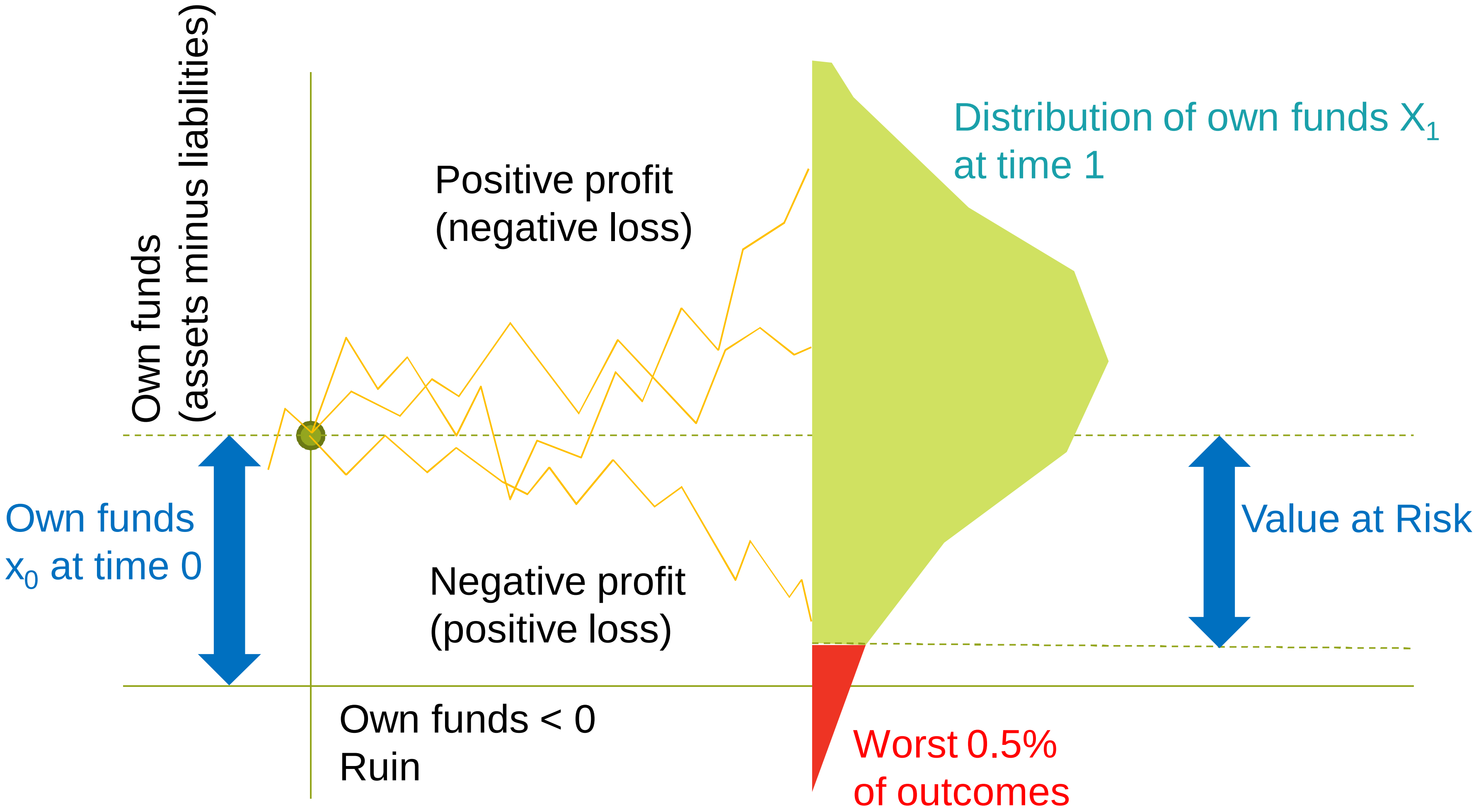
From the “Solvency II Directive” 2009/138/EC

Article 101: Calculation of the Solvency Capital Requirement

1. The Solvency Capital Requirement shall be calculated in accordance with paragraphs 2 to 5.
2. The Solvency Capital Requirement shall be calculated on the presumption that the undertaking will pursue its business as a going concern.
3. The Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses. **It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5 % over a one-year period.**
4. ...

Value at Risk

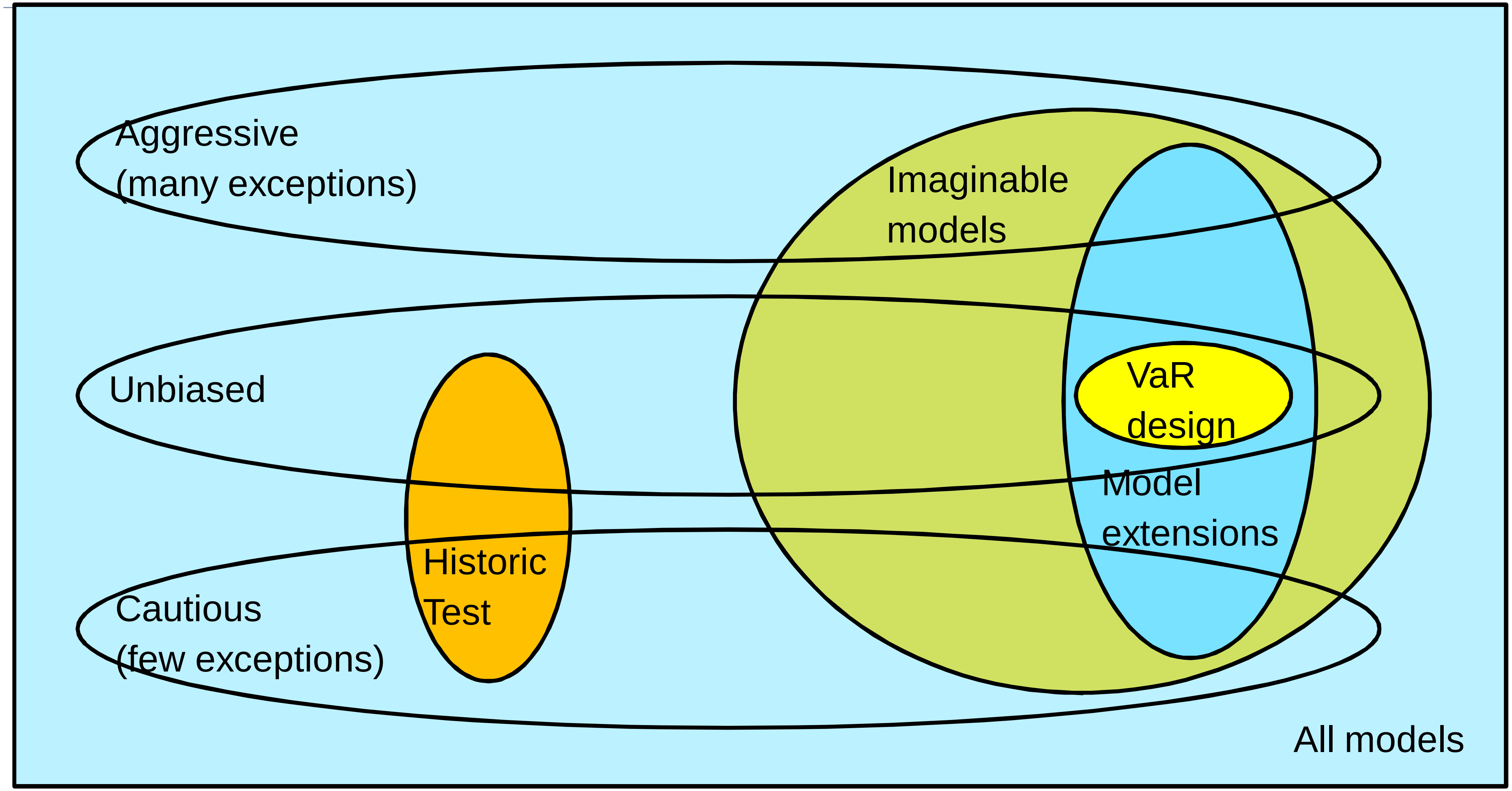
VaR ≤ Own Funds(0) iff (1-α) quantile OF(1) ≥ 0



The standard definition requires knowledge of the "true" distribution:
This is the conceptual problem we address here.

A world without a unique “true” model

Degree of prudence depends on {underlying model, VaR forecast method}



Why there are always aggressive models

Another aspect of the problem of induction

- Given an estimation methodology, there are always some models for which that methodology is aggressive
  - Models where volatility suddenly explodes at the next point
  - Models with discontinuous tail behaviour
  - Nuclear safety example, extrapolating meltdown risk from “slips and falls” operational loss data
- Our proposed approach is to define a “null hypothesis”, H_0, that is, a set of (subjectively) reasonable models for which the forecast methodology is (by design) either unbiased or cautious

Model-Agnostic Quantile Forecasts

Expansions for sums large numbers of iid random variables

Cornish-Fisher expansion (derived from Central Limit Theorem 2nd & 3rd order terms)

$$F^{-1}\{\Phi(z)\} = mean + \left(z + \frac{z^2 - 1}{6} skew + \frac{z^3 - 3z}{24} kurt - \frac{2z^3 - 5z}{36} skew^2 + \ldots \right) \times stdev$$

$$F^{-1}(0.995) = mean + (2.5758 + 0.9391 \times skew + 0.3901 \times kurt - 0.5917 \times skew^2 + \ldots) \times stdev$$

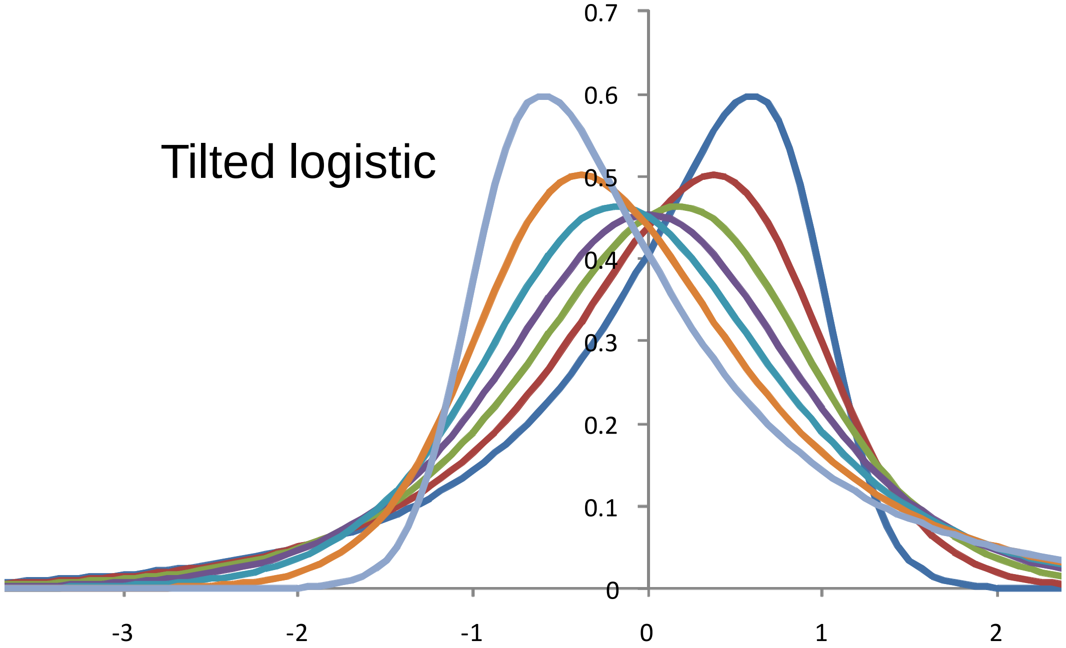
We want to modify this based on a data sample.
Inspired by Cornish-Fisher, we try expressions of the form

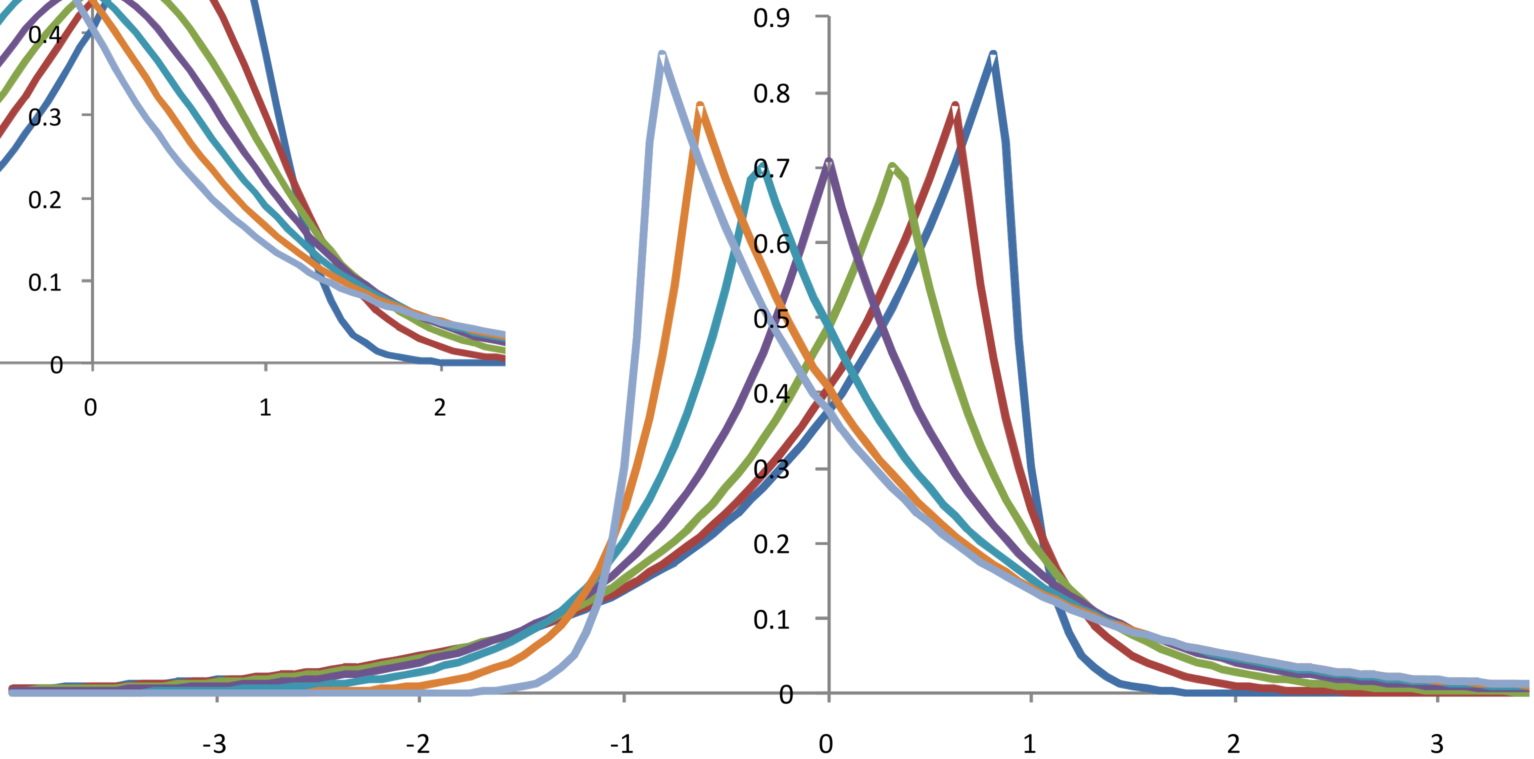
$$Q = mean + (w_2 + w_3 \times skew + w_4 \times kurt - w_5 \times skew^2 + \ldots) \times stdev$$

Use classical moment estimators for mean, stdev etc rather than "true" parameters.
Special case: "multiple of standard deviation" rule: $w_3 = w_4 = w_5 = 0$
The coefficients w_j depend on confidence level which we have set at 99.5%.

Our H_0: Normal, Logistic, Laplace and Esscher

Distributions are all standardised to mean=0,stdev=1

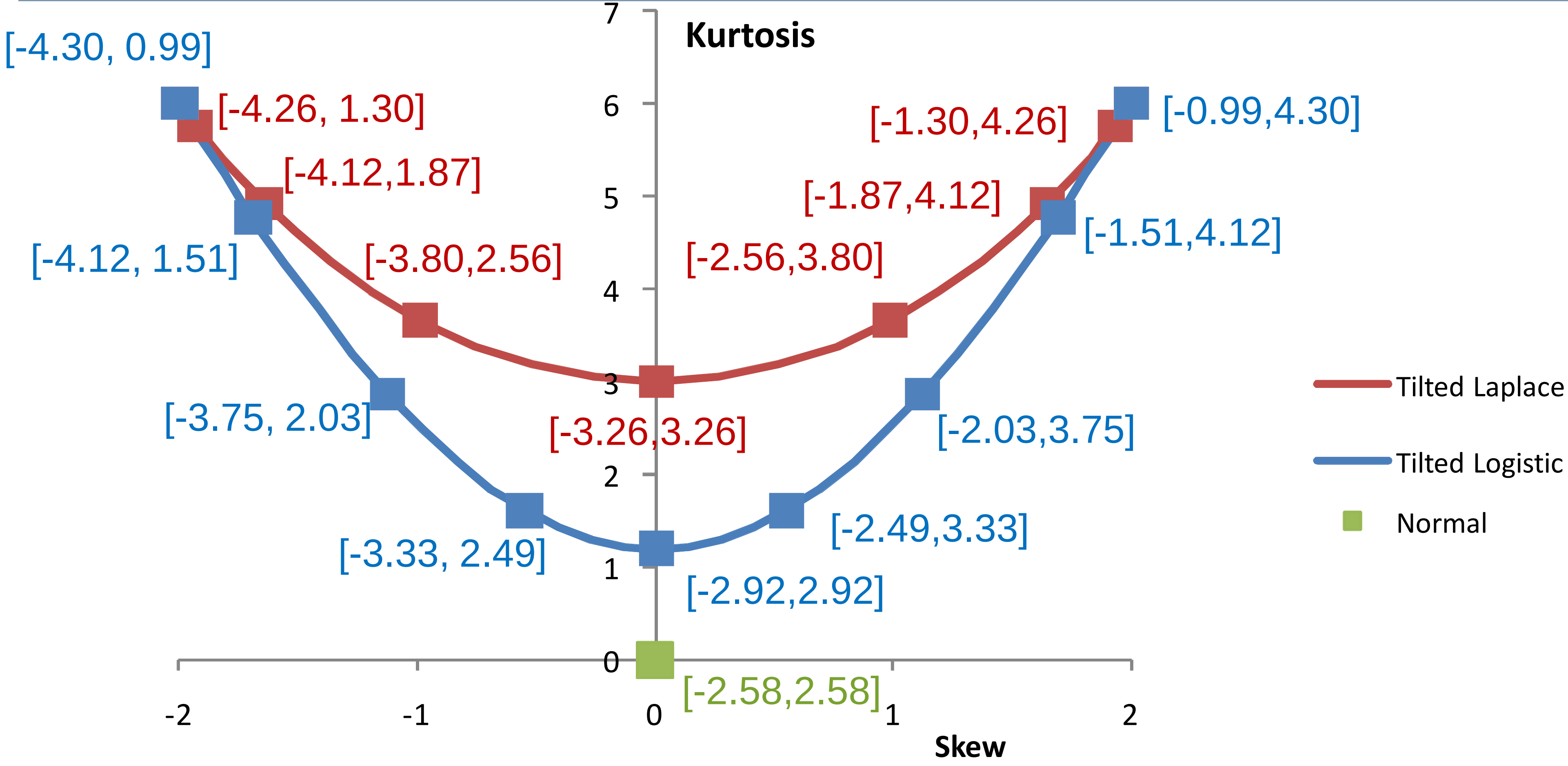




Esscher tilting parameter can take values in (-1,1). We consider multiples of ¼.

If there were no model error or estimation error ...

"Process error" showing 0.5%-ile and 99.5%-ile of standardised distribution



Combining Models with Model-Agnostic Forecasts

- Inputs
  - Probability law P_i in H_0
  - Training random sample $x_1, x_2, \ldots x_t$
  - Quantile forecast $Q(x_1, x_2, \ldots x_t)$: this is a random variable
  - Next observation x_{t+1} independent of history
- Feasibility constraints
  - $P_i\{x_{t+1} \leq Q\}$ is at least $\alpha = 0.995$, Q is stochastic
  - We might like equality for all H_0 but if H_0 is large we cannot easily achieve this
  - The difference is a (new) measure of model risk

Feasible Quantile Forecasts for a Finite Null Hypothesis

- The modified quantile forecast is

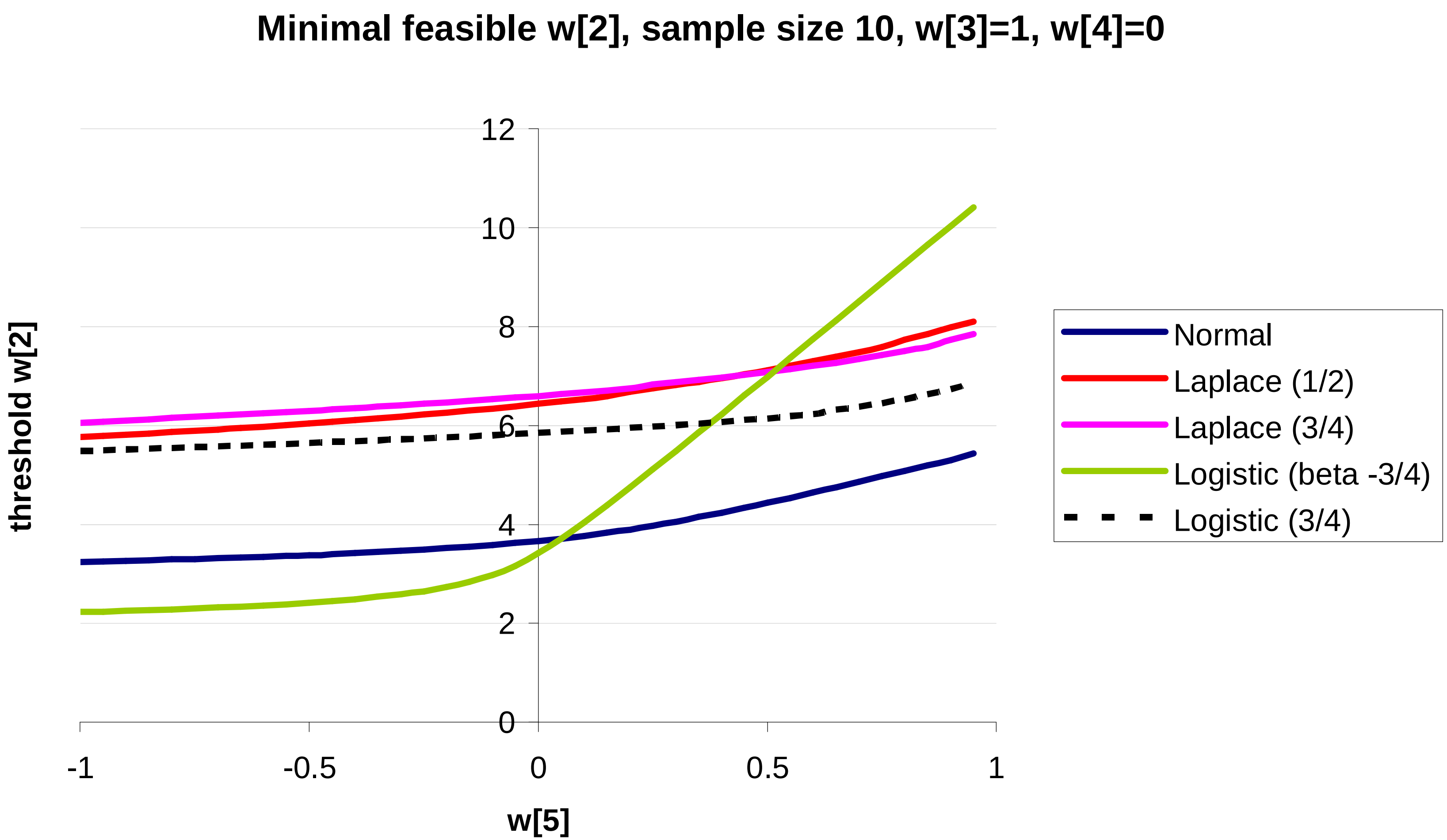
$$Q = mean + \left(w_2 + w_3 \times skew + w_4 \times kurt - w_5 \times skew^2 + \ldots\right) \times stdev$$

- Let us fix w_3, w_4, w_5 and consider w_2
- Under law P_i, set w_2 as the α-quantile of

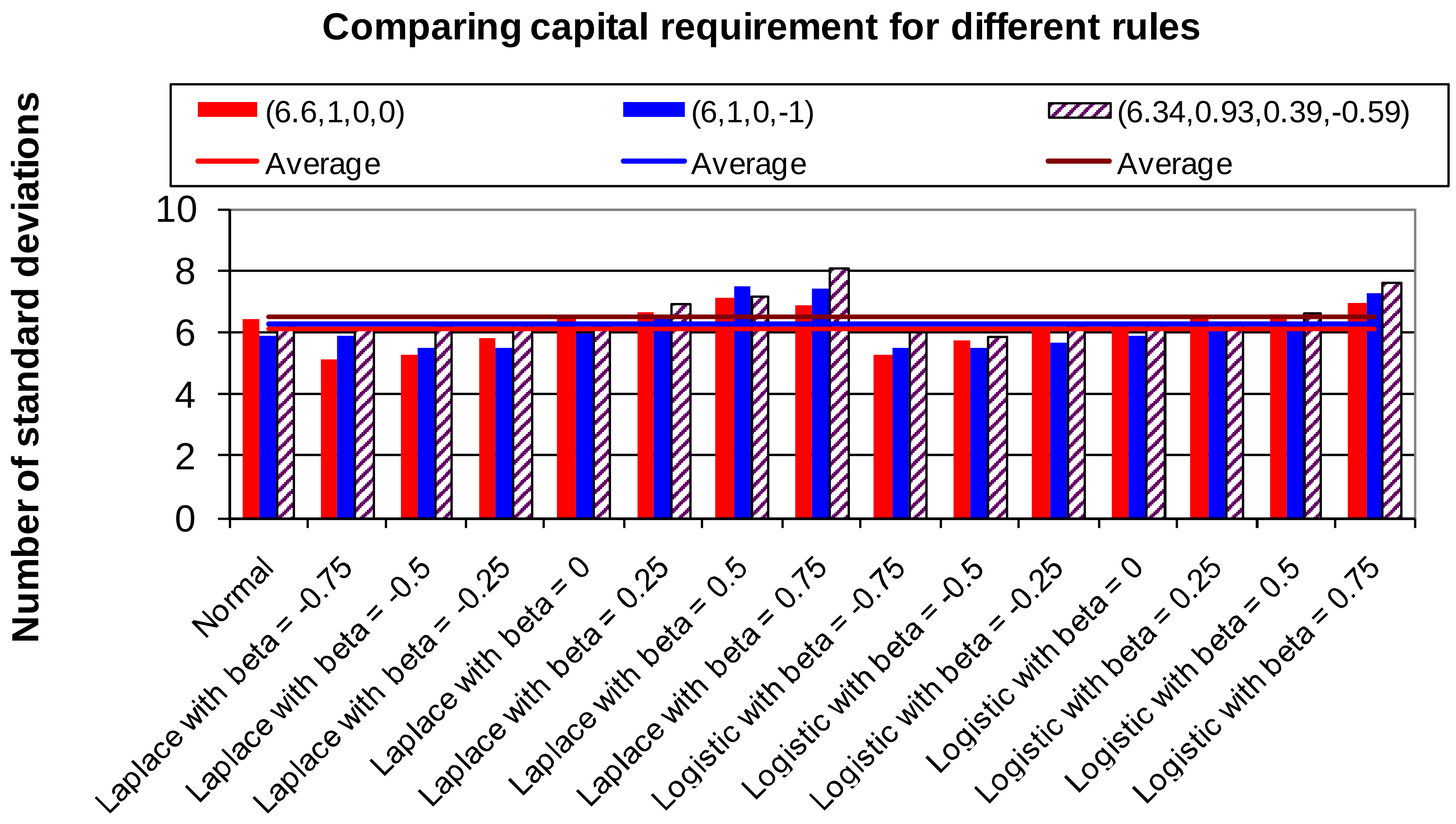
$$\frac{x_{t+1} - mean}{stdev} - w_3 \times skew - w_4 \times kurt + w_5 \times skew^2$$

- Then take the largest w_2 across all the P_i
- The resulting $\{w_2, w_3, w_4, w_5\}$ satisfy the feasibility condition
- $P_i\{x_{t+1} \leq Q\}$ is at least $\alpha = 0.995$ for all i

Monte Carlo Investigation: Sample Size = 10



Trade off model risk and estimation risk



What is a “good” Quantile Forecast?

- Quantile estimates drive capital
- Capital has a cost: to minimise that cost, minimise **E**(Q)

Extreme Quantile
Feasibility
Tower Law
Jensen Effect

$$\mathbf{P}\{x_{t+1} \le \mathbf{E}_i(x_{t+1})\} \le \alpha \le \mathbf{P}_i\{x_{t+1} \le Q\} = \mathbf{E}_i\{F_i(Q)\} \le F_i\{\mathbf{E}_i(Q)\}$$

Inverse cdf F_i^{-1}

$$\mathbf{E}_i(x_{t+1}) \le F_i^{-1}(\alpha) \le F_i^{-1}(\mathbf{P}_i\{x_{t+1} \le Q\}) \le \mathbf{E}_i(Q)$$

process error
Frankland et al (2008)

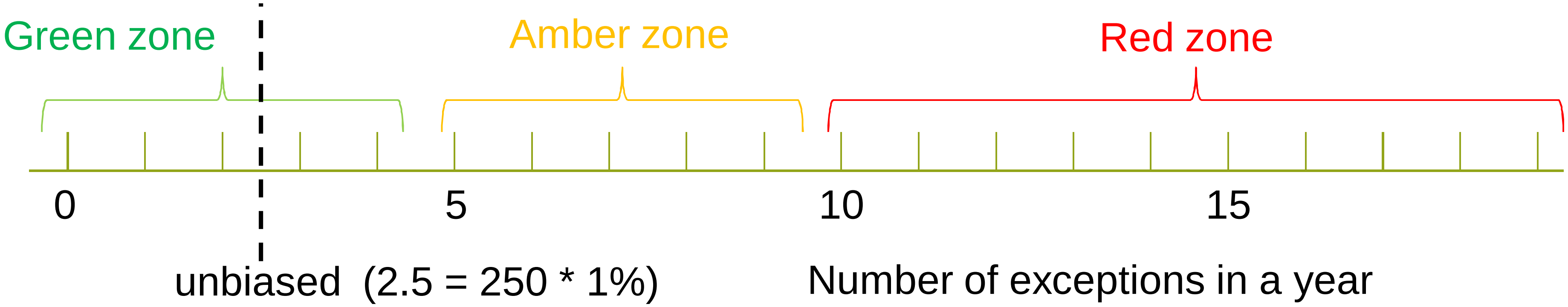
model error

estimation error
Gerrard & Tsanakas (2009)

Expected capital required under $\mathbf{P}_i$

Bank Model Validation under Basel
How do you know your model is right?

- Banks have different rules:10 day VaR at 99% Confidence
  – Look back over last year (250 trading days, overlapping periods each looking 10 days back) in which both VaR and profit are updated



- What does this process test?
  – The "back test" includes implicit tests of model and parameter error as well as outcomes
  – Although it won't test risks that didn't materialise in the last year

Validation Approaches in Insurance

Three Approaches that Don't Work

A. Check the documentation and formulas against best statistical practice.

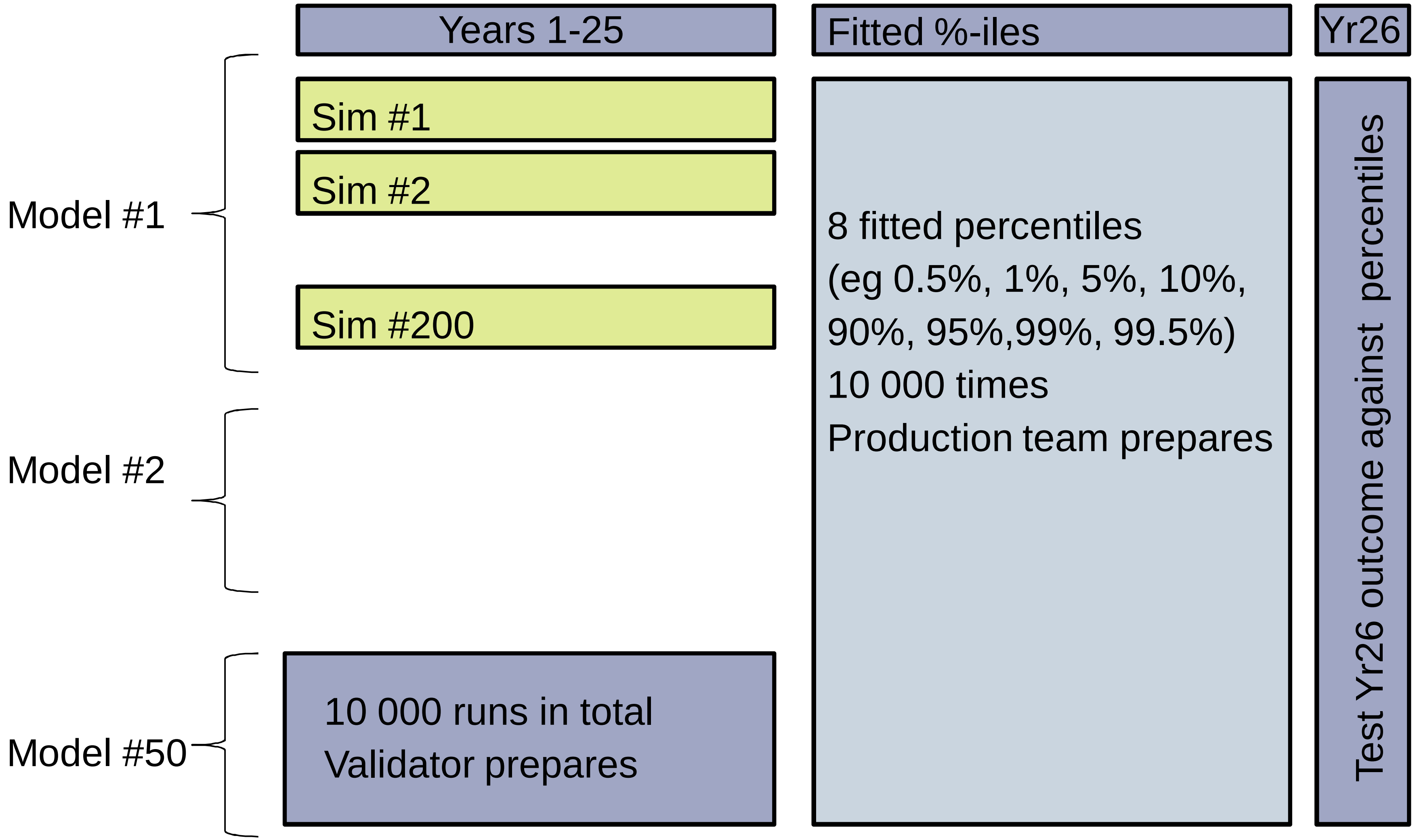
B. Compare insurers, and, (as with ICAS) invite insurers with the most aggressive assumptions to reconsider them on a risk by risk basis.

C. Require back-testing as with VaR models under Basel.

- But documentation is very lengthy and there's a shortage of real experts to conduct in-depth reviews.
- But now more difficult because you are comparing probability distribution forecasts rather than stress tests so its not clear who is being most prudent. This is only a test of relative numerical conformity rather than confirmation of the 1-in-200 standard.
- Under Basel II, VaR is calculated at 99% confidence over 10 days. Allowing overlapping intervals and 250 trading days over a year, a correct model should produce 2.5 exceptions. Based on 1-year 99.5% VaR, you would need 500 years of test data for insurers

Monte Carlo Calibration Test

Is this a practical test of quantile estimates?



Conclusions and Discussion

- We interpret EU legal definition of “Solvency Capital Requirements” in terms of forecast profit/loss percentiles
- Forecasts need to take account of
  - Process error (widely investigated and understood)
  - Estimation error (there is much published research but currently rarely applied in practice)
  - Model error (we propose a new probability approach to this difficult problem)
- Rather than defending individual model parameters, it is better to construct a good process and then audit that the process has been followed.