# Valuation with meromorphic Lévy processes


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[1] Part of this talk is joint work with Alexey Kuznetsov, Andreas Kyprianou and Juan Carlos Pardo

# Outline

1. Setting & two types of problems
2. Short intro into meromorphic Lévy processes (mLP's)
3. A new Monte Carlo simulation technique for (a.o.) ruin problems with mLP's
4. An approximation technique for American option pricing with mLP's

# Outline

## What are Lévy processes?

- A (one dimensional, real valued) Lévy process $X$ is a Markov process starting from $0$ with:
    - stationary, independent increments (like Brownian motion)
    - paths $t \mapsto X_t$ which are right continuous and have left limits (i.e. jumps are possible)

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  where $\text{CPP}_k$'s independent compound Poisson processes
- Jump structure described by Lévy measure $\Pi$:

$$\Pi(A) = \mathbb{E}[\#\{t \in [0,1] \,|\, \Delta X_t \in A\}] \text{ for Borel sets } A \subset \mathbb{R}.$$

  Note: $\#\{t \in [0,1] \,|\, \Delta X_t > \varepsilon\} < \infty$, yet

$$\sum_{t\in[0,1],\ \Delta X_t \leq \varepsilon} |\Delta X_t| \text{ might be infinite!}$$

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  where $\text{CPP}_k$'s independent compound Poisson processes
- Law of $X$ determined by characteristic exponent $\Psi$:

$$\begin{aligned}\Psi(\theta) &:= -\frac{1}{t}\log \mathbb{E}[e^{i\theta X_t}] \\ &= ai\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|\leq 1\}})\Pi(dx)\end{aligned}$$

  where $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $\Pi$ Lévy measure

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- Recall classic Cramer-Lundberg model. Imagine an insurance company with:
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- Quantities of interest (a.o.): ruin time and overshoot (=deficit at ruin):

$$T_{\text{ruin}} = \inf\{t > 0 \,|\, X_t < 0\} \quad \text{and} \quad X_{T_{\text{ruin}}}$$

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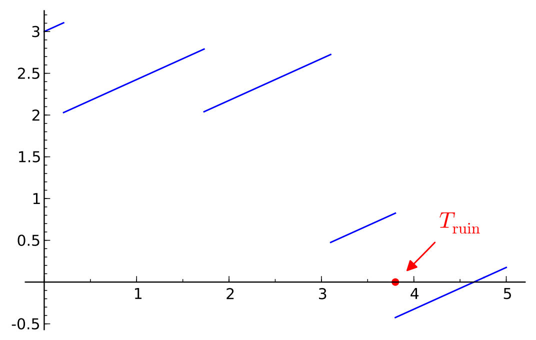


A typical path of $X$ with $x = 3$, $b = 1/2$, $Y_i \sim \exp(1)$

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  - much more data fitting possibilities
  - more complicated jump structure (incoming claims)
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  - Brownian part: aggregated 'high intensity' small premiums/claims (might be set to $0$ as well)
- In this talk: introduce a new simulation method (so-called Wiener-Hopf Monte Carlo simulation method) for a.o. ruin time and overshoot which can be used with meromorpic Lévy processes e.g.

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- Important question: what is the price of an American option in this model? That is, how to find

$$V(T,x) = \sup_{\tau} \mathbb{E}_x[e^{-r(\tau\wedge T)} f(X_{\tau\wedge T})]$$

where $\tau$ a stopping time, $T > 0$ expiry date, $r > 0$ interest rate, $f$ the payoff function, $\mathbb{P}_x$ means $X$ starts from $x$

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  - classic example: American put with $f(x) = (K - e^x)^+$
- In this talk: introduce an algorithm to approximate $V(T,x)$ which can be used with meromorpic Lévy processes e.g.

# Outline

## Meromorphic Lévy processes (mLP's)

$$\Psi(\theta) = -\log \mathbb{E}[e^{\mathrm{i}\theta X_t}]/t$$

- A mLP $X$ is defined as having:
  - a meromorphic $\Psi$ with poles in $\{-\mathrm{i}\rho_n, \mathrm{i}\hat{\rho}_n\}_{n\geq 1}$
  - for all $q \geq 0$, $\theta \mapsto q + \Psi(\theta)$ has roots in points $\{-\mathrm{i}\zeta_n, \mathrm{i}\hat{\zeta}_n\}_{n\geq 1}$
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  - plus some additional technicalities
- Which is essentially equivalent to $X$ having any $\sigma$, any $a$ and Lévy measure $\Pi$ of the form

$$\Pi(dx) = \left( \mathbf{1}_{\{x<0\}} \sum_{n\geq 1} \hat{c}_n e^{\hat{\rho}_n x} + \mathbf{1}_{\{x>0\}} \sum_{n\geq 1} c_n e^{-\rho_n x} \right) dx$$

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- Very broad class! One more explicit example is the $\beta$-class with Lévy measure of the form

$$\Pi(dx) = \left( \mathbf{1}_{\{x<0\}} \gamma_1 \frac{e^{\alpha_1 \beta_1 x}}{(1-e^{\beta_1 x})^{\lambda_1}} + \mathbf{1}_{\{x>0\}} \gamma_2 \frac{e^{-\alpha_2 \beta_2 x}}{(1-e^{-\beta_2 x})^{\lambda_2}} \right) dx$$

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- With $\mathbf{e}(q)$ exp. distributed with mean $1/q$, indep. of $X$ and

$$\overline{X}_t = \sup_{s \le t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{s \le t} X_s$$

we have

$$\phi_q^+(\mathrm{i}z) := \mathbb{E}\left[e^{-z\overline{X}_{\mathbf{e}(q)}}\right] = \prod_{n \ge 1} \frac{1 + z/\rho_n}{1 + z/\zeta_n},$$

$$\phi_q^-(-\mathrm{i}z) := \mathbb{E}\left[e^{z\underline{X}_{\mathbf{e}(q)}}\right] = \prod_{n \ge 1} \frac{1 + z/\hat{\rho}_n}{1 + z/\hat{\zeta}_n}$$

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- Fourier inversion may be applied to deduce the laws of $\overline{X}_{\mathbf{e}(q)}$ and $\underline{X}_{\mathbf{e}(q)}$. For example, in the $\beta$-class we get

$$\mathbb{P}(\overline{X}_{\mathbf{e}(q)} \in dx) = \left(\sum_{n\ge1} k_n \zeta_n e^{\zeta_n x}\right) dx$$

and similar for $\underline{X}_{\mathbf{e}(q)}$.

## Main points about mLP's

- It is a broad class of Lévy processes of which we know both their characteristics $\sigma$, $a$, $\Pi$ and the laws of $\overline{X}_{\mathbf{e}(q)}$ and $\underline{X}_{\mathbf{e}(q)}$. (This is not common)

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- Knowing the laws of $\overline{X}_{\mathbf{e}(q)}$ and $\underline{X}_{\mathbf{e}(q)}$ is used in the methods in this talk. Note: as

$$X_{\mathbf{e}(q)} \stackrel{d}{=} \underline{X}_{\mathbf{e}(q)} + \overline{X}_{\mathbf{e}(q)}$$

  the law of $X_{\mathbf{e}(q)}$ is a mixture of exponentials on the negative and positive axis

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## The Wiener-Hopf Monte Carlo simulation technique (WHMC)

- Let $X$ be an mLP and let $I_q$ and $S_q$ be independent rv's with

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- From Wiener-Hopf theory we know:

$$X_{\mathbf{e}(q)} \stackrel{d}{=} S_q + I_q$$

and as a consequence

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- Q2: how can we extend this to obtain (approximate) samples from quantities like

$$T_{\text{ruin}} = \inf\{t > 0 \,|\, X_t < 0\} \quad \text{and} \quad X_{T_{\text{ruin}}}$$

for our ruin problem?

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Idea[2]: use a 'stochastic time grid'. That is to say:

- Let

$$\mathbf{g}(n,q) := \sum_{i=1}^{n} \mathbf{e}^{(i)}(q) \text{ for } n \geq 0,$$

  where $(\mathbf{e}^{(i)}(q))$ iid, $\mathbf{e}^{(i)}(q) \sim \exp(q)$

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- Idea:
  1. we can approximate any $t > 0$ by setting $q = n/t$ and let $n \to \infty$. Indeed by the law of large numbers we have $\mathbf{g}(n, n/t) \to t$ a.s. as $n \to \infty$
  2. exploit the homogeneity of $X$ to use (1) above on all grid intervals $[\mathbf{g}(k-1,q), \mathbf{g}(k,q)]$ to get an expression for $(X_{\mathbf{g}(n,q)}, \overline{X}_{\mathbf{g}(n,q)})$

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$$\mathbf{g}(n,q) = \sum_{i=1}^{n} \mathbf{e}^{(i)}(q) \text{ for } n \geq 0$$

More precisely, let $(I_q^{(i)})$ (resp. $(S_q^{(i)})$) be iid copies of $I_q$ (resp. $S_q$).

Then for $n = 2$:

$$\begin{aligned}
(X_{\mathbf{g}(2,q)}, \overline{X}_{\mathbf{g}(2,q)}) &= (X_{\mathbf{g}(2,q)}, \max\{\overline{X}_{0,\mathbf{g}(1,q)}, \overline{X}_{\mathbf{g}(1,q),\mathbf{g}(2,q)}\}) \\
&\stackrel{(d)}{=} (X_{\mathbf{g}(1,q)} + Y_{\mathbf{g}(1,q)}, \max\{\overline{X}_{0,\mathbf{g}(1,q)}, X_{\mathbf{g}(1,q)} + \overline{Y}_{\mathbf{g}(1,q)}\})
\end{aligned}$$

↑ by stat. indep. incr., $Y$ indep. copy of $X$

$$\stackrel{(d)}{=} \left(I_q^{(1)} + S_q^{(1)} + I_q^{(2)} + S_q^{(2)}, \max\left\{S_q^{(1)}, I_q^{(1)} + S_q^{(1)} + S_q^{(2)}\right\}\right)$$

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$$\mathbf{g}(n,q) = \sum_{i=1}^{n} \mathbf{e}^{(i)}(q) \text{ for } n \geq 0$$

Generalised for any $n \geq 1$ we get:

$$(X_{\mathbf{g}(n,q)}, \overline{X}_{\mathbf{g}(n,q)}) \stackrel{d}{=} (V(n,q), J(n,q))$$

where $V(0,q) = J(0,q) = 0$,

$V(n,q) := V(n-1,q) + S_q^{(n)} + I_q^{(n)}$ and $J(n,q) := \max\{J(n-1,q), V(n-1,q) + S_q^{(n)}\}$.

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Using that $\mathbf{g}(n, n/t) \to t$ a.s. our first **main result** follows:

$$(V(n,n/t), J(n,n/t)) \stackrel{d}{\to} (X_t, \overline{X}_t) \text{ as } n \to \infty$$

**Hence**, for $X$ an mLP, we can produce $N$ samples from $(V(n,n/t), J(n,n/t))$ for big $n, N$ and use

$$\mathbb{E}[f(X_t, \overline{X}_t)] \approx \frac{1}{N} \sum_{i=1}^{N} f(V^{(i)}(n,n/t), J^{(i)}(n,n/t))$$

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- Using the same arguments as before it follows that

$$\Big((X_{\mathbf{g}(0,n/t)}, \overline{X}_{\mathbf{g}(0,n/t)}), \ldots, (X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)})\Big) \stackrel{d}{=} \Big((V(0,n/t), J(0,n/t)), \ldots, (V(n,n/t), J(n,n/t))\Big)$$

## The Wiener-Hopf Monte Carlo simulation technique (WHMC)

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- Using that we may write $\tau_u = \inf\{t > 0 \,|\, \overline{X}_t > u\}$, with the definitions

$$\hat{k}^{(n)} := \inf\{k \in \{0,\ldots,n\} \,|\, \overline{X}_{\mathbf{g}(k,n/t)} > u\} \text{ and}$$

$$\kappa^{(n)} := \inf\{k \in \{0,\ldots,n\} \,|\, J(k,n/t) > u\}$$

it can be shown that

$$\tau_u \wedge t = \inf\{t > 0 \,|\, \overline{X}_t > u\} \wedge t \approx \frac{t}{n}(\hat{k}^{(n)} \wedge n) \stackrel{d}{=} \frac{t}{n}(\kappa^{(n)} \wedge n)$$

and similarly

$$X_{\tau_u \wedge t} \approx X_{\mathbf{g}(\hat{k}^{(n)}, n/t)} \stackrel{d}{=} V(\kappa^{(n)} \wedge n, n/t)$$

## The Wiener-Hopf Monte Carlo simulation technique (WHMC)

- Hence we arrive at

$$\left(\frac{t}{n}(\kappa^{(n)} \wedge n), V(\kappa^{(n)} \wedge n, n/t)\right) \xrightarrow{d} (\tau_u \wedge t, X_{\tau_u \wedge t} - u)$$

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- If the technicalities are confusing, recall in practice all you need to do is:
  - pick a mLP
  - write down the corresponding $I_q$ and $S_q$
  - for the path functional that you are interested in, write down the corresponding expression in terms of $V(., n/t)$ and $J(., n/t)$
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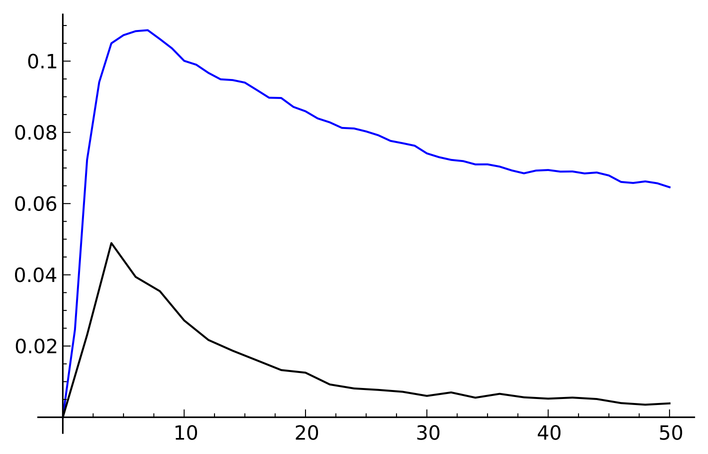
  - write a bit of computer code to run your Monte Carlo simulations
- Advantages this method over standard Monte Carlo (i.e. a random walk approach)?
  - standard MC requires knowing the law of $X_h$, in general not available & numerical Fourier inversion of $\Psi$ necessary
  - standard MC is well known to perform poorly for quantities involving running maximum/first hitting times

## Example of WHMC simulation technique

Error (exact value - simulated value) for the simulation of $t \mapsto \mathbb{P}(\tau_1 \leq t)$ where $X$ is a BM, blue for standard Monte Carlo and black for WHMC




Here $n = 25$ and we used $10^4$ samples

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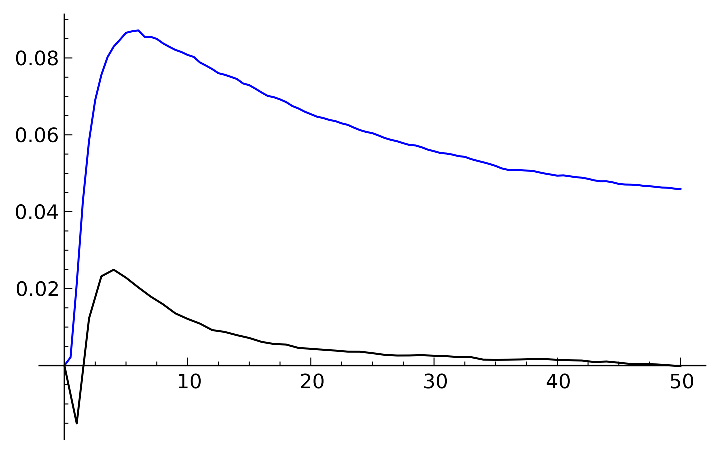


Here $n = 50$ and we used $10^5$ samples

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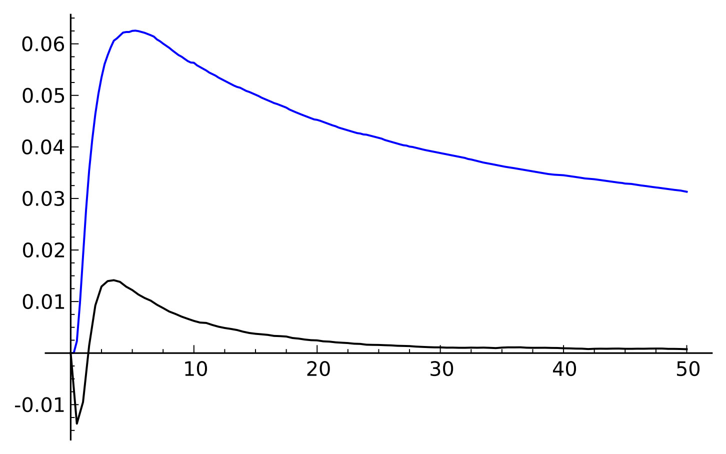


Here $n = 100$ and we used $10^6$ samples

# Outline

## American option pricing with mLP's

- Recall the problem: $f$ is a cts. and bounded payoff function. Determine for $T > 0$ and $x > 0$ the value function $V$:

$$V(T,x) = \sup_{\tau} \mathbb{E}_x \left[ e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right]$$

  where $\tau$ is a stopping time. I.e. $V$ is the value function of the American option with payoff function $f$ in a market driven by a mLP $X$.

[3] related to Canadisation technique by Peter Carr

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- Idea[3]:
  - consider again the 'stochastic grid' from before:

$$0 = \mathbf{g}(0, n/T) < \mathbf{g}(1, n/T) < \ldots < \mathbf{g}(n, n/T)$$

    and recall that by the law of large numbers $\mathbf{g}(n, n/T) \to T$ as $n \to \infty$
  - perform in a clever way backwards induction over this grid, relying on $\mathbf{g}(k, n/T) - \mathbf{g}(k-1, n/T) \sim \exp(n/T)$ and the fact that we know the law of $X$ at exp. distributed times

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- In order to make this backwards induction as nice as we can we need to
  - allow only for stopping *at* the grid points $\{0 = \mathbf{g}(0, n/T), \mathbf{g}(1, n/T), \ldots, \mathbf{g}(n, n/T)\}$ (i.e. not in between)
  - adjust the discounting factor

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## Setup of the algorithm

$$V(T,x) = \sup_{\tau} \mathbb{E}_x \left[ e^{-r(\tau\wedge T)} f(X_{\tau\wedge T}) \right], \quad \mathbf{g}(k, n/T) := \sum_{i=1}^{k} \mathbf{e}^{(i)}(n/T)$$

In formulae this reads as follows.

- $$\sigma_{\mathsf{discr}} \in \{0 = \mathbf{g}(0, n/T), \mathbf{g}(1, n/T), \ldots, \mathbf{g}(n, n/T)\}$$

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  is the new 'averaged' discount factor
- We define for $n \geq 1$ and $k = 0, \ldots, n$ the functions $v_k^{(n)}$:
  $$v_k^{(n)}(x) = \sup_{\sigma_{\text{discr}}} \mathbb{E}_x \left[ D(\sigma_{\text{discr}} \wedge \mathbf{g}(k, n/T)) f(X_{\sigma_{\text{discr}} \wedge \mathbf{g}(k,n/T)}) \right],$$
  i.e. (compare to $V$ above) the optimal value if you are only allowed to stop at the first $k$ grid points and with adjusted discount factor

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This is a useful setup because:

- We can prove that $v_n^{(n)}(x) \to V(T,x)$ as $n \to \infty$. Intuition: $\mathbf{g}(n, n/T) \to T$ and the stochastic grid becomes more and more dense in $[0,T]$ as $n \to \infty$

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$$v_0^{(n)}(x) = f(x), \quad v_k^{(n)}(x) = \max\left\{ f(x), e^{-r/n} \mathbb{E}_x \left[ v_{k-1}^{(n)}(X_{\mathbf{e}(n/T)}) \right] \right\}$$

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- which we can use to find pretty explicit formulae for the $v_k^{(n)}$'s! (Thanks to the fact that when $X$ is an mLP the law of $X_{\mathbf{e}(n/T)}$ is a mixture of exponentials on the negative and positive axis).

# Practical example

$$V(T,x) = \sup_{\tau} \mathbb{E}_x \left[ e^{-r(\tau \wedge T)} f(X_{\tau \wedge T}) \right], \quad \mathbf{g}(k, n/T) := \sum_{i=1}^{k} \mathbf{e}^{(i)}(n/T)$$

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- Suppose that $f(x) = (K - e^x)^+$ (classic American put) and $X$ an mLP. Then the algorithm (1) above yields the following structure for any $n \geq 1$ and $k = 0, \ldots, n$:

$$v_0^{(n)}(x) = (K - e^x)^+, \quad \text{for } k = 1, \ldots, n:$$

$$v_k^{(n)}(x) = \begin{cases} (K - e^x)^+ & \text{if } x \leq x_k^{(n)} \\ \text{messy lin. comb. of terms of the form } a x^b e^{cx} & \text{if } x > x_k^{(n)} \end{cases}$$

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- Terrible formulae, but quite explicit (and hence computer friendly, yet computer implementation is still work in progress)

## References


- *Randomisation and the American put.* Peter Carr
- *Meromorphic Lévy processes and their fluctuation identities.* Andreas Kyprianou, Alexey Kuznetsov & Juan Carlos Pardo
- *A Wiener-Hopf Monte Carlo simulation technique for Lévy process.* Andreas Kyprianou, Alexey Kuznetsov, Juan Carlos Pardo & KvS
- *A note on extending the Wiener-Hopf Monte Carlo simulation technique for Lévy process to hitting times and overshoots.* KvS (In progress)
- *A variation of the Canadisation algorithm for optimal stopping with Lévy processes.* KvS (In progress)